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The Equation $y' = fy$ in $\mathbb{C}_p$ when f is Quasi-Invertible.

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SUMMARY - Let K be a complete algebraically closed extension of C_p. Let D be a clopen bounded infraconnected set in K, let $H(D)$ be the Banach algebra of the analytic elements on D, let $f \in H(D)$ and let $\mathcal{S}(f)$ be the space of the solutions of the equation $y' = fy$ in $H(D)$. We construct such a set D provided with a T-filter $\mathcal{F}$ such that there exists a quasi-invertible $f \in H(D)$ such that $\mathcal{S}(f)$ has non zero elements g which approach zero along $\mathcal{F}$. In extending this construction we show that for every $t \in \mathrm{N}$, we can make a set D and an $f \in H(D)$ such that $\mathcal{S}(f)$ has dimension t. That answers questions suggested in previous articles.

I. Introduction and theorems.

Let K be an ultrametric complete algebraically closed field, of characteristic zero and residue characteristic $p \neq 0$.

Let D be an infraconnected bounded clopen set in K and let $H(D)$ be the Banach algebra of the Analytic Elements on D (i.e., $H(D)$ is the completion of the algebra $R(D)$ for the uniform convergence norm on D) [E_1, E_2, E_3, K_1, K_2, R].

Recall that a set D in K is said to be infraconnected it for every $a \in D$ the mapping $x \to |x - a|$ has an image whose adherence in R is an interval; then $H(D)$ has no idempotent different from 0 and 1 is and only if D is infraconnected [E_2] On the other hand, an open set D is infraconnected if and only if $f' = 0$ implies $f = ct$ for every $f \in H(D)$ [E_6]. Let $f \in H(D)$; we denote by $\mathcal{E}(f)$ the differential equation $y' = fy$ (where $y \in H(D)$) and by $\mathcal{S}(f)$ the space of the solutions of $\mathcal{E}(f)$.

In [E_7] we saw that $\mathcal{S}(f)$ has dimension 1 as soon as it contains

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a g invertible in $H(D)$. If $H(D)$ has no divisor of zero, $\mathcal{S}(f)$ doesn't have dimension greater than one.

In [E_8] we saw that if the residue characteristic of K is zero, then $\mathcal{S}(f)$ never has dimension greater than one.

But when the residue characteristic p is different from zero, in [E_9] we saw that there does exist infraconnected clopen bounded sets with a T-filter $\mathcal{F}$[E_4] and an element f annulled by $\mathcal{F}$ such that the solutions of $\mathcal{E}(f)$ are also annulled by $\mathcal{F}$. Thanks to such T-filters, for every $n \in \mathbb{N}$ we could construct infraconnected clopen bounded sets D with $f \in H(D)$ such that $\mathcal{S}(f)$ has dimension n, and we even constructed sets D with $f \in H(D)$ such that $\mathcal{S}(f)$ is isomorphic to the space of the sequences of limit zero.

Thus [E_8] suggested that a situation where the solutions of $\mathcal{E}(f)$ were not invertible in $H(D)$ should be associated to a non quasi-invertible element f, and so should be spaces $\mathcal{S}(f)$ of dimension greater than one.

(Recall that f is said to be quasi-invertible in $H(D)$ if it factorizes in the form $P(x)g(x)$ where P is a polynomial the zeros of which are in D and g is an invertible element of $H(D)$) [E_1, E_2, E_3, E_4].

Here we will prove this connection does not hold in constructing an infraconnected clopen bounded set D with a T-filter $\mathcal{F}$ and a quasi-invertible element $f \in H(D)$ such that $\mathcal{E}(f)$ has solutions strictly annulled by $\mathcal{F}$.

Next, for every fixed integer t, an extension of that construction will provide us with a set D and a quasi-invertible $f \in H(D)$ such that $\dim \mathcal{S}(f) = t$.

THEOREM 1. *There exist an infraconnected clopen bounded set D with a T-filter $\mathcal{F}$ and quasi-invertible elements $f \in H(D)$ such that $\mathcal{E}(f)$ has solutions strictly annulled by $\mathcal{F}$ and $\mathcal{S}(f)$ has dimension* 1.

