Rendiconti del Seminario Matematico della Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 88 (1992), p. 55-81

<http://www.numdam.org/item?id=RSMUP_1992__88__55_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 88 (1992)

On the Interior Regularity of Weak Solutions of Degenerate Elliptic Systems (the Case $1 < p < 2$).

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1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ ($n \geqslant 2$) be a domain, and let $N \geqslant 2$. We consider the following system of PDE's:

$$(1.1) \quad -\frac{\partial}{\partial x_\alpha} a_i^\alpha(x, Du) = b_i(x, u, Du) \quad \text{in } \Omega\,(^1) \quad (i = 1, \ldots, N),$$

where $u = \{u^1, \ldots, u^N\}$ and $Du = \{D_\alpha u^i\}$ (= matrix of first order derivatives of u; $D_\alpha v^i = \partial v^i / \partial x_\alpha$).

We consider the following conditions on a_i^α:

$$(1.2) \quad a_i^\alpha, \quad \frac{\partial a_i^\alpha}{\partial x_\beta} \text{ are Carathéodory functions on } \Omega \times \mathbb{R}^{nN},$$

$$(1.3) \quad (a_i^\alpha(x, \xi) - a_i^\alpha(x, \eta)(\xi_\alpha^i - \eta_\alpha^j)) \geqslant$$
$$\geqslant c_1(1 + |\xi|^2 + |\eta|^2)^{(p-2)/2} |\xi - \eta|^2 \quad \forall x \in \Omega, \ \forall \xi, \ \eta \in \mathbb{R}^{nN},$$

$$(1.4) \quad |a_i^\alpha(x, \xi)| + \left| \frac{\partial a_i^\alpha}{\partial x_\beta}(x, \xi) \right| \leqslant c_2(1 + |\xi|^{p-1}) \ \forall (x, \xi) \in \Omega \times \mathbb{R}^{nN},$$

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(1) Throughout the paper, Greek and Latin subscripts take independently the value $1, \ldots, n$ and $1, \ldots, N$, respectively. A repeated subscript implies summation over $1, \ldots, n$ resp. $1, \ldots, N$.

resp.

$$(1.4') \quad |a_i^\alpha(x, \xi) - a_i^\alpha(x, \eta)| \leqslant$$

$$\geqslant c_3 (1 + |\xi|^2 + |\xi - \eta|^2)^{(p-2)/2} |\xi - \eta|^2 \quad \forall x \in \Omega, \ \forall \xi, \ \eta \in \mathbb{R}^{nN},$$

where c_1, c_2 and c_3 are positive constants, and $1 < p < 2$ is a fixed real $(\alpha, \beta = 1, \ldots, n; \ i = 1, \ldots, N)$.

The conditions on b_i are as follows,

(1.5) *b_i is a Carathéodory function on $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$;*

controlled growth:

$$(1.6) \quad |b_i(x, u, \xi)| \leqslant d_1 (1 + |u|^{n(p-1)/(n-p)} + |\xi|^{p-1})$$

$$\forall (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \qquad (d_1 = \text{const} > 0),$$

resp.

natural growth:

$$(1.7) \quad \forall M > 0 \ \exists d_2 = d_2(M) = \text{const} > 0\colon \ |b_i(x, u, \xi)| \leqslant d_2 (1 + |\xi|^p),$$

$$\forall (x, u, \xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}, \quad |u| \leqslant M, \quad (i = 1, \ldots N).$$

Remarks. 1) Obviously, hypothesis (1.4′) implies both the Lipschitz continuity of the function $\xi \to a_i^\alpha(x, \xi)$ and the bound on $|a_i^\alpha(x, \xi)|$ in (1.4).

2) Assume that the function $\xi \to a_i^\alpha(x, \xi)$ possesses partial derivatives such that

$$(1.3_1) \quad \frac{\partial a_i^\alpha}{\partial \xi_\beta^j}(x, \xi)\, \eta_\alpha^i \eta_\beta^j \geqslant c_4 (1 + |\xi|^2)^{(p-2)/2} |\eta|^2 \quad \forall x \in \Omega, \forall j, \ \eta \in \mathbb{R}^{nN},$$

$$(1.4_1) \quad \left| \frac{\partial a_i^\alpha}{\partial \xi_\beta^j}(x, \xi) \right| \leqslant c_5 (1 + |\xi|^2)^{(p-2)/2} \qquad \forall (x, \xi) \in \Omega \times \mathbb{R}^{nN}$$

where c_4, $c_5 = \text{const} > 0$ $(\alpha, \beta = 1, \ldots, n; \ i, j = 1, \ldots, N)$. The latter conditions are frequently used in the literature (cf. e.g. [1], [3], [4]). It is readily seen that (1.3_1) and (1.4_1) imply (1.3) and (1.4′), respectively. ∎

In recent time, the regularity of weak solutions to elliptic systems with degeneration of type (1.3) has been intensively studied, the main attention being devoted to variational problems. In [1], [2], the authors prove the partial Hölder continuity of minimizers of functionals with growth properties which correspond to the above type of degeneration. Related results within the framework of differential forms are obtained in [8], [9], where the case $1 < p < 2$ is reduced to the case $p > 2$ via a duality argument. The papers [5], [6] (cf. pp. 126-127 therein) and [12] are concerned with the Hölder continuity of weak solutions to special cases of the above type of elliptic systems.

The aim of the present paper is two-fold. Firstly, we extend the results from [10] to the more general system (1.1) and prove simultaneously the higher integrability results of that paper without any restriction on p. Secondly, with respect to second order systems with natural growth nonlinearities, we prove the same differentiability result on weak solutions as in [3] (even with a slightly higher integrability of the second order derivatives) only using (1.5) and (1.7), i.e. without any differentiability condition upon b_i. In this respect, our results improve those in [3]. ■

2. Differentiability of weak solutions (controlled growth).

Let $W^{m,s}(\Omega)$ $(m = 1, 2, \dots;\ 1 \leq s \leq \infty)$ denote the usual Sobolev space. Set $L^s(\Omega; \mathbb{R}^N) = [L^s(\Omega)]^N$, $W^{m,s}(\Omega; \mathbb{R}^N) = [W^{m,s}(\Omega)]^N$ etc.

In what follows, we introduce the notion of weak solution to (1.1), regardless of whether or not the solution under consideration is subject to any boundary condition.

Let a_i^α and b_i $(\alpha = 1, \dots, n;\ i = 1, \dots, N)$ satisfy conditions (1.2), (1.4) and (1.5), (1.6), respectively. A (vector) function $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ is called a *weak solution* to (1.1) if

$$\int_\Omega a_i^\alpha(x, Du)\, D_\alpha \varphi^i\, dx = \int_\Omega b_i(x, u, Du)\, \varphi^i\, dx, \qquad \forall \varphi \in W^{1,p}(\Omega; \mathbb{R}^N),\ \ \operatorname{supp}(\varphi) \subset \Omega. \tag{2.1}$$

Let

$$B_r = B_r(x_0) = \{x \in \mathbb{R}^n \colon |x - x_0| < r\}.$$

Our first differentiability results is

THEOREM 1. *Let* (1.2), (1.3), (1.4) *and* (1.5), (1.6) *be satisfied. Let* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *be any weak solution to* (1.1).

Then

$$\text{(2.2)}\qquad \begin{cases} u \in W^{2,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^N), \\ \left(p \leqslant q < 2 \textit{ arbitrary if } n = 2,\ q = \dfrac{np}{n+p-2} \textit{ if } n \geqslant 3\right), \end{cases}$$

$$\text{(2.3)}\qquad (1 + |Du|^2)^{(p-2)/4} |D^2 u| \in L^2_{\mathrm{loc}}(\Omega)\ (^2)$$

and

$$\text{(2.4)}\qquad \int_{B_r} (|D^2 u|^p + (1 + |Du|^2)^{(p-2)/2} |D^2 u|^2)\, dx \leqslant$$

$$\leqslant c \left\{ \frac{1}{r^{p'}} \int_{B_{3r}} (1 + |Du|^p)\, dx + \frac{1}{r^p} \int_{B_{3r}} |Du - \lambda|^p dx + \right.$$

$$\left. + \int_{B_{3r}} (1 + |u|^{np/(n-p)} + |Du|^p)\, dx \right\} (^3)$$

for all $\lambda \in \mathbb{R}^{nN}$ *and all balls* $B_{3r} \subset \Omega$ ($c = \mathrm{const} > 0$ *independent of* r).

