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Some results in viscoelasticity theory via a simple perturbation argument

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Some Results in Viscoelasticity Theory Via a Simple Perturbation Argument.

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1. Introduction.

The goal of this note is to explain a perturbation argument contained in [4] and to show how it provides existence and uniqueness for the theory of viscoelasticity « without initial conditions ».

Let Ω be a smooth, bounded open set of $\boldsymbol{R}^n$, $(n\geqslant 1)$. The equations of motion of a linear viscoelastic body are given by:

$$(1.1)\quad \varrho(x)(u^i)_{tt}-\frac{\partial}{\partial x_j}\left\{A_{ijkl}(x)\frac{\partial u^k}{\partial x_l}-\int_{-\infty}^{t}G_{ijkl}(x,t-s)\frac{\partial u^k}{\partial x_l}(x,s)\,ds\right\}=f_i(x,t)$$

$$\forall i=1,2,\ldots,n\,,\ \forall t\in(-\infty,0)\,,\quad \text{a.e. } x \text{ in } \Omega$$

together, for instance, with the boundary condition:

$$(1.2)\qquad u(x,t)=\big(u_1(x,t),u_2(x,t),\ldots,u_n(x,t)\big)=0\,,\qquad x\in\partial\Omega\,,\ t\in(-\infty,0)\,.$$

Here $n=3$, but the theory applies for any n. u is the displacement of the body and $f=(f_1,f_2,\ldots,f_n)$ the applied external force

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which we suppose to be given. For simplicity, we choose to study the motion up to time $t = 0$, of course, our results would apply up to any time T.

This kind of problem has been attacked by numerous people (see for instance [1], [4], [6], [10], [12], [15], [18]). Usually, however, the displacement u is supposed to be known up to some fixed time $t_0 < 0$. So, in this case, if we transfer the term

$$\frac{\partial}{\partial x_j}\left\{\int_{-\infty}^{t_0} G_{ijkl}(x, t-s)\frac{\partial u^k}{\partial x_l}(x, s)\,ds\right\}$$

to include it into f, the problem reads:

$$(1.3)\qquad \varrho(x)(u^i)_{tt}-\frac{\partial}{\partial x_j}\left\{A_{ijkl}(x)\frac{\partial u^k}{\partial x_l}-\int_{t_0}^{t} G_{ijkl}(x, t-s)\frac{\partial u^k}{\partial x_l}(x, s)\,ds\right\} = f_i(x, t)$$

$$\forall i = 1, 2, \ldots, n\,,\quad \forall t \in (t_0, 0)\,,\ \text{a.e. } x \text{ in } \Omega\,.$$

Of course together with (1.3), (1.2), u is prescribed at time t_0. One has then to solve an initial value problem (see [6]).

This approach has some physical limitation and considering the fading memory of the kernel this may lead to prefer the formulation (1.1), (1.2). In [10], [12] Fichera studied the quasi-static version of this problem, i.e. the case where ϱ equal 0 in (1.1). We would like to investigate here the dynamical case.

First, we will transform the problem, rewriting it in an abstract setting. The main ingredients to do so will be explained in section 2 where we will solve the problem in the case $G = 0$. In section 3 we will explain our perturbation argument. Finally, in section 4 we will apply our results to (1.1), (1.2).

Note that our solution u, as well as our external force f, will be assumed to have an exponential decay at $-\infty$. This will guarantee existence and uniqueness. The necessity of such a decay has been established in [4]. Indeed, considering a unidimensional version of (1.1), (1.2), one can show that uniqueness may fail, for instance for $f = 0$, if the decay of the solution is not supposed to be strong enough. We refer the reader to [4] for details.

If V is a Banach space, we denote by $W^{1,\infty}((-\infty, 0); V)$, $W^{2,\infty}$ $\cdot((-\infty, 0); V)$ the set of functions defined on $(-\infty, 0)$ with values in V, one or two times differentiable in the distributional sense with derivatives in $L^\infty((-\infty, 0); V)$. All the equations that we will write involving such functions are, of course, supposed to hold in the distributional sense and almost everywhere.