More precisely, we will concretely construct such a set D and $f \in H(D)$ in Proposition B.

THEOREM 2. *Let $t \in \mathbb{N}$. There exist an infraconnected clopen bounded set D and quasi-invertible elements $f \in H(D)$ such that* $\dim(\mathcal{S}(f)) = t$.

Theorem 2 will also be proven by a concrete construction.

REMARK. We are not able to construct an infraconnected clopen bounded set D with a quasi-invertible $f \in H(D)$ such that $\mathcal{S}(f)$ has infinite dimension. By then, the following conjecture seems to be likely.

CONJECTURE. *If f is quasi-invertible, $\mathcal{S}(f)$ has finite dimension.*

The following Proposition A will demonstrate Theorem 1 by showing how to obtain the set D, the T-filter $\mathcal{F}$, and the element f.

PROPOSITION A. *Let $(b_m)_{m \in \mathbb{N}}$ be a sequence in $d^-(0,1)$ such that $|b_m| < |b_{m+1}|$, and let $(p_m)_{m \in \mathbb{N}}$ be a sequence of integers in the form p^{q_m} where q_m is a sequence of integers satisfying*

$$\lim_{m \to \infty} q_m = +\infty , \tag{1}$$

$$|p_1| > |p_m| \quad \text{whenever } m \geqslant 2 , \tag{2}$$

$$\lim_{m \to \infty} \left| \frac{b_m}{b_{m+1}} \right|^{p_{m+1}} = 0 . \tag{3}$$

Let R be $\geqslant 1$, and let $D = d(0,R) \setminus \left(\bigcup_{m=1}^{\infty} d^-(b_m, |b_m|) \right)$. For each $m \in \mathbb{N}^*$ let

$$h_m = \prod_{j=1}^{m} \frac{1}{(1 - x/b_j)^{p_j}} \in R(D) .$$

Then the sequence (h_m) converges in $H(D)$ to a limit h that is strictly annulled by the increasing T-filter $\mathcal{F}$ of center 0 of diameter 1, and $h \in \mathcal{S}(\mathcal{F})$.

The series $\sum_{m=1}^{\infty} p_j/(b_m - x)$ converges is $H(D)$ to a limit f quasi-invertible in $H(D)$ and h is a solution of $\mathcal{E}(f)$.

II. The proof of Proposition A

The proof of proposition will use the following Lemma B.

LEMMA B. *Let q and n be two integers such that $C < n \leqslant p^q$. Then $|C^n_{(p^q)}| \leqslant p^{-q}/|n|$.*

PROOF. If n is a multiple of some p^h, then $p^q - n$ is obviously multiple of p^h. Let b the bijection from $\{1, \ldots, n\}$ onto $\{(p^q - n + 1), \ldots, p^q\}$ defined by $b(j) = p^q - j + 1$. By the last sentence, when j is divisible by p^h, $b(j+1)$ is also divisible by p^h hence $|b(j+1)| \leqslant |j|$ therefore $|(p_q - 1)(p_q - 2) \ldots (p^q - n + 1)| \leqslant |(n-1)!|$ and finally $|C^h_{p^q}| \leqslant p^{-q}/|n|$.

PROOF OF PROPOSITION A. Since $\lim_{m\to\infty} |b_m/b_{m+1}|^{p_{m+1}} = 0$ we have $\lim_{m\to\infty}(p^{q_{m+1}}\log|b_{m+1}/b_m|) = +\infty$. Thus we can easily define a sequence of integers l_m such that $\lim_{m\to\infty}(q_m - l_m) = +\infty$ and $\lim_{m\to\infty}(p^{l_{m+1}}\log|b_{m+1}/b_m|) = +\infty$. We put $t_m = p^{l_m}$, $\omega_m = |p_m/t_m|$, $\varepsilon_m = |b_{m-1}/b_m|^{t_m}$. Then we have $\lim_{m\to\infty}\omega_m = \lim_{m\to\infty}\varepsilon_m = 0$.

