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# Torsional Instabilities of Greenhill Type in Elastic Rods.

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ABSTRACT - The equilibrium and stability of an elastic rod subjected to a wrench is studied. Stability conditions are given for the class of helical and straight solutions, generalizing the classical results of Greenhill.

## 1. Introduction.

The problem of stability of a Kirchhoff rod subjected to a wrench has been investigated first by Greenhill [7] in 1883 and later by many others (vid. Love [8], Antman [3] and the references quoted therein). In this note we make use of a general theory of nonlinearly elastic rods developed by Antman in [1-3], as restricted to the hyperelastic case, as well as of some results of Ericksen [4, 5]; our purpose is to investigate the stability of certain classes of solutions.

Both Antman and Ericksen generalize the classical theory of Kirchhoff by allowing for axial extension, shearing of the cross-section with respect to the axis, and certain cross-sectional deformations; moreover, they accept nonlinear constitutive equations. This richer structure gives way to a wider variety of solutions, with classical results still included as particular cases. We here study the problem of stability of helical solutions, under the action of dead loads (cfr. Antman [2] and Ericksen [4]). We use a semi-inverse method modelled after Ericksen [5], and specify the load to be a wrench, *i.e.*, a force of magnitude $F$ acting along the axis of the helix and a couple of magnitude $M$ parallel to such axis.

As both straight and helical solutions can be maintained under the same wrench, we are faced with a problem of nonuniqueness of equilib-

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rium states. By choosing a suitable class of perturbed solutions, and accepting an energy criterion for stability, we give a formal condition of stability for solutions branching from a trivial (*i.e.*, straight) solution. We then restrict attention to helices, and assume the material to be transversely isotropic about the rod axis; under these hypothesis we obtain explicit stability conditions depending on the force $F$ and the couple $M$, as well as the geometrical-kinematic variables of a helix, namely, pitch, radius, length and extension. Such conditions generalize the classical results attributed to Greenhill by Love [8], which, on the other hand, can also be found directly, as shown in Section 3.

## 2. Equilibrium equations for a hyperelastic rod.

A rod is a material curve given by a vector-valued function of a material coordinate $s$:

(2.1) $$\boldsymbol{r} = \boldsymbol{r}(s, t), \qquad 0 \leqslant s \leqslant l,$$

and equipped with a pair of vector-valued functions:

(2.2) $$\boldsymbol{d}_\alpha = \boldsymbol{d}_\alpha(s, t), \qquad \alpha = 1, 2,$$

satisfying the condition:

(2.2′) $$\boldsymbol{d}_1 \times \boldsymbol{d}_2 \cdot \boldsymbol{r}' \neq 0,$$

where a prime denotes differentation with a respect to $s$. If $s$ is the arc length parameter of the curve in the reference configuration, $l = \int_0^l ds$ is the total length of the curve. The plane passing through the point of coordinate $s$ on the curve and containing $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ is called the *section* of the rod at $s$.

For a hyperelastic rod, the density of strain-energy is specified by the scalar-valued function

(2.3) $$W = W(\boldsymbol{r}', \boldsymbol{d}_\alpha, \boldsymbol{d}'_\alpha;\ s).$$

The total stored energy is

(2.4) $$E = \int_0^l W\, ds;$$

under the assumption that only end loads are applied to the rod, we can obtain the equilibrium equations (cfr. [4])

$$\left(\frac{\partial W}{\partial \boldsymbol{r}'}\right)' = \mathbf{0}, \qquad \left(\frac{\partial W}{\partial \boldsymbol{d}_\alpha'}\right)' - \frac{\partial W}{\partial \boldsymbol{d}_\alpha} = \mathbf{0}. \tag{2.5}$$

Of course, in order to have a well-posed boundary-value problem, these equations musy be supplemented by suitable boundary conditions.

The differential system (2.5) has some firts integrals. One of these reads off directly from $(2.5)_1$:

$$\frac{\partial W}{\partial \boldsymbol{r}'} = \text{const} \tag{2.6}$$

and is interpreted as the balance of forces. Balance of moments, as shown in [4], has the form

$$\boldsymbol{r} \times \frac{\partial W}{\partial \boldsymbol{r}'} + \boldsymbol{d}_\alpha \times \frac{\partial W}{\partial \boldsymbol{d}_\alpha'} = \text{const}. \tag{2.7}$$

Finally, if $W$ does not depend explicitly on $s$, we obtain (cfr. [4]) that

$$W - \boldsymbol{r}' \cdot \frac{\partial W}{\partial \boldsymbol{r}'} - \boldsymbol{d}_\alpha' \cdot \frac{\partial W}{\partial \boldsymbol{d}_\alpha'} = \text{const}. \tag{2.8}$$

If we put

$$\boldsymbol{n} = \frac{\partial W}{\partial \boldsymbol{r}'} \quad \text{and} \quad \boldsymbol{m} = \boldsymbol{d}_\alpha \times \frac{\partial W}{\partial \boldsymbol{d}_\alpha'},$$

equations (2.6) and (2.7) take the well-known form

$$\boldsymbol{n}' = \mathbf{0}, \qquad (\boldsymbol{r} \times \boldsymbol{n} + \boldsymbol{m})' = \mathbf{0}. \tag{2.9}$$

Let the reference configuration of the rod be straight and prismatic. Then $\boldsymbol{r}'$ has constant value $\boldsymbol{e}_3$ in the reference configuration, with $\boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1$. We introduce a referential orthonormal basis $\{\boldsymbol{e}_i\}$: $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$, $i, j = 1, 2, 3$. (Here and henceforth Greek indices take values on 1, 2; Latin indices on 1, 2, 3; the summation convention holds).

