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Kasch Bimodules.

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0. Introduction.

Let A and R be two rings with a non zero identity. Denote by $\mathcal{T}_A$ the full subcategory of Mod-A consisting of all submodules of the finitely generated modules in Mod-A and by ${}_R\mathcal{T}$ the analogous subcategory of R-Mod. Observe that $\mathcal{T}_A$ and ${}_R\mathcal{T}$ are finitely closed.

Let A be a Kasch ring, *i.e.* a ring with a perfect duality [K]. It is easy to check that the functors $\mathrm{Hom}_A(-, A_A)$ and $\mathrm{Hom}_A(-, {}_AA)$ induce a duality between $\mathcal{T}_A$ and ${}_A\mathcal{T}$.

On the other hand, by well known results of Azumaya [Az], if A_A and ${}_RR$ are artinian, then every duality between two finitely closed subcategories of Mod-A and R-Mod, both containing all finitely generated modules, induces a duality between $\mathcal{T}_A$ and ${}_R\mathcal{T}$.

This paper is devoted to the study of the bimodules ${}_RK_A$ with the property that the functors $\mathrm{Hom}_A(-, K_A)$ and $\mathrm{Hom}_R(-, {}_RK)$ give rise to a duality between $\mathcal{T}_A$ and ${}_R\mathcal{T}$. Such a bimodule will be called a *Kasch bimodule*. Since $\mathcal{T}_A$ and ${}_R\mathcal{T}$ are finitely closed subcategories of Mod-A and R-Mod respectively, the above duality is a special case of Morita duality, thus ${}_RK_A$ is a Morita bimodule, *i.e.* ${}_RK_A$ is faithfully balanced and K_A and ${}_RK$ are (injective) cogenerators. Moreover A and B are semiperfect rings.

This paper is subdivided into three parts. In the first part we give the following characterization:

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Let ${}_RK_A$ be a Morita bimodule; then ${}_RK_A$ is a Kasch bimodule if and only if ${}_RK$ and K_A are finitely generated, if and only if A_A and ${}_RR$ are l.c.d. (i.e. linearly compact in the discrete topology) and with essential socle. Moreover, if ${}_RK_A$ is a Kasch bimodule and if A_A is a cogenerator of Mod-A, *then A is a Kasch ring.*

In the second part relations between Kasch bimodules and semiperfect rings are investigated. In particular, we obtain a generalization of Azumaya's Theorem 6 in [Az] in the following way:

Let A be a ring. Then there exist a ring R and a Kasch bimodule ${}_RK_A$ if and only if:

1) *A_A is l.c.d. with essential socle;*

2) Mod-A *contains a finitely generated injective cogenerator.*

If A is commutative, then only condition 1) suffices by a very recent result of Ánh [An].

Moreover, in this part we give the following form of Morita equivalence:

If $F: \mathcal{T}_A \to \mathcal{T}_R$ is an equivalence, then F extends to a Morita equivalence between Mod-A *and* Mod-R.

This fact will be substantially used in the sequel.

In the third part we obtain our principal results:

Let ${}_RK_A$ be a Kasch bimodule and let $\boldsymbol{B}(R)$ and $\boldsymbol{B}(A)$ be the basic rings of R and A. Then there exists a uniquely determined Kasch bimodule ${}_{\boldsymbol{B}(R)}H_{\boldsymbol{B}(A)}$ where ${}_{\boldsymbol{B}(R)}H$ is a minimal injective cogenerator of $\boldsymbol{B}(R)$-Mod *and similarly for $H_{\boldsymbol{B}(A)}$.*

Using this fact, we get our main result which can be summarized as follows:

Let ${}_RK_A$ be a Kasch bimodule with R and A basic rings and assume that A_A is a cogenerator of Mod-A. *Then*

$$ {}_RK \cong {}_RR, \qquad K_A \cong A_A \qquad \text{and} \qquad R \cong A. $$

We end this introduction with some notations and conventions. All rings considered in this paper have a non-zero identity and all modules are unital. For every ring R we shall denote by Mod-R (resp. R-Mod) the category of all right (resp. left) modules over R. The symbol M_R (${}_RM$) is used to emphasize that M is a right (left) module. All categories and functors will be additive. Every subcategory of a given category will be full and closed with respect to isomorphic objects.