As the holes of D are in the form $d^-(b_m, |b_m|)$ it is easily seen that

$$\left\| \frac{1}{1 - x/b_j} \right\|_D \leqslant 1. \tag{4}$$

Let us consider $|h_{m+1}(x) - h_m(x)|$ when $|x| \geqslant |b_m|$. We have

$$|h_m(x)| \leqslant \prod_{j=1}^{m-1} \frac{1}{|1 - x/b_j|}|^{p_j} \leqslant \varepsilon_m \tag{5}$$

and in the same way $|h_{m+1}(x)| \leqslant \varepsilon_m$ hence

$$|h_{m+1}(x) - h_m(x)| \leqslant \varepsilon_m\,. \tag{6}$$

Now let us consider $h_{m+1}(x) - h_m(x)$ when $|x| < |b_m|$ and let us put

$$u(x) = \frac{1}{\left(1 - \dfrac{x}{b_{m+1}}\right)^{p_{m+1}}} - 1 = -\frac{\sum_{j=1}^{p_{m+1}} \binom{p_{m+1}}{j}\left(-\dfrac{x}{b_{m+1}}\right)^j}{\left(1 - \dfrac{x}{b_{m+1}}\right)^{p_{m+1}}}\,.$$

Then it is clear that $|u(x)| \leqslant \max_{1\leqslant j\leqslant p_m} \left|\binom{p_{m+1}}{j}\right| \cdot \left|\dfrac{b_m}{b_{m+1}}\right|^j$ and then for $1\leqslant j\leqslant t_{m+1}$, as $|j|\geqslant|t_{m+1}|$, we obtain $\left|\binom{p_{m+1}}{j}\right| \leqslant \left|\dfrac{p_{m+1}}{t_{m+1}}\right|$ by Lemma B.

Now for $j > t_{m+1}$ we see that $\left|\dfrac{b_m}{b_{m+1}}\right|^j \leqslant \left|\dfrac{b_m}{b_{m+1}}\right|^{t_{m+1}} = \varepsilon_m$ and then every term $\binom{p_{m+1}}{j}\left(-\dfrac{x}{b_{m+1}}\right)^j$ is upper bounded by $\max(\omega_{m+1}, \varepsilon_m)$ and therefore $|u(x)| \leqslant \max(\omega_{m+1}, \varepsilon_m)$ whenever $x \in D \cap d(0, |b_m|)$.

Finally by (6) we see that $\|h_{m+1} - h_m\|_D \leqslant \max(\omega_{m+1}, \varepsilon_m)$ hence the sequence h_m converges in $H(D)$ to the convergent infinite product

$$h(x) = \prod_{j=1}^{\infty} \frac{1}{(1 - x/b_j)^{p_j}}.$$

By (3) and by the definition of D it is easily seen that the increasing filter $\mathcal{F}$ of center 0, of diameter 1, is a T-filter and it is the only one T-filter on D [E$_4$].

On the other hand, by (5) we have $|h(x)| \leqslant \varepsilon_m$ whenever $x \in D \setminus d^-(0, |b_m|)$ and therefore h is clearly annulled by $\mathcal{F}$, and it is strictly annulled by $\mathcal{F}$ (because $\mathcal{F}$ is the only T-filter on D), and $h(x) = 0$ whenever $x \in \mathcal{P}(\mathcal{F})$ hence $h \in \mathfrak{J}_0(\mathcal{F})$.

Now let us consider the series $\sum_{j=1}^{\infty} p_j/(b_j - x)$. Since $\lim_{m\to\infty} |p^m| = 0$, by (4) we see that series series converge to a limit $f \in H(D)$. Moreover, it is easily seen that $\lim_{\substack{|x|\to 1^- \\ x\in D}} |p_j/(b_j - x)| = |p_j|$ for every $j \in \mathbf{N}^*$, hence, by (2), we have $\lim_{\substack{|x|\to 1^- \\ x\in D}} |f(x)| = p_1$, hence f is not annulled by $\mathcal{F}$.

Since $\mathcal{F}$ is the only T-filter, f is then quasi-invertible.

At last, we shortly verify that h is solution of $\mathcal{E}(f)$.