We deal with the equilibrium problem consisting of (2.5) and the geometrical boundary conditions

$$\boldsymbol{r}(0) = \mathbf{0}, \qquad \boldsymbol{r}(l) \times \boldsymbol{e}_3 = \mathbf{0}; \tag{2.10}$$

moreover a wrench is applied (cfr. [2, 8]), *i.e.*, a pair of a force $\boldsymbol{F}$ and a couple $\boldsymbol{M}$, with

$$\boldsymbol{F} = F\boldsymbol{e}_3, \qquad \boldsymbol{M} = M\boldsymbol{e}_3, \tag{2.11}$$

$F$ and $M$ two assigned constants. Notice that the choice of the $\boldsymbol{d}_\alpha$'s remains arbitrary.

In view of (2.6) and (2.7), the equations of equilibrium may be equivalently written as

$$\frac{\partial W}{\partial \boldsymbol{r}'} = F\boldsymbol{e}_3; \qquad \boldsymbol{r} \times \frac{\partial W}{\partial \boldsymbol{r}'} + \boldsymbol{d}_\alpha \times \frac{\partial W}{\partial \boldsymbol{d}'_\alpha} = \boldsymbol{r}(l) \times F\boldsymbol{e}_3 + M\boldsymbol{e}_3. \tag{2.12}$$

These equations admit straight, helical and circular solutions; as Ericksen has shown in [4], these are the only uniform states which solve the system (2.6), (2.7).

We assume the rod to be homogeneous, but this assumption by itself does not help much as far as the problem of establishing existence and/or multiplicity of solutions is concerned. To obtain explicit results, we must introduce further assumptions on the function $W$ and, in addition, study rods with fairly simple geometry.

For instance, if we restrict ourselves to the generalized Kirchhoff's theory of rods (namely, if we stipulate that $\boldsymbol{d}_\alpha \cdot \boldsymbol{d}_\beta = \delta_{\alpha\beta}$), our analysis is easied by the results of Antman [2]. We can also compare our results on stability with the classical results of Greenhill [7]. Indeed, the independent kinematical unknowns are now six, and equations (2.12) are sufficient to determine the equilibrium configurations, while equation (2.8) can be seen as a constitutive restriction. Following Antman [2], we introduce an orthonormal basis $\{\boldsymbol{d}_i\}$, with $\boldsymbol{d}_3 = \boldsymbol{d}_1 \times \boldsymbol{d}_2$ and relate this basis with respect to the fixed basis $\{\boldsymbol{e}_i\}$ by means of the Euler angles $\psi$, $\theta$, $\phi$ (see Love [8]). As Antman has proved, the same terminal load can maintain deformations with both straight and helical axis. We introduce the strain variables

$$y^i = \boldsymbol{r}' \cdot \boldsymbol{d}^i, \qquad u^i = \frac{1}{2}\varepsilon^{ijh}\boldsymbol{d}'_j \cdot \boldsymbol{d}_h, \tag{2.13}$$

where $\varepsilon^{ijh}$ is the alternating tensor. We need not to record here the relations between $u^i$ and the Euler angles, to be found in Antman [2], Section 2.

The equations of motion have the form (2.9); the constitutive relations for $\boldsymbol{n}$ and $\boldsymbol{m}$ are assumed to satisfy certain conditions of strict monotonicity, coercitivity and isotropy (see [1, 2]).

It proves convenient to define:

$$(2.14)\qquad \begin{cases} y = -\cos\phi y_1 + \sin\phi y_2, & u = \psi'\sin\theta, \\ m = -\cos\phi m_1 + \sin\phi m_2, & m_\alpha = \boldsymbol{m}\cdot\boldsymbol{d}_\alpha, \\ n = F\sin\theta, \qquad p = F\cos\theta, & q = \boldsymbol{m}\cdot\boldsymbol{d}_3 = m_3. \end{cases}$$

It follows from these definitions that $y$, $y_3$ and $m$ satisfy the constitutive relations

$$(2.15)\quad y = Y(n, p, q, u), \qquad y_3 = Z(n, p, q, u), \qquad m = M(n, p, q, u);$$

in addition $q = \boldsymbol{m}(l)\cdot\boldsymbol{d}_3(l) = \text{const}$, $u = \text{const}$.