Recall that a subcategory $\mathcal{B}_A$ of Mod-A is finitely closed if $\mathcal{B}_A$ is closed under taking submodules, homomorphic images and finite direct sums. We assume that the reader is familiar with the elementary properties of linearly compact rings and modules in the sense of Leptin [L]. We denote by $\mathbf{Z}$ the ring of integers, by $\mathbf{Q}$ the field of rationals and by $\mathbf{N}$ the set of positive integers.

1. Finitely generated injective cogenerators.

1.1. Let A and R be two rings. A bimodule ${}_RK_A$ is called a *Morita bimodule* if ${}_RK_A$ is faithfully balanced and both K_A and ${}_RK$ are injective cogenerators of Mod-A and R-Mod respectively. This definition is motivated by the following well known result of Morita [Mo]: *The bimodule ${}_RK_A$ is a Morita bimodule if and only if the subcategories $\mathcal{B}_A$ of* Mod-A *and ${}_R\mathcal{B}$ of R*-Mod *consisting of K-reflexive modules are finitely closed and contain all finitely generated modules.*

If ${}_RK_A$ is a Morita bimodule, then by a known result of Müller [Mü] A_A, ${}_RR$, K_A and ${}_RK$ are l.c.d. modules (*i.e.* linearly compact in the discrete topology). In particular, the rings A and R are semiperfect.

1.2. Let ${}_RK_A$ be a Morita bimodule. Then the subcategory $\mathcal{T}_A$ of Mod-A consisting of all submodules of the finitely generated modules is contained in $\mathcal{B}_A$. Similarly ${}_R\mathcal{T} \subseteq {}_R\mathcal{B}$. In general the Morita bimodule ${}_RK_A$ does not induce a duality between $\mathcal{T}_A$ and ${}_R\mathcal{T}$, although $\mathcal{T}_A$ and ${}_R\mathcal{T}$ are finitely closed subcategories of Mod-A and R-Mod respectively.

1.3. DEFINITION. *a*) Let ${}_RK_A$ be a Morita bimodule. We say that ${}_RK_A$ is a *Kasch bimodule* if the functors $H_1 = \mathrm{Hom}_A(-, K_A)$ and $H_2 = \mathrm{Hom}_R(-, {}_RK)$ give a duality between $\mathcal{T}_A$ and ${}_R\mathcal{T}$.

b) Let A be a ring. We say that A has a *right Kasch duality* if there exist a ring R and a Kasch bimodule ${}_RK_A$.

c) Recall that a ring A is said to be a *Kasch ring* if both A_A and ${}_AA$ are injective cogenerators of Mod-A and A-Mod respectively.

We shall see in the sequel that if A is a Kasch ring, then the bimodule ${}_AA_A$ is a Kasch bimodule.

1.4. PROPOSITION. *Let ${}_RK_A$ be a Morita bimodule. The following conditions are equivalent:*

(*a*) *${}_RK_A$ is a Kasch bimodule;*

(*b*) *K_A and ${}_RK$ are finitely generated;*

(*c*) A_A *and* ${}_RR$ *are finitely cogenerated;*

(*d*) A_A *and* ${}_RR$ *are l.c.d. with essential socle.*

PROOF. $(b) \Leftrightarrow (c)$ It is well known that the assignment $I \mapsto \mathrm{Ann}_{{}_RK}(I)$ defines a lattice antiisomorphism between the lattice $\mathcal{L}(A_A)$ of right ideals of A and the lattice $\mathcal{L}({}_RK)$ of submodules of ${}_RK$.

$(c) \Leftrightarrow (d)$ is evident.

$(a) \Rightarrow (b)$ K_A is a submodule of a finitely generated module $F_A \in$ Mod-A. Since K_A is injective, K_A is a direct summand of F_A. Thus K_A is finitely generated.

