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Zero-One Matrices with an Application to Abelian Groups.

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SUMMARY - An $n \times n$ matrix E is called a 0,1-matrix if each entry of E is either a 0 or a 1. In this case we can view E as either an integer valued matrix, or a matrix over Z_2, the integers mod 2. Matrices of this type, enjoying other properties as well, have recently cropped up in the study of torsion-free abelian group theory. Our aim is to study properties of these matrices in a setting unencumbered by this group theory. As a consequence we are able to answer a question posed in [FM].

1. A 0,1-matrix E is called admissible in [FM] provided $|E_k| = \det E_k \neq 0$ for each k, where E_k is E with its k^{th} colums replaced by the vector $\underline{1}$ containing only 1's. We will say that F is equivalent to E if one can complement (by interchanging 1's and 0's) certain columns of E to get F. It is easy to check that admisibility is preserved under this equivalence. This is because if E' is equivalent to E after the i^{th} column only of E was complemented, then $|E_j'| = -|E_j|$ when $j \neq i$, and $|E_i'| = |E_i|$. The admissible matrices play a significant role in abelian group theory, a role which will be summarized in the second section.

We will consider two conditions imposed on a matrix E over Z_2:

(α) Each row sum of E, computed in Z_2, is the same, and

(β) E is equivalent to an invertible matrix over Z_2.

Clearly, both conditions are preserved under our equivalence relation. We will compare these conditions to the property of being admis-

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sible. We will call a matrix E over Z_2, admissible mod 2, if for all k the Z_2-determinant of E_k, $|E_k|_2$, is not zero where E_k is as defined above. Of course, if E is admissible mod 2 then E is admissible when viewed as a matrix with integer entries.

PROPOSITION 1. *Let E be an $n \times n$ matrix over Z_2 and E^* the classical adjoint of E (over Z_2). Then E is admissible mod 2 if and only if* $E^*\bar{1} = \bar{1}$.

PROOF. The k^{th} entry of $E^*\bar{1}$ is $M_{1k} + M_{2k} + \ldots + M_{nk}$ where $M_{ik} = i, k^{\text{th}}$ cofactor ($=$ minor) of E. But this sum is just the cofactor expansion of $|E_k|_2$ along its k^{th} column. Hence, $|E_k|_2 = 1$, (i.e. $|E_k|_2 \neq 0$) for all k if and only if $E^*\bar{1} = \bar{1}$.

We will show that E satisfies both (α) and (β) if and only if E is admissible mod 2. In case E satisfies (α) we often refer to E as having row parity. Clearly E has row parity if and only if $\bar{1}$ is an eigenvector for E over Z_2. In case $E\bar{1} = \bar{0}$, E has even row parity, and if $E\bar{1} = \bar{1}$, then E has odd row parity. We will use $\bar{n}$ to denote $\{1, 2, \ldots, n\}$ when no confusion is possible.

THEOREM 2. *E is admissible mod 2 if and only if (α) and (β) hold for E.*

PROOF. The j^{th} column of E is the characteristic function on some index set $I \subseteq \bar{n}$. As such we will call the support of the j^{th} column of E, I.

If E is admissible mod 2 and I is the support of the 1^{st} column of E, let E' be the matrix resulting from complementing the 1^{st} column of E. Then, the support of the 1^{st} column of E' is $I' = \bar{n} \setminus I$. By performing cofactor expansion of $|E_1|_2$, $|E|_2$ and $|E'|_2$ along their first columns, we see that $|E_1|_2 = 1 = |E|_2 + |E'|_2$. If $|E|_2 = 0$ then $|E'|_2 = 1$ so that E is equivalent to an invertible matrix. Also, by Proposition 1, $EE^*\bar{1} = E\bar{1} = (\det E)\bar{1}$, so that E has row parity.

Conversely, it is enough to assume that E is invertible. From this and because of (α), $E\bar{1} = \bar{1}$. Then $E^*E\bar{1} = E^*\bar{1} = (\det E)\bar{1} = \bar{1}$, and E is admissible mod 2 by Proposition 1. ∎

EXAMPLE 3. It can be checked that $E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ is admissible, but E does not have row parity so it is not admissible mod 2.

Row parity is easily checked. Any $n \times n$ 0,1-matrix E is equivalent to a matrix $E' = \begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix}$ where F is an $(n-1) \times (n-1)$ 0,1-matrix and $I \in Z_2^{n-1}$. Hence to check that (β) holds for E we need only compute $|F|_2$, which is clearly preferable to the computation of n determinants for admissiblity mod 2.