Under the present circumstances, the following theorems hold.

i) If $\sin\theta = 0$, and if $p$ and $q$ (*i.e.*, $F$ and $M$) are prescribed arbitrarily, then the governing equations have a unique solution. This deformation has a straight axis along $\boldsymbol{e}_3$, with sections perpendicular to it, and has a constant twist.

ii) Let the constitutive functions $Y$ and $Z$ satisfy the mild growth conditions

$$\frac{Y(n, p, q, u)}{|u|} \to 0, \qquad \frac{Z(n, p, q, u)}{|u|} \to 0, \qquad \text{as } |u| \to \infty,$$

and let $F \neq 0$ and $q$ be arbitrary. Then, there is a number $\delta > 0$ depending on $F$ and $q$ such that there is a deformation with a nondegenerate helical axis for any prescribed $\theta$ satisfying $0 < \sin\theta < \delta$.

iii) Let $q \neq 0$ be arbitrary. Then there is a number $\delta > 0$ depending on $q$ such that there is a deformation with a nondegenerate helical axis maintained by zero terminal forces for any prescribed $\theta$ satisfying $0 < \sin\theta < \delta$.

These theorems are demonstrated in a more general form in [2], where it is also shown that a deformation with helical axis can be maintained solely by a force or a couple along the axis $\boldsymbol{e}_3$, and where the case of deformations with circular axis is also studied. By summary, we have that, for any prescribed load $F$ and $M$, there exists a helical solution with a fixed angle $\theta$; the straigh (= degenerate) solution is included, for $\theta = 0$.

## 3. Kirchhoff's problem.

In this section we want to show that one can recover Love's results on equilibrium and stability for an inextendible rod, in the simple case

$\boldsymbol{F} = \mathbf{0}$, namely, that there exist helical solutions; that the helical form can be maintained by terminal couple alone; and that, when twisting couple exceeds a critical value, the straight twisted rod becomes unstable.

Let us consider a rod, straight and not prismatic in the reference unstressed state, under a twisting couple $\boldsymbol{M} = M\boldsymbol{e}_3$. (If the rod were prismatic in the unstressed state, a wrench would be required to maintain a helical form; see Love [8], p. 415). The inextensibility condition reads: $(\boldsymbol{r}'\cdot\boldsymbol{r}')^{1/2} = \text{const}$. The equilibrium equations (2.12) become

$$\boldsymbol{m} = \boldsymbol{d}_\alpha \times \frac{\partial W}{\partial \boldsymbol{d}'_\alpha} = M\boldsymbol{e}_3 . \tag{3.1}$$

Taking the scalar product of (3.1) by $\boldsymbol{e}_3$, we obtain:

$$d_{i3}\, m^i = M . \tag{3.2}$$

where

$$d_{i3} = \boldsymbol{d}_i \cdot \boldsymbol{e}_3 , \qquad \boldsymbol{m} = m^i \boldsymbol{d}_i ,$$

or, alternatively, by scalar product with $\boldsymbol{d}_i$:

$$m_i = d_{i3} M . \tag{3.3}$$

More explicitely, (3.3) reads

$$\begin{cases} m_1 = -\cos\phi \sin\theta M , \\ m_2 = \sin\phi \sin\theta M , \\ m_3 = \cos\theta M . \end{cases} \tag{3.4}$$

In terms of the strain variables (2.13) $u^i$ and $y^i$, the strain energy density $W = W(u, y)$ gives rise to the constitutive equations

$$m_i = \frac{\partial W}{\partial u^i} , \qquad n_i = \frac{\partial W}{\partial y^i} , \tag{3.5}$$

and, by (3.3)

$$\frac{\partial W}{\partial u^i} = d_{i3} M . \tag{3.6}$$

We accept the classical Kirchhoff-Love assumption and let $W$ be a quadratic function of the variables $u^i$ only:

$$W = \frac{1}{2} \sum_{i=1}^{3} c_i (u^i)^2 , \qquad c_1 = c_2 = c . \tag{3.7}$$

Accordingly, (3.6) takes the aspect

(3.8) $$c_i u_i = M d_{i3}, \qquad u_i = \delta_{ih} u^h .$$

Then, as $\boldsymbol{m}' = \mathbf{0}$, and using (3.7), we have

$$m_3 = \text{const} \neq 0$$

or, equivalently,

$$u_3 = \text{const} \neq 0 \quad \text{and} \quad \theta = \text{const} \qquad \left(0 \leqslant \theta \leqslant \frac{1}{2}\pi\right).$$

If $\theta$ vanishes, we have only twist, no bending: the deformed configuration is straight; if $\theta = (1/2)\pi$, we have only bending, no twist: the deformed configuration is circular. Finally, using the relations referred to above between $u_i$ and the Euler angles, we can see that the solutions we are interested have the form:

i) helical solutions: $\theta = \text{const}$, $\psi' = M/c = \text{const}$,
$\phi' = (1/c_3 - 1/c) M \cos\theta$,

ii) straight solutions: $\theta = 0$, $\psi' + \phi' = M/c_3 = \text{const}$.

Now we wish to study the stability problem of straight deformed configurations versus helical perturbations. For a prescribed end load, a solution will be called stable if it minimizes the total energy in the class of neighbouring helical and straight deformations.