$(b) \Rightarrow (a)$ Let $M \in \mathcal{T}_A$ There exists an exact sequence

$$0 \to M \to F \tag{1}$$

in Mod-A such that F is finitely generated, *i.e.* there exists an exact sequence $A^n \to F \to 0$. Dualizing we get $0 \to H_1(F_A) \to {}_RK^n$, hence $H_1(F_A) \in {}_R\mathcal{T}$. Dualizing (1) we obtain the exact sequence

$$H_1(F_A) \to H_1(M) \to 0,$$

therefore $H_1(M) \in {}_R\mathcal{T}$. □

1.5. COROLLARY. *Let A be an arbitrary ring. Then ${}_AA_A$ is a Kasch bimodule if and only if A is a Kasch ring.*

1.6. The following proposition shows a further connection between Kasch bimodules and Kasch rings.

1.7. PROPOSITION. *Let ${}_RK_A$ be a Kasch bimodule and assume that A_A is a cogenerator of* Mod-A. *Then A is a Kasch ring.*

PROOF. Mod-A has finitely many non-isomorphic modules since A is semiperfect. On the other hand A_A is a cogenerator of Mod-A by hypothesis. Then, by Theorem 12.5.2 from [K], A_A is an injective cogenerator of Mod-A. Since A_A is l.c.d. with essential socle by Proposition 1.4, we can apply Theorem 2.10.*a*) from [DO] (due to Claudia Menini, see also [Me]) to conclude that ${}_AA$ is quasi-injective and consequently injective. Again by Theorem 12.5.2 from [K], every simple module in A-Mod is isomorphic to a left ideal of A. Thus ${}_AA$ is an injective cogenerator of A-Mod. Hence A is a Kasch ring. □

The following corollary is a consequence of a well known property of Kasch rings.

1.8. COROLLARY. *Under the hypotheses of the above proposition, A induces a duality between* $\mathcal{T}_A$ *and* ${}_A\mathcal{T}$ *defined by means of the linear forms. Consequently* ${}_R\mathcal{T}$ *and* ${}_A\mathcal{T}$ *are equivalent categories.*

1.9. PROPOSITION. *Let A be right artinian ring. Then A admits a Morita duality with a ring R if and only if A admits a Kasch duality with R. In this case* $\mathcal{T}_A$ *coincides with the category of finitely generated modules in* Mod-*A and similarly for* ${}_R\mathcal{T}$.

PROOF. By a result essentially due to Azumaya [Az] (see *e.g.* Tachikawa [T], Theorem 3.8), A has a right Morita duality if and only if Mod-A has a finitely generated injective cogenerator.

1.10. COROLLARY. *Every commutative artinian ring admits a Kasch duality with itself.*

PROOF. It is enough to show this for a local commutative artinian ring with maximal ideal $\mathfrak{m}$. By a well known result of Matlis [Ma], the module $K_A = E(A/\mathfrak{m})$, the injective envelope of $A/\mathfrak{m}$, is a Loewy module with finite Loewy length and finite Loewy invariants. Therefore K_A is a module with finite length in the usual sense. Moreover $\mathrm{End}(K_A) = = A$.

1.11. EXAMPLE. Let ${}_RK_A$ be a Kasch bimodule. In general, the rings R and A need not be self-injective, as the following example shows.

Let A be a commutative local artinian ring with maximal ideal $\mathfrak{m}$, and set $K_A = E(A/\mathfrak{m})$. Assume that A is self-injective and observe that Soc(A) is essential in A, so that A_A is an injective cogenerator of Mod-A. We will show that, in this case, Soc(A) must be simple. Indeed, let n be the length of Soc(A). Then $A = E(\mathrm{Soc}(A)) = K^n$. Dualizing we get $K = \mathrm{Hom}_A(A, K) = \mathrm{Hom}_A(K^n, K) = A^n = K^{n^2}$ and this is possible if and only if $n = 1$, so that Soc(A) is simple.

This is not true in general: let k be a field and set $A = k \ltimes k^2$, the trivial extension of the module k^2 by the ring k.

1.12. EXAMPLE. Let ${}_RK_A$ be a Kasch bimodule. In general Soc(A_A) and Soc(${}_RR$) do not contain a copy of each simple module in Mod-A or in R-Mod respectively. Consider the following example.