LEMMA 4. *There are* $\prod_{j=0}^{m-1}(2^m - 2^j)$ *invertible* $m \times m$ *matrices over* Z_2.

PROOF. To form an invertible $m \times m$ matrix, we must select $X_1 \in Z_2^m \setminus \{0\}$ for the first column, $X_2 \in Z_2^m \setminus \operatorname{span}\{X_1\}$ for the second, $X_3 \in Z_2^m \setminus \operatorname{span}\{X_1, X_2\}$ for the third, and so on. There are $(2^m - 1) \cdot (2^m - 2) \dots (2^m - 2^{m-1})$ ways for this selection to occur.

It is desiderable to know just how many admissible mod 2 matrices there are. Let $\mathcal{E} = \{E \mid E \text{ is } n \times n, \text{ admissible mod } 2 \text{ and invertible}\}$. Since $\mathcal{E}$ is finite and is closed under multiplication, $\mathcal{E}$ is a group. Let $\mathcal{F}$ be the subgroup of $\mathcal{E}$ consisting of those $E \in \mathcal{E}$ with $E = \begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix}$, as above.

THEOREM 5. (i) $|\mathcal{F}| = \prod_{j=0}^{n-2}(2^{n-1} - 2^j)$.

(ii) $|\mathcal{E}| = 2^{n-1}\prod_{j=0}^{n-2}(2^{n-1} - 2^j)$.

(iii) *There are* $2^n \prod_{j=0}^{n-2}(2^{n-1} - 2^j)$ *admissible* mod 2 *matrices.*

PROOF. Any $E \in \mathcal{F}$ can be expressed as $E = \begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix}$ with F an $(n-1) \times (n-1)$ invertible matrix uniquely determined by E. Since E has row parity, $I + F\bar{1} = \bar{1}$, since the first row of E has parity 1, so that $I = (F\bar{1})'$ (the complement of $F\bar{1}$) is determined by F. Conversely, any $(n-1) \times (n-1)$ invertible matrix F determines the matrix $\begin{bmatrix} 1 & 0 \\ I & F \end{bmatrix} \in \mathcal{F}$ where $I = (F1)'$, so the computation of $|\mathcal{F}|$ follows from lemma 4.

Any $E \in \mathcal{E}$ is equivalent to a matrix in $\mathcal{F}$. Now suppose that $E \in \mathcal{F}$, and that E' is a matrix equivalent to E as the result of complementing the j^{th} column (only) of E. An in the proof of Theorem 2, $1 = |E|_2 + |E'|_2$ so that $E' \notin \mathcal{E}$. If E'' is matrix resulting from complimenting only one column of E', then as before $|E''|_2 + |E'|_2 = 1$ and $E'' \in \mathcal{E}$. It follows that if $E^{(s)}$ results from E by complimenting some s columns of E, then $E^{(s)} \in \mathcal{E}$ if and only if s is even.

Let a_n = number of subsets of $2^{\overline{n}}$ containing an even number of elements. We hage just shown that $|\mathcal{E}| = a_n |\mathcal{F}|$. Set $b_n = 2^n - a_n$, and define $\delta\colon 2^{\overline{n}} \to Z_2$ by letting $\delta(T)$ = remainder of card(T) mod 2. Let $S + T$ denote the symmetric difference of S and T so that $2^{\overline{n}}$ is an abelian group under $+$. Since $\operatorname{card}(S+T) = \operatorname{card}(S) + \operatorname{card}(T) - 2\operatorname{card}(S \cap T)$, $\delta(S+T) = \delta(S) + \delta(T)$ and δ is a homomorphism. Hence $a_n = b_n = 2^n/2 = 2^{n-1}$.

If $\mathcal{E}'$ is the set of admissible mod 2 matrices with zero determinant, then the map sending $E \in \mathcal{E}$ to the matrix E' formed by complimenting the first column of E, is a bijection. Thus, there are $2|\mathcal{E}|$ admissible mod 2 matrices. ■

2. In this section we will attempt to convey the role that the matrices $E \in \mathcal{E}$ play in abelian group theory without involving the group theory.

The use of admissible matrices in classifying a certain class of Butlwe groups (specifically, the $B(1)$-groups) was initiated in [FM], and investigated further in [GM]. Other results concerning the same class of groups were obtained earlier in [AV] and [Ri]. For a deeper involvement of the group theory, see the listed references.

The set of isomorphism classes of subgroups of the rationals form a distributive lattice Δ. Moreover, any finite distributive lattice T is isomorphic to a sublattice of Δ ([R] or [GU]). Let us fix an isomorphism. Then for any collection $\tau_1, \dots, \tau_n \in T$, the n-tuple $\tau = (\tau_1, \dots, \tau_n)$ determines a certain abelian group $G = G[\tau_1, \dots, \tau_n]$. The description of G is not relevant here but the interested reader should consult the cited references (in fact, G is only determined up to quasi-isomorphism: see below).