If, in addition, (1.4′) *is satisfied, then*

$$\text{(2.5)}\qquad \int_{B_r} (|D^2 u|^p + (1 + |Du|^2)^{(p-2)/2} |D^2 u|^2)\, dx \leqslant$$

$$\leqslant c \left\{ \frac{1}{r^2} \int_{B_{3r}} (1 + |Du|^2 + |Du - \lambda|^2)^{(p-2)/2} |Du - \lambda|^2 dx + \right.$$

$$\left. + \int_{B_{3r}} (1 + |u|^{np/(n-p)} + |Du|^p) + |Du - \lambda|^p)\, dx \right\},$$

for all $\lambda \in \mathbb{R}^{nN}$ *and all balls* $B_{3r} \subset \Omega$ ($c = \mathrm{const} > 0$ *independent of* r).

(2) $D^2 u = \{D_\alpha D_\beta u^i\}$ denotes the matrix of second order derivatives of u.
(3) $p' = p/(p-1)$.

REMARK. Let be u as in Theorem 1. From (2.2) we obtain by virtue of Sobolev's imbedding theorem

$$u \in C^{2(1-1/q)}(\Omega; \mathbb{R}^N) \quad \text{if } n = 2, \tag{2.6}$$

$$u \in C^{1-1/p}(\Omega; \mathbb{R}^N) \quad \text{if } n = 3. \tag{2.7}$$

Next, suppose that conditions (1.2), (1.4), (1.3_1), (1.4_1) and (1.5), (1.6) are satisfied.

Define $w^{i\alpha} = D_\alpha u^i$ ($\alpha = 1, \ldots, n$; $i = 1, \ldots, N$). Then (2.2) and (2.3) imply $w \in W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^{nN})$ and $(1 + |w|^2)^{(p-2)/4} |Dw| \in L^2_{\text{loc}}(\Omega; \mathbb{R}^{nN})$, and we infer from (2.1) by integration by parts that

$$\int_\Omega A^{\alpha\beta}_{ij\gamma\nu}(x, w) D_\beta w^{j\nu} D_\alpha \varphi^{i\gamma} dx + \int_\Omega B^{\alpha\beta}_i(x, w) D_\alpha \varphi^{i\beta} dx = 0$$

for all $\varphi \in C^1(\Omega; \mathbb{R}^{nN})$, $\operatorname{supp}(\varphi) \subset \Omega$, where

$$A^{\alpha\beta}_{ij\gamma\nu}(x, \xi) = \frac{\partial a^\alpha_i}{\partial \xi^j_\beta}(x, \xi)\, \delta_{\gamma\nu},$$

$$B^{\alpha\beta}_i(x, \xi) = \frac{\partial a^\alpha_i}{\partial x_\beta}(x, \xi) + \frac{1}{n} b_i(x, u(x), \xi)\, \delta_{\alpha\beta}.$$

($\alpha, \beta, \gamma, \nu = 1, \ldots n$; $i, j = 1, \ldots, N$; $\delta_{\gamma\nu}$ resp. $\delta_{\alpha\beta}$ = Kronecker's delta ($x \in \Omega$, $u \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{nN}$)). If, in addition, we suppose that the growth of b_i is independent of u it is readily verified that $A^{\alpha\beta}_{ij\gamma\nu}$ and $B^{\alpha\beta}_i$ fulfill the conditions in [11]. Hence Du is partial Hölder continuous. ■

Let $e_\beta = \{0, \ldots, 0, 1, 0, \ldots, 0\}$ [1 at the β-th place] ($\beta = 1, \ldots, n$), and define

$$\Delta_h v(x) = \Delta_h^{(\beta)} v(x) = \frac{1}{h}[v(x + he_\beta) - v(x)].$$

The following two results are well-known:

(i) *Let* $v \in W^{1,s}(B_R)$ $(1 \leqslant s < \infty)$. *Then, for any* $0 < r < R$,

$$\int_{B_r} |\Delta_h v|^s dx \leqslant \int_{B_R} |D_\beta v|^s dx, \qquad \forall 0 < |h| < R - r.$$

(ii) *Let* $v \in L^s(B_R)$ $(1 < s < \infty)$ *satisfy*

$$\int_{B_r} |\Delta_h v|^s dx \leqslant C_0 = \text{const} \qquad \forall 0 < |h| < h_0 \leqslant R - r.$$

Then $D_\beta v \in L^s(B_r)$ *and*

$$\int_{B_r} |D_\beta v|^s \, dx \leqslant C_0. \qquad \blacksquare$$

PROOF OF THEOREM 1. Let B_r be any ball such that $B_{3r} \subset \Omega$. Let $\zeta \in C^\infty(\mathbb{R}^n)$ be a cut-off function for B_{2r}: $\zeta \equiv 1$ on B_r, $\zeta \equiv 0$ in $\mathbb{R}^n \setminus B_{2r}$ and $0 \leqslant \zeta \leqslant 1$, $|D\zeta| \leqslant c_0/r$, $|D^2\zeta| \leqslant c_0/r^2$ in $\mathbb{R}^n$ ($c_0 = \text{const} > 0$ independent of r).

We prove (2.2)-(2.4). Let $\lambda \in \mathbb{R}^{nN}$ be arbitrary. The function

$$\varphi = \Delta_{-h}(\zeta^2 \Delta_h (u - \lambda \cdot x)) \,(^4), \qquad 0 < |h| < r,$$

is admissible in (2.1). We obtain by virtue of (1.3)

$$(2.8) \qquad c_1 \int_{B_{2r}} (1 + |Du(x + he_\beta)|^2 + |Du(x)|^2)^{(p-2)/2} |\Delta_h Du|^2 \zeta^2 \, dx \leqslant$$

$$\leqslant \frac{1}{h} \int_{B_{2r}} [a_i^\alpha(x + he_\beta, Du(x + he_\beta)) - a_i^\alpha(x + he_\beta, Du(x))] \zeta^2 \Delta_h D_\alpha u^i \, dx =$$

$$= -\int_{B_{2r}} \left(\int_0^1 \frac{\partial a_i^\alpha}{\partial x_\beta}(x + the_\beta, Du(x + he_\beta)) \, dt \right) \zeta^2 \Delta_h D_\alpha u^i \, dx +$$

$$+2 \int_{B_{2r}} a_i^\alpha(x, Du) \Delta_{-h}(\zeta D_\alpha \zeta \Delta_h (u - \lambda \cdot x)^i) \, dx -$$

$$- \int_{B_{2r}} b_i(x, u, Du) \Delta_{-h}(\zeta^2 \Delta_h (u - \lambda \cdot x)^i) \, dx = I_1 + I_2 + I_3$$

$(0 < |h| < r)$.

(4) $\lambda = \{\lambda_\alpha^i\}$, $\lambda \cdot x = \{(\lambda \cdot x)^1, \ldots, (\lambda \cdot x)^N\}$ where $(\lambda \cdot x)^i = \lambda_\alpha^i x_\alpha$.

To estimate I_1 we make use of (1.4). By the aid of Hölder's inequality,

$$I_1 \leqslant \varepsilon \int\limits_{B_{2r}} |\Delta_h Du|^p \zeta^p \, dx + c\varepsilon^{1/(1-p)} \int\limits_{B_{3r}} (1 + |Du|^p) \, dx \, (^5)$$

($\varepsilon > 0$ arbitrary).