Let $\mathbf{Z}_2$ be the field with two elements and set

$$A = \begin{bmatrix} \mathbf{Z}_2 & \mathbf{Z}_2 \\ 0 & \mathbf{Z}_2 \end{bmatrix},$$

the ring of 2×2 upper triangular matrices over $\mathbf{Z}_2$. Then A is both left and right artinian and both Mod-A and A-Mod have a finitely generated injective cogenerator (for instance, an injective cogenerator of Mod-A is $\mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})$). Thus, by Proposition 1.9, A admits a left and a right Kasch duality.

Consider the maximal two-sided ideal

$$\mathfrak{m} = \begin{bmatrix} 0 & \mathbf{Z}_2 \\ 0 & \mathbf{Z}_2 \end{bmatrix}.$$

Clearly there is a monomorphism of right A-modules $A/\mathfrak{m} \to A_A$ if and only if there exists $a \in A$, $a \neq 0$, such that $a\mathfrak{m} = 0$. Let us see that such an element does not exist. In fact, let $a = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ with $x, y, z \in \mathbf{Z}_2$. Then $a\mathfrak{m} = 0$ yields

$$0 = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad 0 = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & y \\ 0 & z \end{bmatrix},$$

hence $x = y = z = 0$. Consequently $a = 0$.

1.13. Example of a non artinian commutative Kasch ring. Let R be a noetherian commutative local ring with maximal ideal $\mathfrak{m}$ and assume that R is complete in its $\mathfrak{m}$-adic topology. Set $E = E(R/\mathfrak{m})$: by well known results of Matlis [Ma], $R \cong \mathrm{End}_R(E)$. Consider the trivial extension

$$A = R \ltimes {}_R E_R .$$

Clearly A is a commutative local l.c.d. ring and moreover A is subdirectly irreducible. Then, by an important result of Ánh (cf. [An], Theorem 7), A is self-injective. Therefore A_A is an injective cogenerator of Mod-A.

This example generalizes the one in ex. (11) in [K], Chap. 12.

1.14. Other examples of the kind given in 1.13 can be obtained in the following way. Let A be any commutative l.c.d. ring. For $a \in A$, $a \neq 0$ and let I be an ideal of A which is maximal with respect to the property $a \notin I$. Then A/I is an l.c.d. ring with simple and essential socle. On the other hand, by the result of Ánh, A/I is self-injective. Thus A/I is a Kasch ring.

2. Kasch bimodules and semiperfect rings.

We begin with a theorem which gives a necessary and sufficient condition for a ring A in order to have a right Kasch duality.

2.1. LEMMA. *Let A be a ring which is l.c.d. on the right, K_A an injective cogenerator of* Mod-A *with essential socle and* $R = \mathrm{End}(K_A)$. *Then the following conditions are equivalent:*

(*a*) K_A *is l.c.d.;*

(*b*) ${}_RK$ *is an injective cogenerator of* R-Mod;

(*c*) ${}_RR$ *is l.c.d.*

PROOF. See [DO], Corollary 5.12. □

2.2. THEOREM. *For any ring A the following conditions are equivalent:*

(*a*) *A has a right Kasch duality;*

(*b*) A_A *is l.c.d. with essential socle and* Mod-A *has a finitely generated injective cogenerator* K_A.

If (b) is fulfilled, then, setting $R = \mathrm{End}(K_A)$, ${}_RK_A$ *is a Kasch bimodule.*

PROOF. $(a) \Rightarrow (b)$ follows from Proposition 1.4. $(b) \Rightarrow (a)$ Set $R = \mathrm{End}(K_A)$; then ${}_RK_A$ is faithfully balanced by Proposition 2.10 of [DO]. The module K_A is l.c.d. since it is a homomorphic image of A^n, for a suitable $n \in \mathbb{N}$. By Lemma 2.1 and by Müller's Theorem 1 in [Mü], ${}_RK_A$ is a Morita bimodule. Finally, since A_A is finitely cogenerated, ${}_RK$ is finitely generated. □

When A is a commutative ring, the above theorem can be improved as follows.