2. An abstract existence and uniqueness result.

Let V and H two Hilbert spaces, $V', H' \simeq H$ their dual spaces. We denote by $\| \ \|$ the norm on V, by $(\, , \,)$ the scalar product on H and by $| \ |$ its norm. $\langle \, , \, \rangle$ denotes the duality bracket between V' and V. Moreover, we suppose that

(2.1) $$V \subset H \simeq H' \subset V' ,$$

V being dense in H and the canonical embedding from V into H being completely continuous. In particular, for some constant c one has

(2.2) $$c \cdot |v| \leqslant \|v\| \qquad \forall v \in V .$$

A typical situation is for instance when

$$V = H_0^1(\Omega) , \qquad H = L^2(\Omega)$$

and Ω is a smooth, bounded open set of $\boldsymbol{R}^n$, $(n \geqslant 1)$. (We refer the reader to [13] for definitions and properties of Sobolev spaces).

Let $a(u, v)$ be a bilinear, symmetric, continuous, coercive form on V, i.e. such that:

(2.3) $$|a(u, v)| \leqslant \beta \|u\| \cdot \|v\| \qquad \forall u, v \in V ,$$

(2.4) $$\alpha \|u\|^2 \leqslant a(u, u) \qquad \forall u \in V , \ \alpha > 0 .$$

Clearly, a defines a unique operator A from V into V' through the formula:

$$a(u, v) = \langle Au, v \rangle \qquad \forall u, v \in V .$$

If we denote by D_A the set

(2.5) $$D_A = \{v \in V : Av \in H\}$$

then, A can be viewed as an unbounded operator from D_A into H and one has, when H is identified with H':

$$(2.6) \qquad a(u, v) = (Au, v) \quad \forall u \in D_A, \ \forall v \in V.$$

Moreover, it is well known that D_A is a Hilbert space when endowed with the norm

$$(|Au|^2 + |u|^2)^{\frac{1}{2}}.$$

In fact, we will find more convenient to use on D_A the norm $|Au|$. These two norms are equivalent on D_A. Indeed, by (2.2), (2.4), (2.6) one has

$$c\cdot\alpha\cdot|u|^2 \leqslant \alpha\cdot\|u\|^2 \leqslant (Au, u) \leqslant |Au|\cdot|u| \qquad \forall u \in D_A.$$

Thus:

$$c\cdot\alpha\cdot|u| \leqslant |Au|$$

from which it results that

$$|Au| \leqslant (|Au|^2 + |u|^2)^{\frac{1}{2}} \leqslant C\cdot|Au|.$$

for some constant C.

DEFINITION. If V is a Banach space, normed by $|\ |$, for $\nu > 0$ we denote by $L^\infty_\nu((-\infty, 0); V)$ the set of bounded, measurable functions v from $(-\infty, 0)$ into V such that for some positive constant C:

$$|v(t)| \leqslant C\cdot\exp[\nu t] \qquad \text{a.e. } t \in (-\infty, 0).$$

Moreover, we define on $L^\infty_\nu((-\infty, 0); V)$ the norm

$$|v|_\nu = \operatorname*{ess\,sup}_{t\in(-\infty,0)} \{|v(t)|\cdot\exp[-\nu t]\}.$$

It is easy to see that $L^\infty_\nu((-\infty, 0); V)$, when equiped with this norm, is a Banach space.

LEMMA 1. Assume $\lambda > 0$. For $f \in L^\infty_\nu((-\infty, 0); \boldsymbol{R})$, there exists a unique $\alpha \in W^{2,\infty}((-\infty, 0); \boldsymbol{R})$ such that

$$(2.7) \qquad \alpha_{tt} + \lambda\cdot\alpha = f \quad \text{in } (-\infty, 0), \qquad \alpha \in L^\infty_\nu((-\infty, 0); \boldsymbol{R}).$$

(The first equation holds in the distributional sense and almost everywhere). Moreover one has:

$$\alpha,\ \alpha_t,\ \alpha_{tt} \in L^\infty_\nu((-\infty, 0); \boldsymbol{R})\,. \tag{2.8}$$