Given an n-tuple $\tau = (\tau_1, \dots, \tau_n)$ with $\tau_i \in T$, and a 0,1-matrix E we can let E operate on τ as follows: Set $\tau_I = \bigwedge_{i \in I} \tau_i$ for any $\phi \neq I \subseteq \overline{n}$. If I_i is the support of the i^{th} column of E, define $\tau E = (\sigma_1, \dots, \sigma_n)$ where $\sigma_i = \tau_{I_i} \vee \tau_{I_i'}$ and $I_i' = \overline{n} \setminus I_i$.

We will now summarize some of the results concerning the groups $G[\tau_1, \dots, \tau_n]$ in terms of τ and our operation τE. Two abelian groups G and H are called quasi-isomorphic if each is isomorphic to a subgroup of finite index in the other, in which case we write $G \sim H$.

THEOREM 6. *Let $\tau = (\tau_1, \dots, \tau_n)$ and $\sigma = (\sigma_1, \dots, \sigma_n)$ with $\tau_i, \sigma_j, \in T$ for all i, j. Furthermore, assume that $\tau \nleq \tau_I \vee \tau_{I'}$ for any proper $I \subset \overline{n}$ except $I = \{i\}$ or $\{i\}'$, and $\sigma_j \nleq \sigma_J \vee \sigma_{J'}$, for any proper $J \subset \overline{n}$ except, $J = \{j\}$ or $\{j\}'$. Let $G = G[\tau_1, \dots, \tau_n]$ and $H = G[\sigma_1, \dots, \sigma_n]$.*

(1) [FM] *$G \sim H$ if and only if $\tau E \geqslant \sigma$, and $\sigma F \geqslant \tau$ for some admissible matrices E and F*

(2) [GM] *$G \sim H$ if and only if $\tau E \geqslant \sigma$ and $\sigma F \geqslant \tau$ for some matrices E and F which are admissible mod 2. In this case, if we choose $E \in \mathcal{E}$, then $F = E^{-1}$ works.*

Given $\tau = (\tau_1, \dots, \tau_n)$ and $G = G[\tau_1, \dots, \tau_n]$, we will say that τ is strongly indecomposable if $\tau_i \nleqslant \tau_I \vee \tau_{I'}$ for all $0 \neq I \subseteq \overline{n}$ except $I = \{i\}$ or $\{i\}'$ for each i. Following [FM], τ will be called regular if $\tau_i = \tau_i \vee \vee \bigwedge_{j \neq i} \tau_j$ for each i, so that $\tau_i = \tau_I \vee \tau_{I'}$ when $I = \{i\}$ or $\{i\}'$. Assuming that τ is regular and strongly indecomposable, they say that $\sigma = = (\sigma_1, \dots, \sigma_n)$ is a representation type of G if σ is regular, strongly indecomposable, they say that $\sigma = (\sigma_1, \dots, \sigma_n)$ is a representation type of G if σ is regular, strongly indecomposable, and $G[\tau_1, \dots, \tau_n] \sim \sim G[\sigma_1, \dots, \sigma_n]$. By Theorem 6(2), and a mild computation, we may replace this last condition with the condition that $\tau E = \sigma$ and $\sigma F = \tau$ for two admissible mod 2 matrices E and F.

Two representation types $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\gamma = (\gamma_1, \dots, \gamma_n)$ are called equivalent if $\sigma = (\gamma_{f(1)}, \dots, \gamma_{f(n)})$ for some f in the permutation group S_n. Fuchs and Metelli ask for an upper bound on the number of nonequivalent representation types of $G[\tau_1, \dots, \tau_n]$ given $\tau = = (\tau_1, \dots, \tau_n)$, in terms on n (problem 3 in [FM]).

THEOREM 7. *Let $\tau = (\tau_1, \dots, \tau_n)$ be strongly indecomposable and regular and let $G = G[\tau_1, \dots, \tau_n]$. There are at most* $\prod_{i=0}^{n-2} (2^{n-1} - 2^i)/n!$ *nonequivalent representation types of G.*

PROOF. Let $\mathcal{R}_\tau$ denote the collection of representation types of G. If $\sigma \in \mathcal{R}_\tau$ then $\sigma = \tau E$ for some admissible mod 2 matrix E. If I is the support of the i^{th} column of E and E' is formed by complementing the i^{th} column of E, then the support of the i^{th} column of E' is I', and for $\delta = \tau E'$, and for $\delta = \tau E'$, $\delta_i = \tau_{I'} \vee \tau_{(I')'} = \tau_I \vee \tau_{I'} = \sigma_i$. Therefore we may assume that $E \in \mathcal{F}$, and theorem 5(i) implies that $\mathcal{R}_\tau$ has at most $\prod_{i=0}^{n-2} (2^{n-1} - 2^i)$ members.