Next, by (i) with $R = 3r$,

$$\int\limits_{B_{2r}} |\Delta_{-h}(\zeta D_\alpha \zeta \Delta_h (u - \lambda \cdot x)^i)|^p \, dx \leqslant \int\limits_{B_{3r}} |D_\beta(\zeta D_\alpha \zeta \Delta_h (u - \lambda \cdot x)^i)|^p \, dx \leqslant$$

$$\leqslant \int\limits_{B_{2r}} \left(\frac{c_0}{r} |\Delta_h Du| \zeta + \frac{2c_0}{r^2} |\Delta_h (u - \lambda \cdot x)| \right)^p dx$$

($0 < |h| < r$). Hence, observing (1.4) and applying once more (i) we obtain, for any $\delta > 0$,

$$I_2 \leqslant \frac{\delta}{r^p} \int\limits_{B_{2r}} |\Delta_h Du|^p \zeta^p \, dx + \frac{\delta}{r^{2p}} \int\limits_{B_{3r}} |Du - \lambda|^p \, dx +$$

$$+ c\delta^{1/(1-p)} \int\limits_{B_{2r}} (1 + |Du|^p) \, dx .$$

Finally, from (1.6) we get by an analogous reasoning,

$$I_3 \leqslant c \left(\int\limits_{B_{2r}} (1 + |u|^{np/(n-p)} + |Du|^p \, dx \right)^{(p-1)/p} \cdot$$

$$\cdot \left(\int\limits_{B_{2r}} |\Delta_{-h}(\zeta^2 \Delta_h (u - \lambda \cdot x))|^p \, dx \right)^{1/p} \leqslant$$

(^{5}) In what follows, by c we denote positive constants which may change there numerical value from line to line, but do not depend on r.

$$\leqslant \varepsilon \int_{B_{2r}} |\Delta_h Du|^p \zeta^p \, dx + \frac{c\varepsilon}{r^p} \int_{B_{3r}} |Du - \lambda|^p \, dx +$$

$$+ c\varepsilon^{1/(1-p)} \int_{B_{2r}} (1 + |u|^{np/(n-p)} + |Du|^p) \, dx$$

($\varepsilon > 0$ arbitrary).

Thus, choosing appropriately ε and δ, we infer from (2.8)

$$\text{(2.9)} \quad \int_{B_r} (|\Delta_h Du|^p + (1 + |Du(x + he_\beta)|^2 + |Du(x)|^2)^{(p-2)/2} |\Delta_h Du|^2) \, dx \leqslant$$

$$\leqslant c \left\{ \frac{1}{r^{p'}} \int_{B_{3r}} (1 + |Du|^p) \, dx + \frac{1}{r^p} \int_{B_{3r}} |Du - \lambda|^p \, dx + \right.$$

$$\left. + \int_{B_{3r}} (1 + |u|^{np/(n-p)} + |Du|^p) \, dx \right\}$$

($c = \text{const} > 0$ independent of r; $0 < |h| < r$).

We now proceed as follows. Firstly, combining (ii) and (2.9) gives $D_\beta Du \in L^p(B_r; \mathbb{R}^{nN})$ ($\beta = 1, \dots, n$) [clearly,

$$\text{(2.10)} \qquad \Delta_h Du \to D_\beta Du \quad \text{weakly in } L^p(B_r; \mathbb{R}^{nN}),$$

as $h \to 0$]. Secondly, by passing to a subsequence if necessary we have .

$$\text{(2.11)} \quad \begin{cases} (1 + |Du(\cdot + he_\beta)|^2 + |Du(\cdot)|^2)^{(p-2)/4} \Delta_h Du \to w_\beta, \\ \text{weakly in } L^2(B_r; \mathbb{R}^{nN}), \end{cases}$$

$$\text{(2.12)} \quad \begin{cases} (1 + |Du(\cdot + he_\beta)|^2 + |Du(\cdot)|^2)^{(p-2)/4} \to (1 + 2|Du(\cdot)|^2)^{(p-2)/4}, \\ \text{strongly in } L^s(B_r) \ (1 < s < \infty), \end{cases}$$

as $h \to 0$ ($w_\beta \in L^2(B_r; \mathbb{R}^{nN})$). Now (2.10)-(2.12) imply

$$(1 + 2|Du|^2)^{(p-2)/4} D_\beta Du = w_\beta \quad \text{a.e. in } B_r, \tag{2.13}$$

and (2.4) follows from (2.9) by taking the lim inf therein.

We establish the integrability properties on u stated in (2.2). To this end, define $U = (1 + |Du|^2)^{p/4}$ a.e. in Ω. By (2.3), $U \in W^{1,2}_{\mathrm{loc}}(\Omega)$, and therefore by Sobolev's imbedding theorem

$$\begin{cases} Du \in L^s_{\mathrm{loc}}(\Omega; \mathbb{R}^{nN}) \ (1 \leqslant s < \infty) & \text{if } n = 2, \\ Du \in L^{np/(n-2)}_{\mathrm{loc}}(\Omega; \mathbb{R}^{nN}) & \text{if } n \geqslant 3. \end{cases} \tag{2.14}$$

Observing that

$$|D^2 u|^\sigma = (1 + |Du|^2)^{\sigma(2-p)/4}[(1 + |Du|^2)^{(p-2)/4} |D^2 u|]^\sigma$$

for all $\sigma > 0$ and a.a. $x \in \Omega$, we infer from (2.3) and (2.14)

$$D^2 u \in L^\sigma_{\mathrm{loc}}(\Omega; \mathbb{R}^{n^2 N}), \quad \forall p \leqslant \sigma < 2 \qquad \text{if } n = 2,$$

$$D^2 u \in L^{np/(n+p-2)}_{\mathrm{loc}}(\Omega; \mathbb{R}^{n^2 N}) \qquad \text{if } n \geqslant 3.$$

Whence (2.2).

Let (1.4′) be satisfied. We prove (2.5). To begin with, we note that (2.8) implies

$$\begin{aligned} \frac{c_1}{4} \int_{B_{2r}} \big(|\Delta_h Du|^p \zeta^p + (1 + |Du(x + he_\beta)|^2 + \\ + |Du(x)|^2)^{(p-2)/2} |\Delta_h Du|^2 \zeta^2 \big)\, dx \leqslant c \int_{B_{3r}} (1 + |Du|^p)\, dx + \\ +2 \int_{B_{2r}} a_i^\alpha(x, Du)\, \Delta_{-h}(\zeta D_\alpha \zeta \Delta_h (u - \lambda \cdot x)^i)\, dx - \\ - \int_{B_{2r}} b_i(x, u, Du)\, \Delta_{-h}(\zeta^2 \Delta_h (u - \lambda \cdot x)^i)\, dx \end{aligned} \tag{2.15}$$