By Corollary of [E$_6$] we know that $h' \in H(D)$ and the sequence h'_m converges to h' in $H(D)'$. On the other hand, it is easily seen that

$$h'_m = \left(\sum_{j=1}^{m} \frac{p_j}{(1 - x/b_j)^{p_j}} \right) h_m = h_m \sum_{j=1}^{m} \frac{p_j}{b_j - x}$$

hence

$$\lim_{m\to\infty} h'_m = h \left(\sum_{j=1}^{\infty} \frac{p_j}{b_j - x} \right) = hf$$

and therefore h is a solution of $\mathcal{E}(f)$, and that ends the proof of Proposition A.

III. The proof of Theorem 2.

LEMMA C. *Let q, n be integers such that $0 < n < q$. Then* $|q!/n!| \leqslant p^{1-(q-n)/p}$.

PROOF. $q!/n!$ has $q-n$ consecutive factors. It is easily seen among these $q-n$ factors, the number of them that are multiple of p, is at least $\mathrm{Int}(q-n)/p)$ and therefore $v(q!/n!) \geq \mathrm{Int}((q-n)/p) > (q-n)/p-1$ and that ends the proof of Lemma C.

LEMMA D. *Let* $R \in [p^{-1/p}, 1[$, *let* $\varepsilon \in]0, 1/p[$ *and let* $\varphi(x) = \sum_{-\infty}^{+\infty} a_n x^n$ *be a Laurent series convergent for* $|x| = R$, *such that* $\sup |a_n| R^n = |a_q| R^q$ *with* $q < 0$. *Then* φ *does not satisfy the inequality*

$$(1) \qquad \left| \frac{\varphi'(x)}{\varphi(x)} - 1 \right| < \varepsilon \quad \textit{for all } x \in C(0, R).$$

PROOF. We suppose φ satisfies (1) and we put $M = |a_q| R^q$. By (1) it is easily seen that

$$(2) \qquad |na_n - a_{n-1}| R^{n-1} \leq \varepsilon M \quad \text{for every } n \in \mathbf{Z}.$$

If $q = -1$, relation (2) gives $|-a_{-1}|/R \leq \varepsilon |a_{-1}|/R$ hence $\varphi = 0$. We will suppose $q < -1$ and we will prove that (3) $|a_n| = |a_q(-n-1)!|/|(-q-1)!|$ for $n = q+1, q+2, \ldots, -2, -1$. Indeed, suppose it has been proven up to the range t with $q \leq t < -1$ and let us prove it at the range $t+1$. By (2) we have

$$(3) \quad |(t+1)a_{t+1} - a_t| R^t \leq \varepsilon |a_q| R^q \quad \text{hence} \quad |(t+1)a_{t+1} - a_t| \leq \frac{\varepsilon |a_q|}{R^{t-q}}$$

hence by (3)

$$(4) \qquad |(t+1)a_{t+1} - a_t| \leq \frac{\varepsilon |a_t| |(-q-1)!|}{R^{t-q} |(-t-1)!|}.$$

Now by Lemma C we know that $|(-q)!/(-t)!| \leq p^{1-(t-q)/p}$. Since $R \geq p^{-1/p}$, we see that $R^{t-q} \geq p^{-(t-q)/p}$; hence $|(-q)!/(-t)!| \leq pR^{t-q}$ and therefore $\varepsilon |(-q)!/(-t)!| \leq R^{t-q}$. Then by relation (4) we have

$$(5) \qquad |(t+1)a_{t+1} - a_t| < |a_t| \quad \text{hence} \quad |(t+1)a_{t+1}| = |a_t|,$$

and therefore

$$|a_{t+1}| = \left| \frac{a_t}{t+1} \right| = \frac{|a_q| |(-t-2)!}{|(-(t+1)!)|}$$

so that relation (3) is proven at the range $t+1$. It is then proven for every n up to -1. Then relation (2) for $n = 0$ gives us $|a_{-1}| R^{-1} \leq \varepsilon |a_q| R^q$, hence by (3) we have $|a_q|/|(-q-1)!| \leq \varepsilon R^{q+1} |a_q|$ and therefore

$$(6) \qquad \varepsilon |(-q-1)!| R^{q+1} \geq 1$$

but we know that $R^{q+1}|(-q-1)!| \leqslant p^{-(q+1)/p} p^{1+(q+1)/p} < 1/\varepsilon$ hence (6) is impossible.

Lemma D is then proven.

The following lemma was given in [S5], in constructing the «Produits Bicroulants» (twice collapsing meromorphic products).