2.3. THEOREM. *Let A be a commutative ring. Then A has a Kasch duality if and only if A is l.c.d. with essential socle.*

PROOF. The condition is necessary by the above theorem. For the sufficiency, note that A, being l.c.d., has a Morita duality by the cited result of Ánh. By theorem 3 of [Mü], A has a Morita duality with itself, so that there is a Morita bimodule ${}_AK_A$. Since A is finitely cogenerated, K_A is finitely generated. □

2.4. PROPOSITION. *Let A and R be two rings, let $\mathcal{T}_A$ be the subcategory of* Mod-A *consisting of all submodules of the finitely generated modules in* Mod-A, *let $\mathcal{T}_R$ be the analogous subcategory of* Mod-R *and suppose that there exists an equivalence*

$$\mathcal{T}_A \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{T}_R .$$

Then the rings A and R are Morita equivalent.

PROOF. Set $P_R = F(A)$; then $A = \mathrm{End}\,(P_R)$, so we have the bimodule ${}_AP_R$. Let $M \in \mathcal{T}_R$ and consider the natural isomorphisms in Mod-A

$$G(M) \cong \mathrm{Hom}_A\,(A, G(M)) \cong \mathrm{Hom}_R\,(P_R, M).$$

Then $G \approx \mathrm{Hom}_R\,(P_R, -)$ (G is naturally equivalent to $\mathrm{Hom}_R\,(P_R, -)$). We show next that $F \approx - \otimes_A P$.

The module P_R is a generator of $\mathcal{T}_R$ in categorical sense, since A_A is a generator of $\mathcal{T}_A$. Take a module $M \in \mathcal{T}_R$ and set

$$T = \sum \{\mathrm{Im} f : f \in \mathrm{Hom}_R\,(P, M)\}.$$

Assume T is a proper submodule of M and consider the canonical projection

$$\pi : M \to M/T.$$

Then $\pi \neq 0$ and since both M and M/T belong to $\mathcal{T}_R$ and P_R is a generator of $\mathcal{T}_R$, there exists a homomorphism $f : P_R \to M$ such that $\pi \circ f \neq 0$, a contradiction, since $\mathrm{Im} f \subseteq T = \ker \pi$. Therefore $T = M$.

Now assume that $M \in \mathcal{T}_R$ is finitely generated. Then by the above argument there exist $n \in \mathrm{N}$ and a surjective homomorphism $P_R^n \to M$. This shows that P_R generates all finitely generated modules in Mod-R, consequently it is a generator of Mod-R. It is well known that in such a case ${}_AP$ is projective and, in particular, flat. This yields $F \approx - \otimes_A P$. In fact, let $L \in \mathcal{T}_A$. Then there exist a finitely generated module $N \in \mathcal{T}_A$ and an exact sequence $0 \to L \otimes_R P \to N \otimes_R P$. On the other hand, $N \otimes_R P$ is a homomorphic image of P_R^n, for some $n \in \mathrm{N}$, so that $L \otimes_R P \in \mathcal{T}_R$, since $\mathcal{T}_R$ is finitely closed. Thus $- \otimes_R P$, restricted to $\mathcal{T}_A$, is the unique adjoint of G. Now set $T = - \otimes_A P$ and $H = \mathrm{Hom}_R\,(P_R, -)$.

Since P_R is a projective object in $\mathcal{T}_R$, it follows that P_R is quasi-projective in Mod-R. Let us prove now that P_R is finitely generated. In fact, let $(V_\lambda)_{\lambda \in \Lambda}$ be a directed family of finitely generated submodules of P_R with $P_R = \varinjlim V_\lambda$. Since morphisms, subobjects and quotients in $\mathcal{T}_R$

are the usual ones, we get

$$A = H(P) = H(\varinjlim V_\lambda) = \varinjlim H(V_\lambda).$$

There exists $\mu \in \Lambda$ such that $H(V_\mu) = A$, then $T(A) = TH(V_\mu) \cong V_\mu$, the isomorphism being canonical. Then $P = V_\mu$ is finitely generated. This shows that P_R is a quasi-progenerator in the sense of Fuller, because it is a finitely generated quasi-projective generator of Mod-R. Now by Theorem 2.6 in [Fu] the functor T defines an equivalence between Mod-A and Mod-R, so that A and R are Morita equivalent. □