PROOF. Set

$$\alpha(t) = \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{t} \sin\sqrt{\lambda}(t-s)\, f(s)\, ds\,. \tag{2.9}$$

First, due to our decay assumption on f, this integral makes sense. Indeed:

$$|\sin\sqrt{\lambda}(t-s)\, f(s)| \leqslant |f(s)| \leqslant C\cdot\exp[\nu s] \tag{2.10}$$

and this last function is integrable on $(-\infty, 0)$. Moreover, by (2.9), (2.10):

$$|\alpha(t)| \leqslant \frac{C}{\sqrt{\lambda}} \int_{-\infty}^{t} \exp[\nu s]\, ds \leqslant \frac{C}{\sqrt{\lambda}\cdot\nu} \exp[\nu t]$$

and thus $\alpha \in L^\infty_\nu((-\infty, 0); \boldsymbol{R})$. Differentiating twice, one gets:

$$\alpha_t(t) = \int_{-\infty}^{t} \cos\sqrt{\lambda}(t-s)\, f(s)\, ds\,.$$

$$\alpha_{tt}(t) = -\sqrt{\lambda}\int_{-\infty}^{t} \sin\sqrt{\lambda}(t-s)\, f(s)\, ds + f(t) = -\lambda\cdot\alpha + f\,.$$

(A simple justification of this differentiation could be obtained by expressing in (2.9)

$$\sin\sqrt{\lambda}(t-s)$$

in terms of sin and cos). Then, by the same argument than above, these integrals make sense and one has (2.8). So, (2.7), (2.8) hold.

Let us prove now that α is the unique solution of (2.7). For this, it is enough to show that the homogeneous equation

$$\alpha_{tt} + \lambda \cdot \alpha = 0 \quad \text{in } (-\infty, 0)\,, \qquad \alpha \in L^\infty_\nu((-\infty, 0); \boldsymbol{R}) \tag{2.11}$$

has only 0 as a solution.

Note that it results from the first equation of (2.11) that α is of class C^2. So, if we multiply this equation by α_t we get:

$$\frac{d}{dt}\left\{\frac{(\alpha_t)^2}{2} + \frac{\lambda \cdot \alpha^2}{2}\right\} = 0\,.$$

Hence:

$$\frac{(\alpha_t)^2}{2} + \frac{\lambda \cdot \alpha^2}{2} = \text{const}\,.$$

But now, due to (2.11), α, $\alpha_{tt} \to 0$ when $t \to -\infty$. Thus, $\alpha_t \to$ const. But this constant must be 0 if one wants to have $\alpha \to 0$ when $t \to -\infty$. This implies $\alpha \equiv 0$, and the proof is complete.

LEMMA 2. Assume that

$$f \in W^{1,\infty}((-\infty, 0); H) \qquad \text{and} \qquad f,\, f_t \in L^\infty_\nu((-\infty, 0); H)\,.$$

Let $u \in W^{2,\infty}((-\infty, 0); H) \cap L^\infty_\nu((-\infty, 0); D_A)$ be solution of

$$u_{tt} + Au = f\,. \tag{2.12}$$

Then, if $u_t \in L^\infty((-\infty, 0); D_A) \cap L^\infty_\nu((-\infty, 0); V)$ one has the estimate:

$$\frac{\alpha}{2}\|u_t\|^2 + \frac{1}{2}|Au|^2 \leqslant \left(|f|_\nu + \frac{|f_t|_\nu}{2\nu}\right)|Au|_\nu \exp[2\nu t] \qquad \text{a.e. } t \in (-\infty, 0)\,. \tag{2.13}$$

$\big($(2.12) holds in the distributional sense, i.e. in $\mathcal{D}'((-\infty, 0); H)$ and a.e. in $t\big)$.