LEMMA E. *Let $\rho, R', R'', R \in R_+$ with $0 < R' < R'' < R$. There exist sequences $(b'_n)_{n \in \mathbb{N}}$ and $(b''_n)_{n \in \mathbb{N}}$ in $\Gamma(0, R', R'')$ with $|b'_n| > |b'_{n+1}|$, $\lim_{n \to \infty} |b'_n| = R'$, $|b''_n| < |b''_{n+1}|$, $\lim_{n \to \infty} |b''_n| = R''$, such that, if we denote by D the set $d(0,R) \setminus \left[\left(\bigcup_{n=1}^{\infty} d^-(b'_n, \rho)\right) \cup \left(\bigcup_{n=1}^{\infty} d^-(b''_n, \rho)\right)\right]$ the algebra $H(D)$ has an element $\varphi \in H(D)$ satisfying $\lim_{\substack{|x| \to R' \\ x \in D}} \varphi(x) = 1$ and $\lim_{\substack{|x| \to R'' \\ x \in D}} \varphi(x) = 0$.*

PROOF OF THEOREM 2. Let $\omega_1, \ldots, \omega_t$ be points in $d(0,1)$ such that $\omega_1 = 0$, $|\omega_i - \omega_j| = 1$ whenever $i \neq j$. Let $r \in]0,1[$ and let $(b_m)_{m \in \mathbb{N}}$ be a sequence in $d^-(0,t)$ such that $|b_m| < |b_{m+1}|$ and $\lim_{m \to \infty} |b_m| = r$ and let $(q_m)_{m \in \mathbb{N}}$ be a sequence of integers such that $q_1 < q_m$ for all $m > 1$, $\lim_{m \to \infty} q_m = +\infty$ and $\lim_{m \to \infty} \prod_{j=1}^{m-1} |b_j / b_m|^{(p^{q_j})} = 0$. Let $T_m = d^-(b_m, |b_m|)$, let $p_m = p^{q_m}$ and let $A = d^-(0,r) \setminus \left(\bigcup_{m=1}^{\infty} T_m\right)$.

It is easily seen that A admits a T-sequence (T_m, q_m) [S1]. Let $\mathcal{T}$ be the increasing T-filter of center 0, of diameter r on A. First we will construct an infraconnected clopen set included in $d(0,1)$, of diameter 1, satisfying the following conditions:

(1) $\Omega \cap d^-(0,r) = A$.

(2) Ω has an increasing T-filter $\mathcal{F}$ of center 0, of diameter 1.

(3) Ω has a decreasing T-filter $\mathcal{G}$ of center 0, of diameter $R \in]r, 1[$.

(4) The only T-filters of Ω are $\mathcal{T}$, $\mathcal{F}$, $\mathcal{G}$.

(5) There exists φ and $\psi \in H(\Omega) \setminus \{0\}$ such that

$$\varphi(x) = 1, \quad \psi(x) = 0 \qquad \text{for } x \in \Omega \cap d(0,R)$$

and

$$\varphi(x) = 0, \quad \psi(x) = 1 \qquad \text{for } x \in \Omega \setminus d^-(0,1).$$

Let $\rho \in]0, f[$. By Lemma E there exist sequences $(\beta'_n)_{n \in \mathbb{N}}$ and

$(\beta''_n)_{n \in \mathbb{N}}$ in $\Gamma(0, R, 1)$ such that

$$R < |\beta'_{n+1}| < |\beta'_n|, \qquad \lim_{n \to \infty} \beta'_n = R,$$

$$|\beta''_n| < |\beta''_{n+1}| < 1, \qquad \lim_{n \to \infty} |\beta''_n| = 1$$

and such that the set

$$\Lambda = d(0,1) \setminus \left[\left(\bigcup_{n=1}^{\infty} d^-(\beta'_n, \rho) \right) \cup \left(\bigcup_{n=1}^{\infty} d^-(\beta''_n, \rho) \right) \right],$$

defines an algebra $H(\Lambda)$ that contains elements φ satisfying $\varphi(x) = 1$ for $|x| \leqslant R$, $\varphi(x) = 0$ for $|x| = 1$. Let us put $\psi = 1 - \varphi$ and let Ω be the set $A \cup (\Lambda \setminus d^-(0, r))$.