(ζ as above; $0 < |h| < r$) (cf. the estimation of I_1 above). Now, $u \in W^{2,p}_{\text{loc}}(\Omega; \mathbb{R}^N)$ is known from the preceding step of the proof. Hence

$$\Delta_{-h}(\zeta D_\alpha \zeta \Delta_h(u - \lambda\cdot x)) \to D_\beta(\zeta D_\alpha \zeta D_\beta(u - \lambda\cdot x)),$$

$$\Delta_{-h}(\zeta^2 \Delta_h(u - \lambda\cdot x)) \to D_\beta(\zeta^2 D_\beta(u - \lambda\cdot x)),$$

weakly in $L^p(B_{2r}; \mathbb{R}^N)$ as $h \to 0$. Taking into account (2.10), (2.11) and (2.13) we obtain from (2.15) by letting tend $h \to 0$

$$\frac{c_1}{4}\int_{B_{2r}} (|D_\beta Du|^p \zeta^p + (1 + 2|Du|^2)^{(p-2)/2}|D_\beta Du|^2 \zeta^2)\, dx \leq \tag{2.16}$$

$$\leq c\int_{B_{3r}}(1 + |Du|^p)\, dx + 2\int_{B_{2r}} a_i^\alpha(x, Du) D_\beta(\zeta D_\alpha \zeta D_\beta(u - \lambda\cdot x)^i)\, dx -$$

$$- \int_{B_{2R}} b_i(x, u, Du) D_\beta(\zeta^2 D_\beta(u - \lambda\cdot x)^i)\, dx =$$

$$= c\int_{B_{3r}}(1 + |Du|^p)\, dx + J_1 + J_2 .$$

We estimate J_1. By (1.4) and (1.4'),

$$|a_i^\alpha(x, Du) - a_i^\alpha(x_0, \lambda)| \leq$$

$$\leq c_2 r(1 + |Du|^{p-1}) + c_3(1 + |Du|^2 + |Du - \lambda|^2)^{(p-2)/2}|Du - \lambda|$$

for a.a. $x \in B_r (= B_r(x_0))$ and any matrix $\lambda \in \mathbb{R}^{nN}$. Thus,

$$J_1 = 2\int_{B_{2r}} (a_i(x, Du) - a_i^\alpha(x_0, \lambda)) D_\beta(\zeta D_\alpha \zeta D_\beta(u - \lambda\cdot x)^i)\, dx \leq$$

$$\leq \frac{c}{r}\int_{B_{2r}} (r(1 + |Du|^{p-1}) + (1 + |Du|^2 + |Du - \lambda|^2)^{(p-2)/2}|Du - \lambda|)\cdot$$

$$\cdot\left(\frac{1}{r}|Du - \lambda| + |D_\beta Du|\zeta\right) dx \leq$$

$$\leq \frac{c_1}{16}\int_{B_{2r}} (|D_\beta Du|^p \zeta^p + (1 + 2|Du|^2)^{(p-2)/2}|D_\beta Du|^2 \zeta^2)\, dx +$$

$$+\frac{c}{r^2}\int_{B_{2r}}(1+|Du|^2+|Du-\lambda|^2)^{(p-2)/2}|Du-\lambda|^2\,dx+$$

$$+c\int_{B_{2r}}(1+|Du|^p+|Du-\lambda|^p)\,dx\,.$$

Analogously, by (1.6),

$$J_2\leq c\int_{B_{2r}}(1+|u|^{n(p-1)/(n-p)}+|Du|^{p-1})\left(\frac{1}{r}|Du-\lambda|+|D_\beta Du|\zeta\right)dx\leq$$

$$\leq\frac{c_1}{16}\int_{B_{2r}}(|D_\beta Du|^p\zeta^p+(1+2|Du|^2)^{(p-2)/2}|D_\beta Du|^2\zeta^2)\,dx+$$

$$+\frac{c}{r^2}\int_{B_{2r}}(1+|Du|^2+|Du-\lambda|^2)^{(p-2)/2}|Du-\lambda|^2\,dx+$$

$$+c\int_{B_{2r}}(1+|u|^{np/(n-p)}+|Du|^p+|Du-\lambda|^p)\,dx\,.$$

Inserting these estimates into (2.16) gives (2.5). ■

3. Higher integrability of $(1+|Du|^2)^{(p-2)/4}D^2u$ (controlled growth).

The Caccioppoli inequality (2.5) is the point of departure to improve the integrability of $(1+|Du|^2)^{(p-2)/4}D^2u$.

THEOREM 2. *Let* (1.2)-(1.4′) *and* (1.5), (1.6) *be satisfied. Let* $u\in W^{1,p}(\Omega;\mathbb{R}^N)$ *be any weak solution to* (1.1).
Then there exists a real $t>2$ *such that*

(3.1) $\quad (1+|Du|^2)^{(p-2)/4}|D^2u|\in L^t_{\mathrm{loc}}(\Omega)\,,$

$$(3.2)\qquad \int_{B_r} \{(1+|Du|^2)^{(p-2)/4}|D^2u|\}^t dx \leqslant$$

$$\leqslant cr^{n(1-t/2)}\left(\int_{B_{3r}}(1+|Du|^2)^{(p-2)/2}|D^2u|^2 dx\right)^{t/2}+$$

$$+c\int_{B_{3r}}(1+|u|^{np/(n-p)}+|Du|^p)^{t(n+2)/2n}dx,$$

for all balls $B_{3r}\subset\Omega$ ($c=\text{const}>0$ *independent of* r).

COROLLARY. *Let the hypothesis of Theorem 2 be satisfied. Then:*

$$(3.3)\qquad D^2u\in L^{\sigma}_{\text{loc}}(\Omega;\mathbb{R}^{4N})\quad \forall 2\leqslant\sigma<t\qquad \textit{if } n=2,$$

$$(3.4)\qquad D^2u\in L^{npt/(2n+t(p-2))}_{\text{loc}}(\Omega;\mathbb{R}^{n^2N})\qquad \textit{if } n\geqslant 3.$$

REMARKS. 1) Let $n=2$. By Theorem 1, $Du\in L^s_{\text{loc}}(\Omega;\mathbb{R}^{2N})$ for all $1\leqslant s<\infty$. Thus, from (3.3) we obtain by virtue of Sobolev's imbedding theorem:

$$Du\in C^{1-2/\sigma}(\Omega;\mathbb{R}^{2N})\qquad \forall 2<\sigma<t.$$

2) Let $n\geqslant 3$. Then $Du\in L^{npt/2(n-t)}_{\text{loc}}(\Omega;\mathbb{R}^{nN})$ (cf. the proof of the Corollary below). Clearly, the higher integrability (3.4) on D^2u improves the one in (2.2).

In particular, let $n=3$. Then $u\in W^{1,3pt/2(3-t)}_{\text{loc}}(\Omega;\mathbb{R}^N)$ implies $u\in$ $\in C^{1-2(3-t)/pt}(\Omega;\mathbb{R}^N)$. This improves (2.7).

Let $n=4$. Suppose that $8/t-2<p<2$. Now we obtain $u\in$ $\in W^{1,\sigma}_{\text{loc}}(\Omega;\mathbb{R}^N)$ where $\sigma=\min\{4p/(2-p), 2pt/(4-t)\}$ (>4). Thus, $u\in$ $\in C^{1-4/\sigma}(\Omega;\mathbb{R}^N)$. ■

PROOF OF THEOREM 2. Define $w=Du$ a.e. in Ω. By Theorem 1, $w\in W^{1,1}_{\text{loc}}(\Omega;\mathbb{R}^{nN})$ and $(1+|w|^2)^{(p-2)/4}|Dw|\in L^2_{\text{loc}}(\Omega)$.

Set

$$q=\frac{2n}{n+2},\qquad s=\frac{nq}{n-q}=2.$$

Let $\Lambda\in\mathbb{R}^{nN}$ denote the matrix according to the Proposition below (cf. Appendix). We let $\lambda=\Lambda$ in (2.5). Combining (2.5) and the Proposition below gives

$$\int_{B_r} (1 + |Du|^2)^{(p-2)/2} |D^2 u|^2 dx \leqslant$$

$$\leqslant \frac{c}{r^2} \left(\int_{B_{3r}} \{(1 + |Du|^2)^{(p-2)/2} |D^2 u|^2\}^{n/(n+2)} dx \right)^{(n+2)/n} +$$

$$+ c \int_{B_{3r}} (1 + |u|^{np/(n-p)} + |Du|^p + |Du - \Lambda|^p) dx$$

($c = \text{const} > 0$ independent of r). Observing that

$$\fint_{B_{3r}} |Du - \Lambda|^p dx \leqslant c \fint_{B_{3r}} (1 + |Du|^p) dx\,(^6)$$

we find

$$(3.5) \qquad \fint_{B_r} (1 + |Du|^2)^{(p-2)/2} |D^2 u|^2 dx \leqslant$$

$$\leqslant c \left(\fint_{B_{3r}} \{(1 + |Du|^2)^{(p-2)/2} |D^2 u|^2\}^{n/(n+2)} dx \right)^{(n+2)/n} +$$

$$+ c \fint_{B_{3r}} (1 + |u|^{np/(n-p)} + |Du|^p) dx$$

($c = \text{const} > 0$ independent of r).