With this fairly naïve notion of stability, and confining our attention to helices with small $\theta$ which form one complete turn (as requested in Love [8]), we can easily recover the results obtained by Greenhill [7], in a different way (and in the general case when $F$ and $M$ do not vanish). In such straight and helical configurations, as to solve our static problem, the total energies are, respectively

$$E_{\text{str}} = \frac{1}{c_3} M l \left(\frac{1}{2} M - \frac{2\pi c}{l}\right), \qquad E_{\text{hel}} = -\frac{1}{2c_3} M^2 l;$$

thus, when $M$ exceeds $2\pi c/l$, the energy of the helical rod becomes greater than the energy of the straight rod, or, rather, according to the notion of stability explained above, the straight twisted rod becomes unstable.

In the next section we adress ourselves to the corresponding problem in the non-linear case, allowing for non-null extensibility. We still make use of the results of Sections 1 and 2, and rely on an energy criterion to obtain general conditions on stability, again restricting ourselves to the class of helical and straight solutions.

## 4. Equilibrium and stability for an extensible rod.

Let us call *trivial* any straight solution of the boundary value problem (2.10), (2.12). In order to study the class of nontrivial solutions in a neighborhood of a trivial one (and even to define this neighborhood), we introduce the perturbation variables $v^i$, $v^i_\alpha$ which appear in the following representation formulae:

$$\text{(4.1)} \qquad \begin{cases} \boldsymbol{r} = \boldsymbol{r}^* + v^\beta \boldsymbol{d}^*_\beta + v^3 \boldsymbol{r}^{*\prime}, \\ \boldsymbol{d}_\alpha = \boldsymbol{d}^*_\alpha + v^\beta_\alpha \boldsymbol{d}^*_\beta + v^3_\alpha \boldsymbol{r}^{*\prime}; \end{cases}$$

(4.1) gives the vector $\boldsymbol{r}$ and $\boldsymbol{d}_\alpha$ in terms of the corresponding vectors $\boldsymbol{r}^*$ and $\boldsymbol{d}^*_\alpha$ in the trivial configuration, as suggested elsewhere, for shell problems, by J. Ericksen.

The perturbation variables $v^i$, $v^i_\alpha$ are the components of the differences $(\boldsymbol{r} - \boldsymbol{r}^*)$ and $(\boldsymbol{d}_\alpha - \boldsymbol{d}^*_\alpha)$ in the natural *trivial* basis $\{\boldsymbol{d}^*_\alpha, \boldsymbol{r}^{*\prime}\}$; if we assign $v^i$ and $v^i_\alpha$ as regular functions of $s$, they determine uniquely, by (4.1), the equation of a configuration as closer to the trivial solution as smaller we take these variables. We are interested in finding lists of $(v^i, v^i_\alpha)$ corresponding to solutions of the boundary value problem (2.10), (2.12), *i.e.*, to equilibrium configurations which are *close* to the trivial solution. We remark that (4.1) are general enough to encompass also helices; on the other hand, Antman's theorems quoted in Section 2 insure the existence of helical solutions.

If we set $\boldsymbol{d}^*_3 = \boldsymbol{r}^{*\prime}$, the metric tensor $a_{ih}$ and the connection coefficients $c_{ij}$ corresponding to the basis $\{\boldsymbol{d}^*_i\}$ are given by

$$\text{(4.2)} \qquad a_{ih} = \boldsymbol{d}^*_i \cdot \boldsymbol{d}^*_h \quad \text{and} \quad c_{ih} = \boldsymbol{d}^{*\prime}_i \cdot \boldsymbol{d}^*_h .$$

Let us introduce the (constant) extension $A$ at the trivial solution and assume $\boldsymbol{d}^*_\alpha$ to be orthonormal and $\boldsymbol{r}^{*\prime} = A\boldsymbol{e}_3$. Then (4.2) become

$$\text{(4.3)} \qquad \begin{cases} a_{\alpha\beta} = \delta_{\alpha\beta}, \quad a_{\alpha 3} = 0, \quad a_{33} = A^2, \\ c_{\alpha\beta} = -c_{\beta\alpha} = \varepsilon_{\alpha\beta} B, \quad c_{i3} = c_{3i} = 0, \end{cases}$$

where $\varepsilon_{\alpha\beta}$ is the bidimensional alternating tensor and $B$ is a constant, representing the twist.

By the assumption of material objectivity, the strain energy density $W$ can be expressed in terms of the scalar and triple scalar products constructed with the vector arguments shown in (2.3):

$$\text{(4.4)} \qquad W = W(v^i, v^{i\prime}, v^i{}_\alpha, v^{i\prime}{}_\alpha;\ A, B).$$

A slightly simpler form can be given to (4.4), introducing the «covariant» derivative:

$$f^i{}_{|s} = f^{i'} + c^j{}_h f^h ,$$

and writing

$$W = W(v^i{}_{|s}, v^i{}_\alpha, v^{i'}{}_{\alpha|s};\ A,\ B). \tag{4.4'}$$

Whenever $\boldsymbol{r}$ and $\boldsymbol{d}$ solve (2.12), the corresponding $v^i$ and $v^i{}_\alpha$ are solutions of the system

$$\left(\frac{\partial W}{\partial v^i{}_{|s}}\right)_{|s} = 0\,, \qquad \left(\frac{\partial W}{\partial v^i{}_{\alpha|s}}\right)_{|s} - \frac{\partial W}{\partial v^i{}_\alpha} = 0\,. \tag{4.5}$$

Obviously the trivial solution $\boldsymbol{r}^*$, $\boldsymbol{d}^*$ corresponds to $v^i \equiv 0$, $v^i{}_\alpha \equiv 0$.