Ω has clearly three T-filter:

the filter $\mathcal{T}$ on A

the increasing filter $\mathcal{F}$ of center 0, of diameter 1 that strictly annulls φ.

the decreasing filter $\mathcal{G}$ of center 0, of diameter R that strictly annulls ψ.

It is easily seen these three T-filters are the only T-filters on Ω, and Ω, φ, ψ are then defined.

Let $f(x) = \left(\sum_{m=1}^{\infty} p^{q_m}/(1 - x/b_m) \right)$ and let $f_1(x) = \varphi(x) f(x) + \psi(x)$.

Then $f_1(x) = f(x)$ when $x \in \Omega \cap d(0, R)$ and $f_1(x) = 1$ when $x \in \Omega \setminus d^-(0, 1)$. We can deduce that f_1 is a quasi-invertible element in $H(\Omega)$. Indeed, by Proposition B, f is not annulled by $\mathcal{T}$ and by $\mathcal{G}$, hence f_1 is not annulled by $\mathcal{T}$ and by $\mathcal{G}$ either; on the other hand, as $f_1(x) = 1$ when $|x| = 1$, $f_1$is not annulled by $\mathcal{F}$; hence f_1 is not annulled by any one of the three T-filters on Ω so that it is quasi-invertible in $H(\Omega)$.

By Proposition B $\mathcal{E}(f_1)$ has a solution $g_1 = \prod_{m=1}^{\infty} 1/(1 - x/b_m)^{p_m}$.

Now, for each $y = 2, \ldots, t$ let $\Omega_j = \omega_j + \Omega = \{x + \omega_j \mid x \in \Omega\}$ and let $f_j \in H(\Omega_j)$ defined by $f_j(x + \omega_j) = f_1(x)$. In Ω_j the equation $\mathcal{E}(f_j)$ has a solution g_j defined by $g_j(x + \omega_j) = g_1(x)$. Let $D = \bigcap_{j=1}^{t} \Omega_j$ and let $f(x) = \prod_{j=1}^{t} f_j(x) \in H(D)$. Obviously, $f(x) = f_j(x)$ when $|x - \omega_j| < 1$ and $f(x) = 1$ when $|\xi - \omega_l| = 1$ for every $l = 1, \ldots, t$. Each one of the f_j is quasi-invertible in $H(D)$ so that f is also quasi-invertible.

Now each g_j $(1 \leq j \leq t)$ is a solution of $\mathcal{E}(f)$. Indeed, when $|x - \omega_j| < 1$ we have $g_j'(x) = f_j(x) g_j(x) = f(x) g_j(x)$ and when $|x - \omega_j| = 1$, $g_j(x) = 0$.

On the other hand, the g_j clearly have supports two by two disjointed, hence they are linearly independent, and that shows $\mathcal{S}(f)$ has dimension $\geq t$.

We will end the proof in showing that $\{g_1, \ldots, g_t\}$ generates $\mathcal{S}(f)$.

Log will denote the real logarithm function of base p. Let v be the valuation defined in K by $v(x) = -\log|x|$ when $x \neq 0$ and $v(0) = +\infty$. When A is an infraconnected set containing 0, and $f \in H(A)$ we put

$$v(f, \mu) = \lim_{\substack{v(x) \to \mu \\ v(x) \neq \mu \\ x \in D}} v(f(x)) \, [\mathrm{E}_2, \mathrm{E}_3, \mathrm{E}_4].$$

For each $j = 1, \ldots, t$, let $D_j = d^-(\omega_j, 1) \cap D$ and $B_j = d^-(\omega_j, R)$; let $D' = D \setminus \bigcup_{j=1}^{t} D_j$. By definition of f we see that $f(x) = 1$ for all $x \in D'$ and $d^-(\alpha, 1) \subset D'$ for every $\alpha \in D'$. Then it is well known that the equation $y' = y$ has no solution y in $H(d^-(\alpha, 1))$ but the zero solution. Let $h \in \mathcal{S}(f)$. For every $\alpha \in D'$, the restriction of h to $d^-(\alpha, 1)$ is a solution of the equation $y' = y$ that belongs to $H(d^-(\alpha, 1))$ hence we see that $h(x) = 0$ for all $x \in D'$. Since D' is equal to $d(0, 1) \setminus \bigcup_{j=1}^{t} d^-(\omega, 1)$ we see that

$$v(h, 0) = +\infty. \tag{6}$$

Now let us consider $h(x)$ when $x \in B_1$.