PROOF. Due to our assumptions one has

$$(Au)_t = Au_t \quad \text{in } H\,.$$

Taking the scalar product of (2.12) and Au_t we get:

$$\frac{d}{dt}\left\{\frac{1}{2}(Au_t, u_t)+\frac{1}{2}(Au, Au)\right\}=(f, Au_t)=(f, Au)_t-(f_t, Au)\,.$$

(Recall that a is symmetric). Integrating between $-\infty$ and t and using the fact that

$$u\in L^\infty_\nu((-\infty, 0); D_A)\,,\qquad u_t\in L^\infty_\nu((-\infty, 0); V)$$

we obtain:

$$\frac{1}{2}(Au_t, u_t)+\frac{1}{2}(Au, Au)=(f, Au)-\int_{-\infty}^{t}(f_t, Au)\,ds\,.$$

Hence by (2.4):

$$\frac{\alpha}{2}\|u_t\|^2+\frac{1}{2}|Au|^2\leqslant|f|\cdot|Au|+\int_{-\infty}^{t}|f_t|\cdot|Au|\,ds\leqslant$$

$$\leqslant|f|_\nu\cdot|Au|_\nu\cdot\exp[2\nu t]+\int_{-\infty}^{t}|f_t|_\nu\cdot|Au|_\nu\cdot\exp[2\nu s]\,ds\leqslant$$

$$\leqslant\left(|f|_\nu+\frac{|f_t|_\nu}{2\nu}\right)|Au|_\nu\cdot\exp[2\nu t]\,.$$

The results follows.

We are now able to prove:

THEOREM 1. Assume that

$$f\in W^{1,\infty}((-\infty, 0); H)\,,\qquad f, f_t\in L^\infty_\nu((-\infty, 0); H)\,.$$

Then, there exists a unique u solution of:

$$(2.14)\qquad\begin{cases} u\in W^{2,\infty}((-\infty, o); H) \\ \qquad\qquad\qquad\cap L^\infty_\nu((-\infty, o); H)\cap L^\infty((-\infty, o); D_A)\,, \\ u_{tt}+Au=f\,. \end{cases}$$

Moreover, one has:

$$u_t \in L^\infty_\nu((-\infty, 0); V), \qquad u \in L^\infty_\nu((-\infty, 0); D_A) \tag{2.15}$$

and

$$|Au|_\nu \leqslant 2\left(|f|_\nu + \frac{|f_t|_\nu}{2\nu}\right). \tag{2.16}$$

PROOF. *a*) Let $v_k \in D_A$ the basis of normalized orthonormal eigenfunctions of A, i.e. the v_k's are such that:

$$Av_k = \lambda_k \cdot v_k , \qquad v_k \in D_A , \qquad |v_k| = 1 . \tag{2.17}$$

It is well known that the v_k's form a basis for H and that the λ_k's are positive. (Recall that we assumed the canonical embedding from V into H to be completely continuous.)

Taking the scalar product of (2.14) and v_k one has clearly:

$$(u, v_k)_{tt} + \lambda_k \cdot (u, v_k) = (f, v_k) .$$

Hence, if $u, f \in L^\infty_\nu((-\infty, 0); H)$, $\alpha_k = (u, v_k)$ is the unique solution in $L^\infty_\nu((-\infty, 0); \boldsymbol{R})$ of

$$(\alpha_k)_{tt} + \lambda_k \cdot \alpha_k = (f, v_k) . \tag{2.18}$$

(See Lemma 1.) Uniqueness follows then from the representation formula

$$u = \sum_{k=1}^{+\infty} (u, v_k) v_k .$$

b) Conversely, denote by α_k the unique solution in $L^\infty_\nu((-\infty, 0); \boldsymbol{R})$ to (2.18), and set

$$u_N = \sum_{k=1}^{N} \alpha_k \cdot v_k , \qquad f_N = \sum_{k=1}^{N} (f, v_k) \cdot v_k .$$

By Lemma 1 one has clearly:

$$\begin{aligned} &u_N \in W^{2,\infty}((-\infty, 0); H) \cap L^\infty_\nu((-\infty, 0); D_A) , \\ &(u_N)_{tt} + Au_N = f_N , \\ &(u_N)_t \in L^\infty((-\infty, 0); D_A) \cap L^\infty_\nu((-\infty, 0); V) . \end{aligned} \tag{2.19}$$