All solutions are extremals of the total energy function (2.4); the existence of such extrema is guaranteed by results recalled in Section 2. If the second variation of the total energy $E$ at a trivial solution is not positive-definite, then a non-trivial solution furnishes $E$ with an absolute minimum.

As customary, in this situation we term the trivial solution *unstable* and the other solution *stable*. Using the expression (2.3) for $W$, the second variation of $E$ at the trivial solution is given by:

$$\delta^2 E = \int_0^l [W_{\boldsymbol{r}'\boldsymbol{r}'}(\delta\boldsymbol{r}')^2 + W_{\boldsymbol{r}'\boldsymbol{d}_\alpha}\delta\boldsymbol{r}'\,\delta\boldsymbol{d}_\alpha + \ldots + W_{\boldsymbol{d}'_\alpha\boldsymbol{d}'_\beta}\delta\boldsymbol{d}'_\alpha\,\delta\boldsymbol{d}'_\beta\,]\,ds \tag{4.6}$$

(here the subscripts on $W$ denote the corresponding partial derivatives evaluated at the trivial solution).

It seems natural to choose: $\delta\boldsymbol{r}' = \boldsymbol{r}' - \boldsymbol{r}^{*\prime}$, $\delta\boldsymbol{d}_\alpha = \boldsymbol{d}_\alpha - \boldsymbol{d}^*_\alpha$, $\delta\boldsymbol{d}'_\alpha = \boldsymbol{d}'_\alpha - \boldsymbol{d}^{*\prime}_\alpha$. Thus, by means of representations (4.1) and (4.4'), the second variation of the total energy is

$$\delta^2 E = \int_0^l [W_{ih}v^i{}_{|s}v^h{}_{|s} + W^\alpha_{ih}v^i{}_\alpha v^h{}_{|s} + \ldots + W^{\alpha'\beta'}_{ih}v^i{}_{\alpha|s}v^h{}_{\beta|s}]\,ds\,. \tag{4.7}$$

Again, we denote derivatives of $W$, with respect to its variables, evaluated at the trivial solution, by subscripts, namely: $W_{ih} = (\partial^2 W/\partial v^i{}_{|s}\partial v^h{}_{|s})^*$, etc.

For our purpose we can choose for $v^i$ and $v^i{}_\alpha$ the corresponding solutions of the problem linearized about the trivial solution. The actual perturbation variables and their derivatives are known functions of $s$; in principle, we can integrate the right-hand side of (4.7), arriving at an expression for $\delta^2 E$ depending on the constitutive and geometrical properties of the rod. Hence the stability of the trivial solution will depend on the sign of $\delta^2 E$.

We remark that in the Kirchhoff case the variables $v^i{}_\alpha$ are not completely free, because they must satisfy the constraint conditions: $\boldsymbol{d}_\alpha \cdot \boldsymbol{d}_\beta = \delta_{\alpha\beta}$, that read:

$$A^2 v^3{}_\alpha v^3{}_\beta + v^\rho{}_\alpha v^\sigma{}_\beta \delta_{\rho\sigma} + v^\rho{}_\alpha \delta_{\rho\beta} + v^\rho{}_\beta \delta_{\alpha\rho} = 0\,; \tag{4.8}$$

still the independent unknowns are six, as remarked in Section 2.

In order to obtain solutions in the case linearized about the trivial solution, we must solve the system (4.5) with a strain energy function (elastic potential) that depends quadratically on its arguments. In other words, we expand the function $W$ of (4.4′) in a neighborhood of $v^i{}_{|s} = 0$, $v^i{}_\alpha = 0$, $v^i{}_{\alpha|s} = 0$ (corresponding to the straight configuration), truncate the expansion at the second order terms, and use the equilibrium equations to be left with a quadratic approximation of $W$.

We can have different explicit forms for the quadratic approximation of $W$, depending on the prevailing requirements of material symmetry. Material symmetries determine the relevant material moduli, and may simplify the expression for $W$, as shown in the next section. Moreover, material symmetries may simplify the right-hand side in the expression (4.7) of the second variation $\delta^2 E$.

## 5. Helical stability.

In this section we restrict ourselves to the case of helical solutions of the equilibrium equations. We know from previous sections that there exist straight (trivial) and helical deformations which are solutions of the boundary value problem (2.12). Now we wish to discuss the relative stability of such equilibrium configurations, according to the definition of stability given above and using the results of Section 4.