2.5. PROPOSITION. *Let the rings A and R be Morita equivalent. Then*

a) if A_A is l.c.d. then also R_R is l.c.d.;

b) if Soc(A_A) *is essential in A_A, then also* Soc(R_R) *is essential in R_R;*

c) if A_A is semiperfect and A_A is a cogenerator of Mod-A, *then also R is semiperfect and R_R is a cogenerator of* Mod-R;

d) if A is a Kasch ring, then also R is Kasch.

PROOF. Let

$$\text{Mod-}A \underset{H}{\overset{T}{\rightleftarrows}} \text{Mod-}R \tag{2}$$

be a Morita equivalence represented by the faithfully balanced bimodule ${}_AP_R = T(A)$ which is a progenerator on both sides.

a) Follows from Lemma 1.7 in [G].

b) Observe that for every $M \in$ Mod-A there exists an isomorphism between the lattice $\mathcal{L}(M)$ of submodules of M and the lattice $\mathcal{L}(T(M))$ of submodules of $T(M)$. This observation proves *b*), since Soc(M) is essential in M if and only if Soc$(T(M))$ is essential in $T(M)$. In particular Soc(P_R) is essential in P_R, since Soc(A_A) is essential in A_A. There exists a surjective homomorphism

$$P_R^n \to R \to 0 \tag{3}$$

($n \in$ N), thus R_R is a direct summand of P_R^n. Therefore Soc(R_R) is essential in R_R, since Soc(P_R^n) is essential in P_R^n.

c) First of all we note that the equivalence (2) preserves finitely generated modules and projective covers, so that R is semiperfect, since every finitely generated module in Mod-R has a projective cover by the respective property of Mod-A. By Theorem 12.5.2 in [K], A_A is a cogenerator of Mod-A if and only if every faithful module in Mod-A is a generator. Since Mod-R has only a finite number of non-isomorphic simple modules, we can apply the same theorem to show that R_R is a cogenerator of Mod-R. In order to do this it suffices to note that the equivalence (2) takes generators to generators and faithful A-modules to faithful R-modules. We give a proof of the latter fact.

A module $M \in$Mod-A is faithful if and only if $\cap(\ker f : f \in \in \mathrm{Hom}_A(A, M)\} = 0$. Thus for a faithful module $M \in$Mod-A

(4) $$\cap \{\ker g : g \in \mathrm{Hom}_R(P_R, T(M)) = 0.$$

To prove that $T(M) \in$Mod-R is faithful, it suffices to show that

$$I = \cap \{\ker h : h \in \mathrm{Hom}_R(R, M)\} = 0.$$

It follows from (3) that $P^n = X \oplus R$ for a submodule X of P^n. Clearly (4) implies

(5) $$\cap \{\ker \varphi : \varphi \in \mathrm{Hom}_R(P^n, T(M)) = 0.$$

Let $r \in I$. To prove that $r = 0$, it suffices to show that the element $(0, r)$ of P^n belongs to the intersection (5). In fact, for $\varphi : P^n \to T(M)$ consider the restriction φ_1 of φ to $\{0\} \oplus R$. Clearly $\varphi(0, r) = \varphi_1(0, r) = 0$ by the choice of r, since φ_1 can be considered as an element of $\mathrm{Hom}_R(R, T(M))$.

d) Follows obviously from *a*), *b*) and *c*). □

2.6. Example of a non artinian non commutative Kasch ring. Let A be a Kasch ring and let $A_n\,(n > 1)$ be the ring of $n \times n$ matrices over A. Then A_n is Morita equivalent to A and so is a Kasch ring by Proposition 2.5, but is not commutative since $n > 1$.