Hence by Lemma 2:

$$\frac{\alpha}{2}\,\|(u_N)_t\|^2+\frac{1}{2}\,|Au_N|^2\leqslant\left(|f_N|_\nu+\frac{|(f_N)_t|_\nu}{2\nu}\right)|Au_N|_\nu\exp\,[2\nu t]\leqslant$$
$$\leqslant\left(|f|_\nu+\frac{|f_t|_\nu}{2\nu}\right)|Au_N|_\nu\exp\,[2\nu t]\,.$$

From which it results easily that:

$$(2.20)\qquad |Au_N|_\nu\leqslant 2\left(|f|_\nu+\frac{|f_t|_\nu}{2\nu}\right),\qquad \|(u_N)_t\|_\nu\leqslant\frac{2}{\sqrt{\alpha}}\left(|f|_\nu+\frac{|f_t|_\nu}{2\nu}\right).$$

Due to (2.2) and the equation of (2.19) we deduce

$$|u_N|_\nu,\;\;|(u_N)|_{tt}|_\nu,\;\;|Au_N|_\nu\leqslant C$$

where C is some constant independent of N.

Thus, there exists a subsequence, still denoted by u_N, such that:

$$\begin{aligned}
u_N &\rightharpoonup u && \text{in } L^\infty((-\infty,0);H)\ \text{weak*}\,,\\
(u_N)_{tt} &\rightharpoonup U && \text{in } L^\infty((-\infty,0);H)\ \text{weak*}\,,\\
Au_N &\rightharpoonup \mathcal{A} && \text{in } L^\infty((-\infty,0);H)\ \text{weak*}\,,\\
f_N &\rightharpoonup f && \text{in } L^\infty((-\infty,0);H)\ \text{weak*}\,.
\end{aligned}$$

Now it is easy to see that if

$$u_N\rightharpoonup u\qquad \text{in } L^\infty((-\infty,0);H)\ \text{weak*}$$

then for any $v\in H$

$$(u_N,v)\to(u,v)\qquad \text{in } \mathcal{D}'(-\infty,0)\,.$$

(This results from the equality

$$\int_{-\infty}^{0}(u_N,v)\cdot\varphi\,dt=\int_{-\infty}^{0}(u_N,\varphi\cdot v)\,dt$$

and the fact that $\varphi\cdot v\in L^1((-\infty,0);H)$).

So, taking the scalar product of (2.19) and $v \in H$ one has in $\mathcal{D}'(-\infty, 0)$

$$\big((u_N)_{tt}, v\big) + (Au_N, v) = (f_N, v)\,.$$

Taking the limit in N we get in $\mathcal{D}'(-\infty, 0)$

$$(U, v) + (\mathcal{A}, v) = (f, v)\,.$$

But, now, by the formula

$$\big((u_N)_{tt}, v\big) = (u_N, v)_{tt}$$

we have also in $\mathcal{D}'(-\infty, 0)$

$$(U, v) = (u, v)_{tt}\,. \tag{2.22}$$

Moreover, for $v \in D_A$ we have:

$$(Au_N, v) = (u_N, Av) \to (u, Av) \quad \text{in } \mathcal{D}'(-\infty, 0)$$

and thus

$$(\mathcal{A}, v) = (u, Av) = (Au, v) \quad \text{in } \mathcal{D}'(-\infty, 0)\,.$$

Since D_A is dense in H this leads to

$$(\mathcal{A}, v) = (Au, v) \quad \text{in } \mathcal{D}'(-\infty, 0) \tag{2.23}$$

for any $v \in H$. Combining (2.21)-(2.23) we obtain

$$(u, v)_{tt} + (Au, v) = (f, v) \quad \text{in } \mathcal{D}'(-\infty, 0)$$

for any $v \in H$. From which is follows easily that u satisfies:

$$u_{tt} + Au = f\,.$$

(See for instance [18], Chap. III, Lemma 1.1). Now, passing to the limit in (2.20) implies clearly (2.15), (2.16).