Let us consider an undeformed straight rod, characterized by the vector functions

$$\bar{\boldsymbol{r}} = s\boldsymbol{e}_3\,, \qquad \bar{\boldsymbol{d}}_\alpha = \boldsymbol{e}_\alpha\,, \qquad \boldsymbol{e}_i \cdot \boldsymbol{e}_h = \delta_{ih}\,. \tag{5.1}$$

Another configuration of the rod can be expressed in terms of perturbing variables $u^i$, $u^i{}_\alpha$ as follows:

$$\boldsymbol{r} = (s + u^3)\boldsymbol{e}_3 + u^\alpha \boldsymbol{e}_\alpha\,, \qquad \boldsymbol{d}_\alpha = (\delta^\beta{}_\alpha + u^\beta{}_\alpha)\boldsymbol{e}_\beta + u^3{}_\alpha \boldsymbol{e}_3\,. \tag{5.2}$$

In order to compare a helical solution with the trivial solution, we take variables in (4.1) to be differences between the values of the variables appearing in (5.2) for the two deformations. The trivial solution can be explicitly written as

$$(5.3) \qquad \boldsymbol{r}^* = As\boldsymbol{e}_3\,, \qquad \boldsymbol{d}^*_\alpha = \mathbf{R}_B(s)\,\boldsymbol{e}_\alpha\,,$$

where $\mathbf{R}_B(s)$ is a rotation matrix, formally given by

$$\mathbf{R}_B(s) = e^{Bs\Omega}, \qquad B \in \Re, \qquad \Omega = -\,\Omega^T,$$

such that $\mathbf{R}_B(s)\boldsymbol{e}_3 = \boldsymbol{e}_3$, and consequently, $\partial\mathbf{R}_B(s)/\partial s = B\mathbf{R}_B(s)\boldsymbol{e}_3\times$. If we denote with capital letters the variables in (5.2) corresponding to this solution, we easily find:

$$(5.4) \quad U^\alpha = 0\,, \qquad U^3 = (A-1)\,s\,, \qquad U^\beta{}_\alpha = R_B{}^\beta{}_\alpha - \delta^\beta{}_\alpha\,, \qquad U^3{}_\alpha = 0\,,$$

where $R_B{}^\beta{}_\alpha$ are the elements of $\mathbf{R}_B(s)$. We point out that $(5.4)_3$, or equivalently $\mathbf{U} = \mathbf{R} - \mathbf{1}$, implies $\mathbf{U}\mathbf{U}^T + \mathbf{U} + \mathbf{U}^T = \mathbf{0}$, that is, just the orthonormality condition $\boldsymbol{d}^*_\alpha \cdot \boldsymbol{d}^*_\beta = \delta_{\alpha\beta}$.

The helical solution, with radius $\rho$, pitch $b$, extension $a$, is given by

$$(5.5) \qquad \boldsymbol{r} = \mathbf{R}_b(s)(\boldsymbol{\rho} + as\boldsymbol{e}_3)\,, \qquad \boldsymbol{d}_\alpha = \mathbf{R}_b(s)\,\boldsymbol{D}_\alpha\,,$$

where: $\boldsymbol{\rho} = \rho^\alpha \boldsymbol{e}_\alpha$, $\rho = \sqrt{\delta_{\alpha\beta}\rho^\alpha\rho^\beta}$, $\mathbf{R}_b(s) = e^{bs\Omega}$, $\boldsymbol{D}_\alpha = \boldsymbol{d}_\alpha(s=0)$. The parameter $a$ is the extension of the helix with respect to its axis; $a$ is related to the stretch $\boldsymbol{r}'\cdot\boldsymbol{r}' = \lambda$ by $\lambda^2 = a^2 + b^2\rho^2$; if the rod is inextensible, $\lambda = 1$; if $\rho \ll 1$ and $b$ is not very large, then $\lambda \cong a$.

By comparison of (5.5) with (5.2), we have

$$(5.6) \qquad \begin{cases} u^\alpha = R_b{}^\alpha{}_\beta\,\rho^\beta\,, & u^3 = (a-1)\,s\,, \\ u^\beta{}_\alpha = R_b{}^\beta{}_\lambda\, d^\lambda{}_\alpha - \delta^\beta{}_\alpha\,, & u^3{}_\alpha = d^3{}_\alpha\,, \end{cases}$$

where $d^i{}_\alpha = \boldsymbol{D}_\alpha \cdot \boldsymbol{e}^i$ and $R_b{}^\beta{}_\lambda$ are the elements of $\mathbf{R}_b$. It follows that the variables introduced in (4.1) can be written as the differences

$$(5.7) \quad \begin{cases} v^\alpha = u^\alpha - U^\alpha = R_b{}^\alpha{}_\beta\,\rho^\beta\,, & v^3 = u^3 - U^3 = (a-A)\,s\,, \\ v^\beta{}_\alpha = u^\beta{}_\alpha - U^\beta{}_\alpha = R_b{}^\beta{}_\lambda\, d^\lambda{}_\alpha - R_B{}^\beta{}_\alpha\,, & v^3{}_\alpha = u^3{}_\alpha - U^3{}_\alpha = d^3{}_\alpha\,. \end{cases}$$

We remark that the scalar products $d^i{}_\alpha$ are to be considered as known quantities once we are given a helical solution: they are obviously related with the Euler angles $\theta$, $\phi$, $\psi$ used by Love and Antman.