3. Reduction of a Kasch bimodule to a Kasch ring.

3.1. Now we need some properties of semiperfect rings (cf. [AF], Chapter 7, §2).

a) A semiperfect ring B is said to be a *basic ring* if $B/J(B) = \prod_{i=1}^{n} D_i$, where the D_i are division rings. Lifting the idempotents of $B/J(B)$ to B, we get the following decomposition of B

(6) $$B = Be_1 \oplus \ldots \oplus Be_n,$$

where $(e_i)_{i=1}^n$ is a complete system of pairwise orthogonal primitive idempotents of B. The left ideals $\{Be_i : i = 1,\dots, n\}$ provide a complete irredundant set of representatives of the projective indecomposable modules in B-Mod. For $i = 1,\dots, n$ $\mathrm{End}_B\,(Be_i) = e_i\,Be_i$ is a local ring.

b) Let R be an arbitrary semiperfect ring. Then R is Morita equivalent to a basic ring B which is uniquely determined up to isomorphism. This basic ring will be called *the basic ring of R* and denoted by $\boldsymbol{B}(R)$. Let

$$F : B\text{-Mod} \to R\text{-Mod}$$

be a Morita equivalence. Then the modules $\{F(Be_i) : i = 1,\dots, n\}$ provide a complete irredundant set of representatives of the projective indecomposable modules in R-Mod.

c) Two semiperfect rings are Morita equivalent if and only if their basic rings are isomorphic.

3.2. Lemma. *Let ${}_RK_A$ be a Morita bimodule and suppose that R is a basic ring. Then K_A is the minimal injective cogenerator of* Mod-A.

Proof. Since ${}_RK_A$ is a Morita bimodule, $\mathrm{Soc}\,(K_A) = \mathrm{Soc}\,({}_RK)$ and this is an essential submodule both in K_A and in ${}_RK$. Applying the functor $\mathrm{Hom}_A\,(-, K_A)$ to the exact sequence

$$0 \to \mathrm{Soc}\,(K) \to K \to K/\mathrm{Soc}\,(K) \to 0$$

we get the exact sequence

$$0 \to \mathrm{Hom}_A\,(K/\mathrm{Soc}\,(K), K_A) \to R \to \mathrm{End}_A\,(\mathrm{Soc}\,(K)) \to 0.$$

Clearly $\mathrm{Hom}_A\,(K/\mathrm{Soc}\,(K), K_A)$ coincides with the ideal of R consisting of the endomorphisms of K_A vanishing on $\mathrm{Soc}\,(K_A)$ *i.e.* having essential kernel. Since K_A is injective, $\mathrm{Hom}_A\,(K/\mathrm{Soc}\,(K), K_A)$ coincides with the Jacobson radical $J(R)$ of R. Therefore

$$\mathrm{End}_A\,(\mathrm{Soc}\,(K_A)) = R/J(R) = \prod_{i=1}^n D_i\,,$$

where the D_i are division rings. This yields that $\mathrm{Soc}\,(K_A)$ is a direct sum of pairwise non isomorphic simple A-modules (cf. [DO], Lemma 7.4). □

3.3. PROPOSITION. *a) If* ${}_RK_A$ *is a Kasch bimodule with* A *and* R *basic rings, then* ${}_RK$ *and* K_A *are the minimal injective cogenerators of* R-Mod *and* Mod-A *respectively.*

b) Let ${}_RK_A$ *be a Kasch bimodule and let* $\tilde{R}$ *and* $\tilde{A}$ *be rings Morita equivalent to* R *and* A *respectively. Then there exists a Kasch bimodule* ${}_{\tilde{R}}H_{\tilde{A}}$. *If* $\tilde{R} = \boldsymbol{B}(R)$ *and* $\tilde{A} = \boldsymbol{B}(A)$ *are the basic rings, then* ${}_{\boldsymbol{B}(R)}H_{\boldsymbol{B}(A)}$ *is uniquely determined by the property of being a minimal injective cogenerator on both sides.*

PROOF. *a*) follows directly from the above lemma.

b) Consider a Morita equivalence

$$F: R\text{-Mod} \to \tilde{R}\text{-Mod}.$$

Then clearly $\mathrm{End}_{\tilde{R}}(F({}_RK)) \cong A$, so we have the bimodule ${}_{\tilde{R}}F({}_RK)_A$. By Proposition 2.5, ${}_{\tilde{R}}\tilde{R}$ is l.c.d. with essential socle and $F({}_RK)$ is a finitely generated injective cogenerator of $\tilde{R}$-Mod. By virtue of Theorem 2.2, ${}_{\tilde{R}}F({}_RK)_A$ is a Kasch bimodule.