3. Existence and uniqueness: a perturbation argument.

One would like to consider here the problem:

$$(3.1)\qquad \begin{cases} u \in W^{2,\infty}((-\infty, o); H) \cap L^\infty_\nu((-\infty, o); D_A)\,, \\ u_{tt} + Au - \int_{-\infty}^{t} G(t-s)u(s)\,ds = f \quad \text{in } (-\infty, o) \end{cases}$$

where for any $s > 0$, $G(s)$ is a continuous operator from D_A into H, the above integral being understood in the usual sense.

More precisely let us prove:

THEOREM 2. Assume that

$$f \in W^{1,\infty}((-\infty, 0); H)\,, \qquad f, f_t \in L^\infty_\nu((-\infty, 0); H)\,.$$

Moreover, assume that $s \to G(s)$ is a C^1 mapping from $(0, +\infty)$ into $\mathcal{L}(D_A, H)$ where $\mathcal{L}(D_A, H)$ denotes the space of continuous linear maps from D_A into H. Then if

$$(3.2)\qquad \int_0^{+\infty} \|G(s)\| \exp[-\nu s]\,ds + \frac{\|G(0)\|}{2\nu} + \frac{1}{2\nu}\int_0^{+\infty} \|G'(s)\| \exp[-\nu s]\,ds < \frac{1}{2}$$

where $\| \ \|$ denotes the norm of operators of $\mathcal{L}(D_A, H)$, there exists a unique solution to the problem (3.1).

PROOF. Consider $v \in L^\infty_\nu((-\infty, 0); D_A)$. Thanks to our assumptions it is easy to check that

$$g(t) \overset{\text{Def}}{=} f(t) + \int_{-\infty}^{t} G(t-s)\,v(s)\,ds \in W^{1,\infty}((-\infty, o); H) \cap L^\infty_\nu((-\infty, o); H)\,.$$

Moreover:

$$g_t = f_t + G(0)\,v(t) + \int_{-\infty}^{t} G'(t-s)\,v(s)\,ds \in L^\infty_\nu((-\infty, 0); H)\,.$$

$\big(G'$ denotes the derivative of G as a mapping from $\boldsymbol{R}$ into $\mathcal{L}(D_A, H)$, the fact that g, g_t belong to $L^\infty_\nu((-\infty, 0); H)$ is clear from estimates similar to those that we will make below for h and $h_t\big)$. Hence, it results from Theorem 1 that there exists a unique u such that:

$$(3.3) \qquad \begin{cases} u_{tt} + Au = \displaystyle\int_{-\infty}^{t} G(t-s)\, v(s)\, ds + f\,, \\ u \in W^{2,\infty}((-\infty, o); H) \cap L^\infty_\nu((-\infty, o); D_A)\,. \end{cases}$$

If we can prove that the map $v \to u$ has a unique fixed point in $L^\infty_\nu((-\infty, 0); D_A)$, we will be done. For this, remark that if $v_1, v_2 \in L^\infty_\nu(-\infty, 0); D_A)$ and if u_1, u_2 denote the corresponding solutions of (3.3) one has:

$$(u_1 - u_2)_{tt} + A(u_1 - u_2) = \int_{-\infty}^{t} G(t-s)(v_1 - v_2)(s)\, ds \overset{\text{Def}}{=} h(t)\,.$$

Since

$$h_t = G(0)(v_1 - v_2)(t) + \int_{-\infty}^{t} G'(t-s)(v_1 - v_2)(s)\, ds$$

one has in H:

$$|h| \leqslant \int_{-\infty}^{t} \|G(t-s)\| \cdot |A(v_1 - v_2)(s)| \exp[-\nu(t-s)] \cdot \exp[\nu(t-s)]\, ds =$$

$$= \Big\{ \int_{-\infty}^{t} |A(v_1 - v_2)(s)| \cdot \exp[-\nu s] \cdot \|G(t-s)\| \cdot \exp[-\nu(t-s)]\, ds \Big\} \cdot \exp[\nu t] \leqslant$$

$$\leqslant |A(v_1 - v_2)|_\nu \cdot \Big\{ \int_{-\infty}^{t} \|G(t-s)\| \cdot \exp[-\nu(t-s)]\, ds \Big\} \cdot \exp[\nu t] =$$

$$= |A(v_1 - v_2)|_\nu \cdot \Big\{ \int_{0}^{+\infty} \|G(s)\| \exp[-\nu s]\, ds \Big\} \cdot \exp[\nu t]$$

and similarly

$$|h_t| \leqslant |A(v_1 - v_2)|_\nu \cdot \Big\{ \|G(0)\| + \int_{0}^{+\infty} \|G'(s)\| \exp[-\nu s]\, ds \Big\} \cdot \exp[\nu t]\,.$$