Next, set

$$g = \{(1 + |Du|^2)^{(p-2)/2} |D^2 u|^2\}^{n/(n+2)} \quad \text{a.e. in } \Omega,$$

$$f = (1 + |u|^{np/(n-p)} + |Du|^p)^{n/(n+2)} \quad \text{a.e. in } \Omega$$

and

$$q_1 = \frac{n+2}{n}.$$

(6) $\fint_{B_\varrho} f dx = (\text{meas}\, B_\varrho)^{-1} \int_{B_\varrho} f dx.$

Then (3.5) reads

$$\fint_{B_r} g^{q_1}\,dx \leqslant c\left(\fint_{B_{3r}} g\,dx\right)^{q_1} + c\fint_{B_{3r}} f^{q_1}\,dx\,.$$

This estimate holds for all balls $B_{3r} \subset \Omega'$, where Ω' is any bounded open set such that $\overline{\Omega}' \subset \Omega$. Therefore, g is locally integrable to a power $t_1 > q_1$ and there holds

$$\left(\fint_{B_r} g^{t_1}\,dx\right)^{1/t_1} \leqslant c\left\{\left(\fint_{B_{3r}} g^{q_1}\,dx\right)^{1/q_1} + \left(\fint_{B_{3r}} f^{t_1}\,dx\right)^{1/t_1}\right\}$$

($c = \text{const} > 0$ independent of r) (cf. [7]). In other words,

$$\int_{B_r} \{(1+|Du|^2)^{(p-2)/2}|D^2u|^2\}^{nt_1/(n+2)}\,dx \leqslant$$

$$\leqslant cr^{n(1-t_1/q_1)}\left(\int_{B_{3r}} (1+|Du|^2)^{(p-2)/2}|D^2u|^2\,dx\right)^{nt_1/(n+2)} +$$

$$+ c\int_{B_{3r}} (1+|u|^{np/(n-p)} + |Du|^p)^{t_1}\,dx\,.$$

We set $t = 2nt_1/(n+2)$ and obtain the statement of the Theorem. ■

Proof of the corollary. We repeat an argument already used in the proof of Theorem 1. For any $\sigma > 0$,

$$(3.6)\qquad |D^2u|^\sigma = (1+|Du|^2)^{\sigma(2-p)/4}[(1+|Du|^2)^{(p-2)/4}|D^2u|^2]^\sigma$$

for a.a. $x \in \Omega$.

Let $n = 2$. We have $Du \in L^s_{\text{loc}}(\Omega;\mathbb{R}^{2N})$ for all $1 \leqslant s < \infty$ (cf. (2.14)), and therefore $(1+|Du|^2)^{\sigma(2-p)/4} \in L^{t/(t-\sigma)}_{\text{loc}}(\Omega)$ for all $2 \leqslant \sigma < t$. Then (3.6) implies (3.3).

Let $n \geq 3$. As above, define $U = (1 + |Du|^2)^{p/4}$ a.e. in Ω. By (3.1), $|DU| \in L^t_{\rm loc}(\Omega)$, and therefore $|Du| \in L^{npt/2(n-t)}_{\rm loc}(\Omega)$.

Setting $\sigma = npt/(2n + t(p-2))$ we obtain

$$\frac{\sigma(2-p)}{2} \frac{t}{(t-\sigma)} = \frac{npt}{2(n-t)},$$

and (3.6) implies (3.4). ■

4. Differentiability of weak solutions (natural growth).

Let a_i^α and b_i ($\alpha = 1, \ldots, n$; $i = 1, \ldots, N$) satisfy conditions (1.2), (1.4) and (1.5), (1.7), respectively. A (vector) function $u \in W^{1,p}(W; \mathbb{R}^N) \cap L^\infty_{\rm loc}(\Omega; \mathbb{R}^N)$ is called a *weak solution* to (1.1) if

$$\int_\Omega a_i^\alpha(x, Du) D_\alpha \varphi^i \, dx = \int_\Omega b_i(x, u, Du) \varphi^i \, dx, \tag{4.1}$$

$$\forall \varphi \in W^{1,p}(\Omega; \mathbb{R}^N) \cap L^\infty_{\rm loc}(\Omega; \mathbb{R}^N), \ \mathrm{supp}(\varphi) \subset \Omega.$$

To make the following discussion more precise, we fix any bounded open set Ω' such that $\overline{\Omega}' \subset \Omega$. The main result of this section is

THEOREM 3. *Let* (1.2), (1.3), (1.4) *and* (1.5), (1.7) *be satisfied. Let* $u \in W^{1,p}(\Omega; \mathbb{R}^N) \cap C^\mu(\Omega; \mathbb{R}^N)$ $(0 < \mu < 1)$ *be a weak solution to* (1.1).

Then $u \in W^{1,p+2}_{\rm loc}(\Omega; \mathbb{R}^N) \cap W^{2,(p+2)/2}_{\rm loc}(\Omega; \mathbb{R}^N)$ *and*

$$\int_{B_r} |D^2 u|^{(p+2)/2} dx \leq c\left(1 + \frac{1}{r^{(p+2)/2}}\right) \int_{B_{3r}} (1 + |Du|^{p+2}) \, dx \tag{4.2}$$

for all balls $B_{3r} \subset \Omega'$, *where the constant* c *depends on* $\|u\|_{C^\mu(\overline{\Omega}'; \mathbb{R}^N)}$. ■

We begin by proving some integral estimates on the weak solution under consideration, which are of interest in itself. In contrast to Section 2, now we consider the difference

$$\tau_h v(x) = \tau_h^{(\beta)} v(x) = v(x + he_\beta) - v(x),$$

($e_\beta = \{0, \ldots, 0, 1, 0, \ldots, 0\}$[1 at the β-th place]).

We have

LEMMA 1. *Let* (1.2), (1.3), (1.4) *and* (1.5), (1.7) *be satisfied. Let* $u \in W^{1,p}(\Omega; \mathbb{R}^N) \cap L^\infty_{\text{loc}}(\Omega; \mathbb{R}^N)$ *be a weak solution to* (1.1).
Then

$$(4.3) \qquad \int_{B_{2r}} (1 + |Du(x + he_\beta)|^2 + |Du(x)|^2)^{(p-2)/2} |\tau_h Du|^2 \zeta^2 dx \leqslant$$

$$\leqslant c\left(1 + \frac{1}{r^2}\right) h^2 \int_{B_{3r}} (1 + |Du|^p)\, dx + c \int_{B_{2r}} (1 + |Du|^p) |\tau_{-h}(\zeta^2 \tau_h u^i)|\, dx$$

for all $0 < |h| < \min\{r, 1\}$ *and all balls* $B_{3r} \subset \Omega'$ ($c = \text{const} > 0$ *independent of* r).