The problem of finding a minimizer of the total energy different from the trivial solution is now reduced to the problem of finding a minimizer in the class of helical solutions, in a neighborhood of the trivial one, *i.e.*, with small variables (5.7). In view of (5.7), the strain energy density becomes

(5.8) $$\overline{W} = \overline{W}(a,\ b,\ \rho;\ A;\ B;\ s)\,;$$

in other words, $\overline{W}$ is the restriction to helices of the function $W$. The parameters $a$, $b$, $\rho$ of helical solutions of the equilibrium problem must satisfy the following conditions:

(5.9) $$\frac{\partial \overline{W}}{\partial a} = F\,, \quad \frac{\partial \overline{W}}{\partial b} = M\,, \quad \frac{\partial \overline{W}}{\partial \rho} = 0\,, \qquad \text{for } s = l\,.$$

These formulae give us the loads in terms of the parameters of the helix:

(5.10) $$F = F(a,\ b,\ \rho)\,, \qquad M = M(a,\ b,\ \rho)\,;$$

conversely, if certain invertibility conditions (not investigated here) are satisfied, these parameters can be expressed in terms of loads:

(5.11) $$a = a(F,\ M)\,, \qquad b = b(F,\ M)\,, \qquad \rho = \rho(F,\ M)\,.$$

As a criterion for stability, we inspect the sign of the second variation (4.7) of the total strain energy, when use is made of (5.7). In this case, $\delta^2 E$ becomes a function of $(a, b, \rho;\ A, B, l)$ and some constitutive parameters. Through (5.11) we can enter the loads in the expression of $\delta^2 E$ and call critical the values $\overline{F}$ and $\overline{M}$ such that $\delta^2 E(\overline{F},\ \overline{M},\ \ldots) = 0$.

We are aware that our present notion of criticality is different, and somewhat more vague, than the usual notion, where a critical load is meant to be the value of the path parameter at which an equilibrium path bifurcates.

In order to obtain explicit results, we restrict our attention to particular cases.

Let us impose periodical boundary conditions, requiring $b = B = 2\pi/l$. Thus, the rod forms one complete turn of the helix (in the trivial case the cross-section makes a complete turn around the $\boldsymbol{e}_3$-axis); the longer the rod, the smaller $b$. In addition, the helices are chosen to be near the trivial solution, *i.e.*, $\rho$ is small as well as the angle between $\boldsymbol{d}_3$ and $\boldsymbol{e}_3$ (recall that $\boldsymbol{d}_3 = \boldsymbol{d}_1 \times \boldsymbol{d}_2$); this implies that also $(d^i{}_\alpha - \delta^i{}_\alpha)$ is small. These assumptions are equivalent to the assumption that $v^i$, $v^i{}_\alpha$ are small; they have been already introduced in Section 3, together with a very simple strain energy density, in order to obtain the classical results in the inextensible case.

Let us now allow the extensibility and, moreover, assume that the strain energy function is invariant under the change:

$$\boldsymbol{d}_1 \to \pm\boldsymbol{d}_1\,, \qquad \boldsymbol{d}_2 \to \pm\boldsymbol{d}_2\,, \qquad s \to \pm s,$$

a condition slightly stronger than transverse isotropy, used by Cohen [10]. The quadratic approximation of the elastic potential is

$$(5.12)\quad W \simeq \lambda_1 \delta_{\alpha\beta} u^{\alpha}{}_{|s} u^{\beta}{}_{|s} + \lambda_2 v^3{}_{\alpha} u^{\alpha}{}_{|s} + \lambda_3 \delta^{\beta}{}_{\alpha} v^{\alpha}{}_{\beta} u^3{}_{|s} + \lambda_4 u^3{}_{|s} u^3{}_{|s},$$

or, in terms of the helical parameters:

$$(5.12')\quad W \simeq \lambda_1 \delta_{\alpha\gamma} (R^{\alpha'}{}_{\lambda} + c^{\alpha}{}_{\beta} R^{\beta}{}_{\lambda})(R^{\gamma'}{}_{\mu} + c^{\gamma}{}_{\delta} R^{\delta}{}_{\mu}) \rho^{\lambda} \rho^{\mu} +$$

$$+ \lambda_2 d^3{}_{\alpha} (R^{\alpha'}{}_{\lambda} + c^{\alpha}{}_{\beta} R^{\beta}{}_{\lambda}) \rho^{\lambda} + \lambda_3 \delta^{\alpha}{}_{\beta} R^{\lambda}{}_{\alpha} (d^{\beta}{}_{\lambda} - \delta^{\beta}{}_{\lambda})(a - A) + \lambda_4 (a - A)^2.$$

As $b = B$, $\mathbf{R}_b = \mathbf{R}_B = \mathbf{R}$. In (5.12) and (5.12') we have taken into account the symmetry assumptions, the expression (5.7), the smallness of $b$ (as it is appropriate for a long rod); moreover, $c^{\alpha}{}_{\beta}$'s are given by (4.3) and the $\lambda_i$'s are generalized elastic moduli.