Now consider a Morita equivalence

$$G: \text{Mod-}A \to \text{Mod-}\tilde{A}$$

and set $H_{\tilde{A}} = G(F({}_RK)_A)$. Clearly

$$\mathrm{End}(H_{\tilde{A}}) \cong \mathrm{End}(F({}_RK)_A) \cong \tilde{R}.$$

It can be shown as above that the bimodule ${}_{\tilde{R}}H_{\tilde{A}}$ is a Kasch bimodule.

The uniqueness of H in the case when $\tilde{R} = \boldsymbol{B}(R)$ and $\tilde{A} = \boldsymbol{B}(A)$ follows from *a*). □

3.4. THEOREM. *Let* ${}_RK_A$ *be a Kasch bimodule with both* A *and* R *basic rings. The following conditions are equivalent:*

(*a*) A_A *is a cogenerator of* Mod-A;

(*b*) A *is a Kasch ring;*

(*c*) ${}_RR$ *is a cogenerator of* R-Mod;

(*d*) R *is a Kasch ring;*

(*e*) ${}_RK \cong {}_RR$, $K_A \cong A_A$ *and* $A \cong R$.

PROOF. $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ follow from Proposition 1.7.

$(b) \Rightarrow (c)$ A is a Kasch ring; by setting $\Sigma(M) = \mathrm{Hom}_A(M, {}_AA)$ for

$M \in A$-Mod, we get a Kasch duality

$$_A\mathcal{T} \xrightarrow{\Sigma} \mathcal{T}_A .$$

Taking the composition with the Kasch duality

$$\mathcal{T}_A \to {}_R\mathcal{T}$$

given by the bimodule $_RK_A$, we get an equivalence $_A\mathcal{T} \to {}_R\mathcal{T}$, hence a Morita equivalence A-Mod$\to R$-Mod, by Proposition 2.4. By Proposition 2.5 *c*), $_RR$ is a cogenerator of R-Mod.

(*d*) $\Rightarrow$ (*e*) Let R be a Kasch ring. We show first that $_RK \cong {}_RR$. By virtue of Proposition 3.3 *a*), it suffices to show that $_RR$ is a minimal injective cogenerator of R-Mod. By (6) of 3.1 *a*), we can write

$$R = Re_1 \oplus \ldots \oplus Re_n ,$$

where $\{Re_i : i = 1, \ldots, n\}$ are pairwise non isomorphic indecomposable modules. Since $_RR$ is injective with essential socle, also Re_i is injective with essential socle. Thus Re_i is the injective envelope of a simple module. Since the modules Re_i are pairwise non isomorphic, $_RR$ is a minimal injective cogenerator of R-Mod and consequently $_RK \cong {}_RR$. Since $_RK_A$ is faithfully balanced, we have also

$$R = \mathrm{End}\,(_RR) \cong \mathrm{End}\,(_RK) = A.$$

This isomorphism implies also $A_A \cong K_A$.

(*e*) $\Rightarrow$ (*a*) is obvious. □

In contrast with the above theorem we prove the following.

3.5. PROPOSITION. *Let A a commutative ring. Then the following conditions are equivalent:*

(*a*) *A admits a Kasch duality;*

(*b*) *A is a subdirect product of a finite number of subdirectly irreducible Kasch rings.*

PROOF. (*a*) $\Rightarrow$ (*b*) By Theorem 2.3 A is l.c.d. with essential socle. On the other hand, by 1.14 A is a subdirect product of subdirectly irreducible Kasch rings. Since Soc(A) is finitely generated, this product can be taken finite.

(*b*) $\Rightarrow$ (*a*) Clearly A is l.c.d. with essential socle, so that Theorem 2.3 applies again. □

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