PROOF. Let ζ be the same cut-off function as in the proof of Theorem 1. Then the function $\varphi = \tau_{-h}(\zeta^2 \tau_h u)$ $(0 \leqslant |h| < r)$ is admissible in (4.1). Analogously as (2.8) above we now obtain

$$c_1 \int_{B_{2r}} (1 + |Du(x + he_\beta)|^2 + |Du(x)|^2)^{(p-2)/2} |\tau_h Du|^2 \zeta^2 dx \leqslant$$

$$\leqslant -h \int_{B_{2r}} \left(\int_0^1 \frac{\partial a_i^\alpha}{\partial x_\beta}(x + the_\beta, Du(x + he_\beta))\, dt \right) \zeta^2 \tau_h D_\alpha u^i dx +$$

$$+2 \int_{B_{2r}} a_i^\alpha(x, Du) \tau_{-h}(\zeta D_\alpha \zeta \tau_h u^i)\, dx - \int_{B_{2r}} b_i(x, u, Du) \tau_{-h}(\zeta^2 \tau_h u^i)\, dx =$$

$$= I_1 + I_2 + I_3.$$

By (1.4), for any $\varepsilon > 0$,

$$I_1 \leqslant c|h| \int_{B_{2r}} (1 + |Du(x + he_\beta))|^{p-1} |\tau_h Du|\, \zeta^2 dx \leqslant$$

$$\leqslant \varepsilon |h|^{2-p} \int_{B_{2r}} |\tau_h Du|^p \zeta^p dx + c\varepsilon^{(1/1-p)} h^2 \int_{B_{3r}} (1 + |Du|^p)\, dx \leqslant$$

$$\leqslant \frac{\varepsilon p}{2} \int_{B_{2r}} (1 + |Du(x+he_\beta)|^2 + |Du(x)|^2)^{(p-2)/2} |\tau_h Du|^2 \zeta^2 \, dx +$$

$$+ \left(\frac{\varepsilon(2-p)}{2} + c\varepsilon^{(1/1-p)} \right) h^2 \int_{B_{3r}} (1 + 2|Du|^p) \, dx .$$

To estimate I_2, we again make use of (1.4) and (i) (cf. p. 59). Then for any $\delta > 0$,

$$I_2 \leqslant c|h| \left(\int_{B_{2r}} (1 + |Du|^p) \, dx \right)^{1/p'} \left(\int_{B_{2r}} |D_\beta(\zeta D_\alpha \zeta \tau_h u)|^p \, dx \right)^{1/p} \leqslant$$

$$\leqslant \delta |h|^{2-p} \int_{B_{2r}} |D_\beta(\zeta D_\alpha \zeta \tau_h u)|^p \, dx + c\delta^{1/(1-p)} h^2 \int_{B_{2r}} (1 + |Du|^p) \, dx \leqslant$$

$$\leqslant (\delta c_0)^{2/p} r^{-(4(p-1))/p} \int_{B_{2r}} (1 + |Du(x+he_\beta)^2 + |Du(x)|^2)^{(p-2)/2} |\tau_h Du|^2 \zeta^2 \, dx +$$

$$+ c(r^{-2} + \delta r^{-2p} + \delta^{1/(1-p)}) h^2 \int_{B_{3r}} (1 + |Du|^p) \, dx \, (^7) .$$

Finally, by (1.7),

$$I_3 \leqslant c \int_{B_{2r}} (1 + |Du|^p) |\tau_{-h}(\zeta^2 \tau_h u)| \, dx$$

(notice that $M = \|u\|_{L^\infty(\Omega'; \mathbb{R}^N)} < \infty$).

Clearly, without loss of generality we may assume that $0 < r \leqslant 1$. Then choosing appropriately ε and δ gives (4.3). ∎

LEMMA 2. *Let* (1.2), (1.3), (1.4) *and* (1.5), (1.7) *be satisfied. Let* $u \in W^{1,p}(\Omega; \mathbb{R}^N) \cap C^\mu(\Omega; \mathbb{R}^N)$ *be a weak solution to* (1.1) *such that* $Du \in L^q_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ $(p \leqslant q \leqslant p + 2)$.

(7) Recall that $|D\zeta| \leqslant c_0/r$, $|D^2\zeta| \leqslant c_0/r^2$.

Then

$$(4.4)\qquad \int_{B_r} (1 + |Du(x + he_\beta)|^2 + |Du(x)|^2)^{(p-2)/2} |\tau_h Du|^2 dx \leqslant$$

$$\leqslant c\left(1 + \frac{1}{r^2}\right) |h|^{q - p + \mu(2 - q + p)/2} \int_{B_{3r}} (1 + |Du|^q)\, dx,$$

for all $0 \leqslant |h| < \min\{r, 1\}$ *and all balls* $B_{3r} \subset \Omega'$ ($c = \text{const} > 0$ *depends on* $\|u\|_{C^\mu(\bar{\Omega}'; \mathbb{R}^N)}$, *but does neither on* h *nor on* r*).*

Before turning to the proof of Lemma 2 we note the following result.

Let $E \subset \mathbb{R}^N$ *be a measurable set. Let* f *and* g *be measurable functions on* E *such that*

$$f \geqslant 0, \qquad g > 0 \quad \text{a.e. on } E,$$

$$f^\sigma, \qquad g^{\sigma(2-p)/(2-\sigma)}, \qquad g^{p-2} f^2 \in L^1(E),$$

where $0 < \sigma < 2$, $1 < p < 2$.
Then

$$(*)\qquad \int_E f^\sigma dx \leqslant \left(\int_E g^{\sigma(2-p)/(2-\sigma)} dx\right)^{(2-\sigma)/2} \left(\int_E g^{p-2} f^2 dx\right)^{\sigma/2}.$$

Indeed, applying Hölder's inequality with $t = 2/\sigma$, $t' = 2/(2 - \sigma)$ gives

$$\int_E f^\sigma dx = \int_E g^{\sigma(2-p)/2} g^{\sigma(p-2)/2} f^\sigma dx \leqslant$$

$$\leqslant \left(\int_E g^{\sigma(2-p)/(2-\sigma)} dx\right)^{(2-\sigma)/2} \left(\int_E g^{p-2} f^2 dx\right)^{\sigma/2}. \qquad \blacksquare$$

Proof of Lemma 2. Obviously,

$$(4.5)\quad |\tau_{-h}(\zeta^2 \tau_h u)| \leqslant 2\|u\|_{C^\mu(\bar{\Omega}'; \mathbb{R}^N)} |h|^\mu \qquad \forall x \in B_{2r}, \qquad 0 \leqslant |h| < r.$$

1) Let $q = p$ Then (4.3) and (4.5) imply (4.4) ([8]).

([8]) Recall that $|h| < 1$.

2) Let $q < p < p+2$. Set $s = (2-q+p)/(2+q-p)$. By (4.5), Hölder's inequality and (i) (cf. p. 59),

$$\int_{B_{2r}} (1+|Du|^p)|\tau_{-h}(\zeta^2\tau_h u)|\,dx \leqslant c\|u\|^s_{C^\mu(\bar{\Omega}';\,\mathbb{R}^N)}|h|^{\mu s}\left(\int_{B_{2r}}(1+|Du|^q)\,dx\right)^{p/q}\cdot$$

$$\cdot\left(\int_{B_{2r}}|\tau_{-h}(\zeta^2\tau_h u)|^{q(1-s)/(q-p)}\,dx\right)^{(q-p)/q}\leqslant$$

$$\leqslant c|h|^{\mu s+4(q-p)/(2+q-p)}r^{2(p-q)/(2+q-p)}\int_{B_{3r}}(1+|Du|^q)\,dx +$$

$$+c|h|^{\mu s+2(q-p)/(2+q-p)}\left(\int_{B_{2r}}(1+|Du|^q)\,dx\right)^{p/q}\cdot$$

$$\cdot\left(\int_{B_{2r}}(|\tau_h Du|\zeta)^{q(1-s)/(q-p)}\,dx\right)^{(q-p)/q} = J_1 + J_2$$

(notice that $(q(1-s))/(q-p) = 2q/(2+q-p) < q$). It remains only to estimate J_2. To this end, we apply inequality $(*)$ to

$$f = |\tau_h Du|\zeta, \qquad g = 1+|Du(\cdot+he_\beta)|+|Du|, \qquad \sigma = \frac{q(1-s)}{q-p}.$$

Then

$$\int_{B_{2r}}(|\tau_h Du|\zeta)^{q(1-s)/(q-p)}\,dx \leqslant$$

$$\leqslant\left(\int_{B_{2r}}(1+|Du(x+he_\beta)|+|Du(x)|)^q\,dx\right)^{(2-\sigma)/2}\cdot$$

$$\cdot\left(\int_{B_{2r}}(1+|Du(x+he_\beta)|+|Du(x)|)^{p-2}|\tau_h Du|^2\zeta^2\,dx\right)^{\sigma/2}.$$

Thus, for any $\varepsilon > 0$,

$$J_2 \leqslant \varepsilon \int_{B_{2r}} (1 + |Du(x + he_\beta)|^2 + |Du(x)|^2)^{(p-2)/2} |\tau_h Du|^2 \zeta^2 \, dx +$$

$$+ c\varepsilon^{q/(p-2)} |h|^{q - p + \mu(2 - q + p)/2} \int_{B_{3r}} (1 + |Du|^q) \, dx .$$

Now, we choose ε sufficiently small and take into account that

$$q - p + \frac{\mu}{2}(2 - q + p) < \mu s + \frac{4(q-p)}{2 + q - p}, \qquad |h| < \min\{r, 1\} .$$

Inserting the estimate on $\int_{B_{2r}} (1 + |Du|^p) |\tau_{-h}(\zeta^2 \tau_h u)| \, dx$ into (4.3) gives (4.4).