Since $D_1 = \Omega \cap d^-(0, 1)$ the three T-filters $\mathcal{T}$, $\mathcal{F}$, $\mathcal{G}$ of Ω are secant to D_1 and they are the only T-filters on D_1. Then $\mathcal{T}$ is the only one T-filter on B_1 because $\mathcal{F}$ and $\mathcal{G}$ are not secant to $d(0, R)$. The algebra $H(B_1)$ has no divisor of zero. Consider the restriction $\tilde{f}_1$ of to D_1 and the restriction $\hat{f}_1$ to B_1. In $H(B_1)$ the space $\mathcal{S}(\hat{f}_1)$ has dimension one by Theorem 3 of [E_7], hence there exists $\lambda_1 \in k$ such that $h(x) = \lambda_1 g_1(x)$ whenever $x \in B_1$.

Since $g_1 \in \mathfrak{Z}_0(\mathcal{T})$, that implies $h(x) = 0$ whenever $x \in \Gamma(0, r, R)$ hence $v(h, -\log R) = +\infty$. We will deduce that $v(h, \mu) = +\infty$ whenever $\mu \in [0, -\log R]$.

Indeed, suppose this is not true. Then h is strictly annulled by an increasing T-filter of center 0, of diameter $> R$, hence h is strictly annulled by $\mathcal{F}$. Since $\lim\limits_{\substack{|x|\to 1^- \\ x\in D}} \varphi(x) = \lim\limits_{\substack{|x|\to 1^- \\ x\in D}} \psi(x) = 1$. there exists $s \in]R, 1[$ such that

$$\left| \frac{h'(x)}{h(x)} - 1 \right| \leqslant \frac{1}{p^2} \quad \text{for } x \in D \cap \Gamma(0, s, 1). \tag{7}$$

On the other hand, it is easily seen that $h(x)$ is equal to a Laurent series in each annulus $\Gamma(0, |b''_n|, |b''_{n+1}|)$ and for every $s < 1$ there exist intervals $[r', r''] \subset]s, 1[$ such that the function $v(h, \mu)$ is strictly decreasing in $[-\log r'', -\log r']$ and such that $h(x)$ is equal to a Laurent series $\sum\limits_{-\infty}^{+\infty} a_n x^n$. Let $\rho \in]r', r''[$, since $v(h, \mu)$ is strictly decreasing in $[-\log r'', -\log r']$ there exists $q < 0$ such that $|a_q| \rho^q = \sup\limits_{n\in \mathbf{Z}} |a_n| \rho^n$. Then h satisfies the hypothesis of Lemma D and relation (7) is impossible. But then $v(h, \mu) = +\infty$ for every $\mu \in [0, -\log r]$ It follows that $h(x) = 0$ for every $x \in \Gamma(0, R, 1)$ because if there existed a point $\alpha \in \Gamma(0, R, 1)$ with $h(\alpha) \neq 0$, α should be the center of an increasing T-filter that would annull h but the unique T-filter of center α is $\mathcal{F}$ and we have just seen that $\mathcal{F}$ does not annull h.

Thus we have now proven that $h(x) = 0$ for all $x \in B_1$ such that $r \leqslant |x| < 1$. Since $g_1(x) = 0$ whenever $x \in \Gamma(0, r, 1)$, the relation $h(x) = \lambda_1 g_1(x)$ is then true in all B_1. In the same way, for each $j = 2, \ldots, t$, we can show there exists $\lambda_j \in K$ such that $h(x) = \lambda_j g_j(x)$ for every $x \in B_j$ and then $h(x) = \sum\limits_{j=1}^{t} \lambda_j g_j(x)$ is true in $\bigcup\limits_{j=1}^{t} B_j$, and of course in D', hence it is true in all D. That finishes proving $\{g_1, \ldots, g_t\}$ is a base of $\mathcal{S}(f)$.

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