Hence

$$\left(|h|_\nu + \frac{|h_t|_\nu}{2\nu}\right) \leqslant |A(v_1 - v_2)|_\nu \cdot$$

$$\cdot \left\{ \int_0^{+\infty} \|G(s)\| \exp[-\nu s]\, ds + \frac{\|G(0)\|}{2\nu} + \frac{1}{2\nu} \int_0^{+\infty} \|G'(s)\| \exp[-\nu s]\, ds \right\}$$

and thus by (2.16) we get that

$$|A(u_1 - u_2)|_\nu \leqslant 2 \left\{ \int_0^{+\infty} \|G(s)\| \exp[-\nu s]\, ds + \right.$$

$$\left. + \frac{\|G(0)\|}{2\nu} + \frac{1}{2\nu} \int_0^{+\infty} \|G'(s)\| \exp[-\nu s]\, ds \right\} \cdot |A(v_1 - v_2)|_\nu \,.$$

Thus if (3.2) holds the map $v \to u$ is a contraction from $L_\nu^\infty((-\infty, 0); D_A)$ into itself and the result follows.

REMARK 1. Provided we make suitable assumptions it is clear that it would be possible to handle a nonlinear term $G(t-s, u(s))$ instead of $G(t-s)u(s)$ in the integral (3.1).

4. Applications.

Let u be the solution to (1.1), (1.2). The framework of Theorem 2 is recovered if one sets:

$$V = (H_0^1(\Omega))^n \subset (L^2(\Omega))^n = H\,, \qquad a(u, v) = \int_\Omega A_{ijkl}(x) \frac{\partial u^k}{\partial x_l} \frac{\partial v^i}{\partial x_j}\, dx\,.$$

We assume here that the A_{ijkl}'s are smooth and satisfy:

(4.1) $\quad A_{ijkl}(x) = A_{klij}(x) = A_{jikl}(x) \qquad \forall i, j, k, l = 1, \ldots, n\,,$ a.e. $x \in \Omega$.

Moreover, we suppose (with the summation convention):

(4.2) $\quad A_{ijkl}(x)\xi_{ij}\xi_{kl} \geqslant \alpha|\xi|^2 \qquad \forall \xi = (\xi_{ij}) \in \boldsymbol{R}^n\,,\ \xi = \xi^T\,,$ a.e. $x \in \Omega$.

where $|\xi|$ denotes the Euclidean norm of the matrix ξ, ξ^T its transpose, and α a positive constant. Note that (4.1) guarantees the symmetry of a.

The density of the body, $\varrho(x)$, satisfies:

$$\varrho(x) \geqslant \varrho > 0 . \tag{4.3}$$

It is well known from the regularity theory of elliptic systems, (see [16], [17]) that

$$D_A = (H_0^1(\Omega) \cap H^2(\Omega))^n . \tag{4.4}$$

Now, if we assume that the G_{ijkl}'s are smooth one has:

$$\frac{\partial}{\partial x_j}\left\{\int_{-\infty}^{t} G_{ijkl}(x, t-s)\frac{\partial u^k}{\partial x_l}(x, s)\,ds\right\} =$$

$$= \int_{-\infty}^{t} G_{ijkl}(x, t-s)\frac{\partial^2 u^k}{\partial x_l\,\partial x_j}(x, s) + \frac{\partial}{\partial x_j} G_{ijkl}(x, t-s)\frac{\partial u^k}{\partial x_l}(x, s)\,ds .$$