3) Let $q = p + 2$. Then

$$\int_{B_{2r}} (1 + |Du|^p) |\tau_{-h}(\zeta^2 \tau_h u)| \, dx \leqslant c|h| \left(\int_{B_{2r}} (1 + |Du|^{p+2}) \, dx \right)^{p/(p+2)} \cdot$$

$$\cdot \left(\int_{B_{2r}} |D_\beta(\zeta^2 \tau_h u)|^{(p+2)/2} dx \right)^{2/(p+2)} \leqslant$$

$$\leqslant \frac{c|h|^2}{r} \int_{B_{3r}} (1 + |Du|^{p+2}) \, dx +$$

$$+ c|h| \left(\int_{B_{2r}} (1 + |Du|^{p+2}) \, dx \right)^{p/(p+2)} \left(\int_{B_{2r}} (|\tau_h Du| \zeta)^{(p+2)/2} dx \right)^{2/(p+2)} .$$

To evaluate the last integral, we again apply ($*$) with f and g as in the preceding case, but $\sigma = (p + 2)/2$. An analogous reasoning as above gives (4.4). ■

COROLLARY. *Let the assumptions of Lemma 2 be satisfied. Then*

$$\int_{B_r} |\tau_h Du|^{2q/(2+q-p)}\,dx \leq \tag{4.6}$$

$$\leq c(1+r^{-2q/(2+q-p)})|h|^{q[q-p+\mu(2-q+p)/2]/(2+q-p)} \int_{B_{3r}} (1+|Du|^q)\,dx,$$

for all $0 \leq |h| < \min\{r, 1\}$ *and all balls* $B_{3r} \subset \Omega'$.

Indeed, let denote

$$f = |\tau_h Du|, \qquad g = 1 + |Du(\cdot + he_\beta)| + |Du|, \qquad \sigma = \frac{2q}{2+p-q}.$$

From (*) we get

$$\int_{B_r} |\tau_h Du|^{2q/(2+q-p)}\,dx \leq \left(\int_{B_r} (1+|Du(x+he_\beta)|+|Du(x)|)^q\,dx\right)^{(2-\sigma)/2} \cdot$$

$$\cdot \left(\int_{B_r} (1+|Du(x+he_\beta)|^2+|Du|^2)^{(p-2)/2}|\tau_h Du|^2\,dx\right)^{\sigma/2}.$$

Then (4.6) follows from (4.3). ■

Finally, set

$$Q_r = Q_r(x^0) = \{x \in \mathbb{R}^n : |x_\alpha - x_\alpha^0| < r \quad (\alpha = 1, \dots, n)\}.$$

Let $v \in L^q(Q_{3r})$ $(1 < q < \infty)$, and suppose that for $\beta = 1, \dots, n$ and a fixed $\sigma > 0$ there holds

$$\int_{Q_r} |\tau_h^{(\beta)} v|^q\,dx \leq c|h|^\sigma \qquad \forall 0 \leq |h| < 2r \ (c = \text{const} > 0).$$

Clearly,

$$\int_{-2r}^{2r} \frac{1}{|h|^{1+q}} \left(\int_{Q_r} |\tau_h^{(\beta)} v|^q\,dx\right) dh < \infty,$$

for all $0 < \theta < \sigma/q$. Thus,

$$(4.7)\quad \begin{cases} v \in W^{\theta,q}(Q_r), \\ \|v\|^q_{W^{\theta,q}(Q_r)} \leq c \sum_{\beta=1}^{n} \int_{-2r}^{2r} \frac{1}{|h|^{1+q\theta}} \left(\int_{Q_r} |\tau^{(\beta)} v|^q \, dx \right) dh, \end{cases}$$

(cf. [3]).

The following result may be also found in [3]. For notational simplicity, we set $Q = Q_r$.

LEMMA 3. *Let* $v \in W^{1+\theta,q}(Q) \cap C^{\mu}(\overline{Q})$ $(1 \leq q < \infty,\ 0 < \mu < 1,\ 0 < \theta < \min\{n/\mu q, 1\})$.

Then

$$t^s \operatorname{meas}\left\{ x \in Q: \left| Dv(x) - \fint_Q Dv\,dy \right| > t \right\} \leq$$

$$\leq c\|Dv\|^{s/(1+\theta)}_{W^{\theta,q}(Q)} \|v\|^{s\theta/(1+\theta)}_{C^{\mu}(\overline{Q})} \qquad \forall t > 0,$$

where

$$s = \frac{nq(1+\theta)}{n - \mu q\theta}.$$

In particular, $Dv \in L^{\sigma}(Q; \mathbb{R}^N)$ *for all* $1 \leq \sigma < s$. ■

We are now in a position to give the

PROOF OF THEOREM 3. Without loss of generality, we may assume that $0 < \mu \leq 1/2$. Hence $\mu < n/(p+2)$, and therefore

$$\min\left\{ \frac{n(2+q-p)}{2\mu q}, 1 \right\} = 1 \qquad \forall p \leq q \leq p+2.$$

Set $q = p$ in (4.6). We obtain

$$\int_{B_r} |\tau_h Du|^p \, dx \leq c\left(1 + \frac{1}{r^p}\right) |h|^{\mu p/2} \int_{B_{3r}} (1 + |Du|^p) \, dx,$$

for all $0 \leq |h| < r$ and all balls $B_{3r} \subset \Omega'$. Thus,

$$u \in W^{1+\theta,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N) \qquad \forall 0 < \theta < \frac{\mu}{2}$$

(cf. (4.7)). Taking into account that $u \in C^{\mu}(\Omega; \mathbb{R}^N)$, we obtain by the aid of Lemma 3

$$Du \in L^{\sigma}_{\text{loc}}(\Omega; \mathbb{R}^{nN}) \qquad \text{for all } 1 \leqslant \sigma < \frac{np(1+\theta)}{n - \mu p \theta}.$$

Define $q_0 = \sup\{q \in [p, \infty): Du \in L^{q}_{\text{loc}}(\Omega; \mathbb{R}^{nN})\}$. The preceding argument implies $q_0 > p$. Assume $q_0 \leqslant p + 2$. Then we fix a real θ_0 such that

$$\frac{n(q_0 - p)}{2(n + q_0)} < \theta_0 < \frac{1}{2}\left(q_0 - p + \frac{\mu}{2}(2 - q_0 + p)\right).$$

The left inequality gives

$$q_0 < f(q_0),$$

where

$$f(q) = \frac{2nq(1+\theta_0)}{n(2 + q - p) - 2\mu q \theta_0}, \qquad p < q \leqslant p + 2.$$

By continuity, there exists a $p < q^* < q_0$ such that

$$\theta_0 < \frac{1}{2}\left(q^* - p + \frac{\mu}{2(2 - q^* + p}\right), \tag{4.8}$$

$$q_0 < f(q^*). \tag{4.9}$$

Now, set $q = q^*$ in (4.6). As above, we conclude

$$u \in W^{1+\theta, 2q^*/(2+q^*-p)}_{\text{loc}}(\Omega; \mathbb{R}^N) \qquad \forall 0 < \theta < \frac{1}{2}\left(q^* - p + \frac{\mu}{2}(2 - q^* + p)\right).$$

Here $\theta = \theta_0$ is admissible (cf. (4.8)). Hence, again using Lemma 3 we find $Du \in L^{\sigma}_{\text{loc}}(\Omega; \mathbb{R}^{nN})$ for all

$$1 \leqslant \sigma < \frac{2nq^*(1+\theta_0)}{n(2 + q^* - p) - 2\mu q^* \theta_0} = f(q^*),$$

in particular, $q_0 < \sigma < f(q^*)$ (cf. (4.9)) which contradicts the definition of q_0.