The equilibrium conditions become

$$(5.13)\quad \begin{cases} \lambda_3 (d^1{}_1 + d^2{}_2 - 2) + 2\lambda_4 (a - A) = F, \\ \dfrac{16\pi^2}{l} \lambda_1 [(\rho^1)^2 - (\rho^2)^2] + \lambda_3 l (a - A)(d^1{}_2 - d^2{}_1) = M, \\ \lambda_2 (d^3{}_2 \rho^1 - d^3{}_1 \rho^2) + \dfrac{4\pi}{l} \lambda_1 [(\rho^1)^2 + (\rho^2)^2] = 0. \end{cases}$$

The values $d^i{}_{\alpha}$ are constant on each helical solution. If the rod is inextensible: $a - A = \sqrt{1 - (4\pi^2/l^2)\rho^2} - 1$.

Thus, we can find values $\rho^1$, $\rho^2$, $a - A$ which, by substitution in (5.7), give us $v^i$ and $v^i{}_{\alpha}$ as solutions of the linearized problem, with $b = B = 2\pi/l$:

$$(5.14)\quad \begin{cases} v^{\alpha} = \delta^{\alpha}{}_{\beta} \rho^{\beta}, & v^3 = (a - A)s, & v^{\alpha}{}_{\beta} = d^{\alpha}{}_{\beta} - \delta^{\alpha}{}_{\beta}, & v^3{}_{\alpha} = d^3{}_{\alpha}; \\ v^{\alpha'} = \varepsilon^{\alpha}{}_{\beta} b \rho^{\beta}, & v^{3'} = a - A, & v^{\beta'}{}_{\alpha} = \varepsilon^{\beta}{}_{\lambda} b (d^{\lambda}{}_{\alpha} - \delta^{\lambda}{}_{\alpha}), & v^{3'}{}_{\alpha} = 0. \end{cases}$$

The required material symmetries force $\partial W/\partial \boldsymbol{d}_1$, $\partial W/\partial \boldsymbol{d}_2$ and $\partial W/\partial \psi_{|s}$ (for any vector function $\psi$ entering $W$) to be odd functions of $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ and $\psi_{|s}$, respectively. Thus (4.7) reads

$$(5.15)\quad \delta^2 E = \int_0^l [W_{\alpha\beta} v^{\alpha}{}_{|s} v^{\beta}{}_{|s} + W_{33} v^3{}_{|s} v^3{}_{|s} + W^{\alpha\beta}_{\rho\sigma} v^{\rho}{}_{\alpha} v^{\sigma}{}_{\beta} + W^{\alpha'\beta'}_{\rho\sigma} v^{\rho}{}_{\alpha|s} v^{\sigma}{}_{\beta|s} +$$

$$+ W^{\alpha}_{3\beta} v^{\beta}{}_{\alpha} v^3{}_{|s} + W^{\alpha}_{\beta 3} v^3{}_{\alpha} v^{\beta}{}_{|s} + W^{\alpha\beta}_{33} v^3{}_{\alpha} v^3{}_{\beta} + W^{\alpha'\beta'}_{33} v^3{}_{\alpha|s} v^3{}_{\beta|s}]\, ds$$

and, by (5.12):

$$(5.16)\quad \delta^2 E = \left[ \frac{16\pi^2}{l^2} W_{\alpha\beta}\, \varepsilon^{\alpha}{}_{\lambda}\, \varepsilon^{\beta}{}_{\mu}\, \rho^{\lambda} \rho^{\mu} + W_{33}(a-A)^2 + \right.$$

$$+ 4\left( W^{\alpha\beta}_{\rho\sigma}\, \delta^{\rho}{}_{\lambda}\, \delta^{\sigma}{}_{\mu} + \frac{4\pi^2}{l^2} W^{\alpha'\beta'}_{\rho\sigma}\, \varepsilon^{\rho}{}_{\lambda}\, \varepsilon^{\sigma}{}_{\mu} \right)(d^{\lambda}{}_{\alpha} - \delta^{\lambda}{}_{\alpha})(d^{\mu}{}_{\beta} - \delta^{\mu}{}_{\beta}) + W^{\alpha}_{3\beta}(a-A)\cdot$$

$$\left. \cdot (d^{\beta}{}_{\alpha} - \delta^{\beta}{}_{\alpha}) + \frac{4\pi^2}{l^2} W^{\alpha}_{\beta 3}\, \varepsilon^{\beta}{}_{\lambda}\, \rho^{\lambda}\, d^{3}{}_{\alpha} + W^{\alpha\beta}_{33}\, d^{3}{}_{\alpha}\, d^{3}{}_{\beta} \right] l = \Phi(\rho^1, \rho^2, a-A, l)\, l\,.$$

Here $\Phi$ depends also on the values of the second derivatives of $W$ evaluated at the trivial solution; therefore, $\Phi$ depends in general on material and geometrical properties reflected into such a solution. Using (5.11), $\Phi$ becomes a well-defined function of $F$, $M$, $l$ and the material constants. A trivial solution is stable whenever $\Phi > 0$; a helical solution close to the straight configuration is stable whenever $\Phi < 0$; the «critical» loads are the values $F$, $M$ which make the function $\Phi$ to vanish (provided that $\Phi$ changes its sign at these values).

Had we considered transversely isotropic rods, we would have obtained completely analogous results; also, strain energy functions of the type proposed by Ericksen [5] could be constructed in terms of variables (5.7), but we choose to stop our analysis here.

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