Thus, from (4.4), it is clear that if we set:

$$G(s)u(x) = -G_{ijkl}(x, s)\frac{\partial^2 u^k}{\partial x_l\,\partial x_j}(x) - \frac{\partial}{\partial x_j} G_{ijkl}(x, s)\frac{\partial u^k}{\partial x_l}(x)$$

we define that way a smooth map from D_A into H and (1.1), (1.2) can be though as (3.1). Moreover, one has clearly for some constant C and if $|\ |$ denotes the norm in H:

$$|G(s)u| \leqslant C\cdot \operatorname*{Sup}_{i,j,k,l}\left\{|G_{ijkl}(\cdot, s)|, \left|\frac{\partial}{\partial x_j} G_{ijkl}(\cdot, s)\right|\right\}\cdot |Au| ,$$

$$|G'(s)u| \leqslant C\cdot \operatorname*{Sup}_{i,j,k,l}\left\{|G'_{ijkl}(\cdot, s)|, \left|\frac{\partial}{\partial x_j} G'_{ijkl}(\cdot, s)\right|\right\}\cdot |Au| .$$

(The $'$ denotes the derivative in s).

So, if we set:

$$\|G(s)\| = \operatorname*{Sup}_{i,j,k,l}\left\{|G_{ijkl}(\cdot, s)|, \left|\frac{\partial}{\partial x_j} G_{ijkl}(\cdot, s)\right|\right\},$$

$$\|G'(s)\| = \operatorname*{Sup}_{i,j,k,l}\left\{|G(\cdot, s)|, \left|\frac{\partial}{\partial x_j} G'_{ijkl}(\cdot, s)\right|\right\}$$

and if we assume

$$(4.5)\qquad \int_0^{+\infty} \|G(s)\| \exp[-\nu s]\, ds + \frac{\|G(0)\|}{2\nu} + \frac{1}{2\nu}\int_0^{+\infty} \|G'(s)\| \exp[-\nu s]\, ds < C(\varrho, \alpha)$$

where $C(\varrho, \alpha)$ is some constant depending on ϱ, α and on the norm chosen on D_A (see the formulae above for $\|G(s)\|$ and $\|G'(s)\|$) it is clear that Theorem 2 applies and so we get existence and uniqueness of a solution to (1.1), (1.2). (In fact, due to ϱ, one would need a slight variant of what we proved, replacing u_{tt} by ϱu_{tt} in the L^2 framework or more generally Lu_{tt} where L is a linear operator in the H-framework. This doesn not create any special difficulty. For instance, in the case where ϱ is a constant it is enough to divide (1.1) by ϱ before to solve the problem and then argue on $(1/\varrho)A$ instead of A).

REMARK 2. Apparently the condition (4.5) seems difficult to match. In fact, if G is smooth, then (4.5) holds for ν large enough (by the Lebesgue convergence theorem).

Assume that f is smooth and satisfies for some $t_0 < 0$

$$f(x, t) \equiv 0 \qquad t \leqslant t_0 ,$$

then, for ν large enough, (4.5) holds (see Remark 2) and f satisfies the assumptions of Theorem 2. By the uniqueness of u we must have

$$u \equiv 0 \qquad t \leqslant t_0 ,$$

and (1.1), (1.2) is equivalent to (1.3), (1.2) together with the initial condition

$$(4.6)\qquad u(x, t_0) = 0 \qquad \text{a.e. } x \in \Omega .$$

Thus, in this case, existence and uniqueness of a solution to the initial boundary value problem (1.3), (1.2), (4.6) is obtained as a particular case of our results. (Recall that we assume here that f is smooth). Conversely, if f is extended to 0 for $t \leqslant t_0$ and if its extension is smooth the solution u to (1.3), (1.2), (4.6) extended by 0 for $t \leqslant t_0$ satisfies (1.1), (1.2).

REMARK 3. In the above analysis it could be possible to write the solution of (3.1) directly in terms of the semi-group associated with this hyperbolic equation (and obtain a formula like in (2.9)) this, however, would introduce some technical refinements that we have tried to avoid here. Also, it is not clear that this approach would be much shorter nor more general. Recall also (see Remark 1) that our method enable us to treat some mild nonlinear cases.

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