Therefore $q_0 > p + 2$. Then we have $Du \in L^{p+2}_{\text{loc}}(\Omega; \mathbb{R}^{nN})$, and (4.6) with $q = p + 2$ gives

$$\text{(4.10)} \quad \int_{B_r} |\tau_h Du|^{(p+2)/2} dx \leqslant$$

$$\leqslant c\left(1 + \frac{1}{r^{(p+2)/2}}\right) |h|^{(p+2)/2} \int_{B_{3r}} (1 + |Du|^{p+2})\, dx,$$

for all $0 \leqslant |h| < r$ and all balls $B_{3r} \subset \Omega'$. By the aid of (ii) (cf. p. 59) we infer from (4.10) that $u \in W^{2,(p+2)/2}_{\text{loc}}(\Omega; \mathbb{R}^N)$. Then (4.2) follows by letting tend $h \to 0$ in (4.10). ■

Appendix.

Let $1 \leqslant p < 2$. Let B_r be a ball in $\mathbb{R}^n$. We have the following.

PROPOSITION. *Let* $1 \leqslant q \leqslant n$. *Let* $w \in W^{1,1}(B_r; \mathbb{R}^m)$ *satisfy* $(1 + |w|^2)^{(p-2)/4} |Dw| \in L^q(B_r)$.

Then $(1 + |w|^2)^{(p-2)/4} w \in L^{q^*}(B_r; \mathbb{R}^m)$ $(q^* = nq/(n-q)$ *if* $1 \leqslant p < n$, $1 \leqslant q^* < \infty$ *if* $n = q)$, *and for all*

$$1 \leqslant s \leqslant \frac{nq}{n-q} \quad \text{if } 1 \leqslant p < n,$$

$$1 \leqslant s < \infty \quad \text{if } n = q,$$

there exisits a $\Lambda = \Lambda(w, r) \in \mathbb{R}^m$ *such that*

$$\left(\int_{B_r} \{(1 + |w|^2 + |w - \Lambda|^2)^{(p-2)/4} |w - \Lambda|\}^s dx\right)^{1/s} \leqslant$$

$$\leqslant c r^{1 + n/s - n/q} \left(\int_{B_r} \{(1 + |w|^2)^{(p-2)/4} |Dw|\}^q dx\right)^{1/q},$$

$$|\Lambda| \leqslant \left(1 + \fint_{B_r} (1 + |w|^2)^{p/4} dx\right)^{2/q},$$

where the constant c *depends on* n, q *and* s *only.*

PROOF. Define $z = (1 + |w|^2)^{(p-2)/4} w$ a.e. in B_r. We show $z \in W^{1,q}(B_r; \mathbb{R}^m)$.

Indeed, it is readily seen that

$$\text{(A1)}\qquad |D_\alpha z| \leqslant \left(2 - \frac{p}{2}\right)(1 + |w|^2)^{(p-2)/4}\,|Dw| \qquad \text{a.e. in } B_r$$

($\alpha = 1, \dots, n$). Hence, $z \in W^{1,1}(B_r; \mathbb{R}^m)$. By Sobolev's imbedding theorem $z \in L^{n/(n-1)}(B_r; \mathbb{R}^m)$.

If $q \leqslant n/(n-1)$ we have finished. Otherwise, suppose that $n/(n-1) < q \leqslant n$. Then $z \in W^{1,\,n/(n-1)}(B_r; \mathbb{R}^m)$ and therefore $z \in L^{n/(n-2)}(B_r; \mathbb{R}^m)$.

We may repeat this procedure. Let $k \geqslant 0$ be the largest integer such that $k < n - n/q$. Then $z \in L^{n/(n-k-1)}(B_r; \mathbb{R}^m)$ and $q \leqslant n/(n-k-1)$. Whence $z \in W^{1,q}(B_r; \mathbb{R}^m)$.

The Sobolev-Poincaré inequality reads

$$\text{(A2)}\qquad \left(\int_{B_r} \left| z - \fint_{B_r} z\,dy \right|^s dx\right)^{1/s} \leqslant c_0\, r^{1 + n/s - n/q}\left(\int_{B_r} |Dz|^q\,dx\right)^{1/q},$$

with $c_0 = $ const depending on n, s and q only.

On the other hand, an elementary calculation gives

$$\text{(A3)}\qquad |(1 + |w|^2)^{(p-2)/4} w - (1 + |\lambda|^2)^{(p-2)/4}\lambda| \geqslant$$

$$\geqslant \frac{1}{2}\int_0^1 (1 + |w + t(\lambda - w)|^2)^{(p-2)/4}\,|\lambda - w|\,dt \geqslant$$

$$\geqslant 2^{-5/4}(1 + |w|^2 + |\lambda - w|^2)^{(p-2)/4}\,|\lambda - w|,$$

for a.a. $x \in B_r$ and all $\lambda \in \mathbb{R}^m$.

Next, define

$$F(\lambda) = (1 + |\lambda|^2)^{(p-2)/4}\lambda, \qquad \lambda \in \mathbb{R}^m.$$

Clearly, F is continuous from $\mathbb{R}^m$ into $\mathbb{R}^m$. Moreover, it is easily verified that

$$(F(\lambda) - F(\eta), \lambda - \eta) > 0, \qquad \forall \lambda,\ \eta \in \mathbb{R}^m,\ \lambda \neq \eta,$$

$$(F(\lambda), \lambda) > (1 + |\lambda|^2)^{(p+2)/4} - 1 \qquad \forall \lambda \in \mathbb{R}^m.$$

Thus, F is a bijection from $\mathbb{R}^m$ onto itself. Hence, there exists exactly one $\Lambda \in \mathbb{R}^m$ such that

$$(A4) \qquad (1 + |\Lambda|^2)^{(p-2)/4}\Lambda = \fint_{B_r} (1 + |w|^2)^{(p-2)/4} w \, dx .$$

Obviously,

$$|\Lambda|^{p/2} \leqslant 1 + \fint_{B_r} (1 + |w|^2)^{p/4} dx .$$

The proof is now easily completed as follows. Letting $\lambda = \Lambda$ in (A.3) (Λ according to (A.4)), we combine (A.1) and (A.2) to obtain

$$\left(\int_{B_r} \{(1 + |w|^2 + |w - \Lambda|^2)^{(p-2)/4} |w - \Lambda|\}^s dx \right)^{1/s} \leqslant$$

$$\leqslant 2^{5/4} \left(\int_{B_r} |(1 + |w|^2)^{(p-2)/4} w - (1 + |\Lambda|^2)^{(p-2)/4} \Lambda|^s dx \right)^{1/s} \leqslant$$

$$\leqslant 2^{5/4} c_0 r^{1 + n/s - n/q} \left(\int_{B_r} |Dz|^q dx \right)^{1/q} \leqslant$$

$$\leqslant c r^{1 + n/s - n/q} \left(\int_{B_r} \{(1 + |w|^2)^{(p-2)/4} |Dw|\}^q dx \right)^{1/q} . \qquad \blacksquare$$

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Manoscritto pervenuto in redazione il 30 aprile 1991.