Let $\mathcal{P} \subseteq \mathcal{E}$ be the collection of all $n \times n$ permutation matrices. The assignment of $f \in S_n$ to $P_f \in \mathcal{P}$ whose i, j^{th} entry is 1 if and only $f(j) = i$, is a group isomorphism. We will show that $\mathcal{P}$ acts on $\mathcal{R}_\tau$.

If $\sigma \in \mathcal{R}_\tau$, then $\sigma = \tau E$ for some $E \in \mathcal{E}$. Set $\delta = \tau(EP)$ and $\mu = \sigma P$ for $P = P_f \in \mathcal{P}$. For each j, since σ is regular, $\mu_j = \sigma_{f(j)} \vee \sigma_{\{f(j)\}'} = \sigma_i \vee \vee \bigwedge_{k \neq i} \sigma_k = \sigma_i$ where $f(j) = i$. But if the i^{th} column of E is I_i, then

$\delta_j = \tau_{I_i} \vee \tau_{I_i'} = \sigma_i$, so $\delta = \sigma P = (\sigma_{f(1)}, \ldots, \sigma_{f(n)})$. Note that δ is strongly indecomposable and regular, and that $\delta P^{-1} = \sigma$.

Now suppose that $\tau = \sigma F$ for some $F \in \mathcal{E}$. We must show that $\delta(P^{-1}F) = \tau = (\delta P^{-1})F$. Let $\rho = \delta(P^{-1}F)$ and suppose that the support of the k^{th} column if F is J_k. Now P^{-1} has a 1 in the i, j^{th} entry if and only if $f^{-1}(j) = i$, so the support of the k^{th} column of $P^{-1}F$ is $\{i \mid i = f^{-1}(j)$ for some $j \in J_k\} = f^{-1}(J_k)$. Hence $\rho_k = \delta_{f^{-1}(J_k)} \vee \delta_{f^{-1}(J_k)'}$. Also, $\tau = (\delta P^{-1})F = (\delta_{f^{-1}(1)}, \ldots, \delta_{f^{-1}(n)})F$ has $\tau_k = \bigwedge_{i \in J_k} \delta_{f^{-1}(i)} \vee \bigwedge_{i \in J_k'} \delta_{f^{-1}(i)} =$ $= \delta_{f^{-1}(J_k)} \vee \delta_{f^{-1}(J_k)'} = \rho_k$. Thus, if $\sigma P = \delta$, then $\delta(P^{-1}F) = \tau$ and $\tau(EP) = \delta$ so that $\delta \in \mathcal{R}_\tau$. If $P = P_f$ and $Q = P_g$ then mimicking the computation given above, we can show that $(\sigma P)Q = \sigma(PQ) = (\sigma_{gf(1)}, \ldots, \sigma_{gf(n)})$ so that $\mathcal{P}$ acts on $\mathcal{R}_\tau$.

If $\sigma \in \mathcal{R}_\tau$ with $\sigma_i \leq \sigma_j$, and $i \neq j$, then $\sigma_i \leq \sigma_j \vee \bigwedge_{k \neq j} \sigma_k = \sigma_{\{j\}} \vee \sigma_{\{j\}'}$, which contradicts the strong indecomposability of σ. Therefore, $\sigma P = \sigma$ for $P \in \mathcal{P}$ if and only if P is the identity matrix. Since $\mathcal{P}$ acts on $\mathcal{R}_\tau$ and the orbit of σ is the equivalent class of σ which contains $n!$ representation types, there are $|\mathcal{R}_\tau|/n!$ inequivalent representation types. ∎

One could show that $\prod_{i=0}^{n-2}(2^{n-1} - 2^i)/n!$ is an integer by looking at the representation $\mathcal{P}_0$ of $\mathcal{P}$ in $\mathcal{F}$. Then show that $\mathcal{P}_0$ acts on $\mathcal{F}$. Clearly this bound is achieved if and only if τE is a representation type of τ for any $E \in \mathcal{E}$, which is an intrinsic property of T and does not depend, in general, solely on n. Of course when $n = 3$, $\prod_{i=0}^{1}(2^2 - 2^i)/6 = 1$ so the bound is tight in this case, regardless of T.

EXAMPLE 8. Let $\tau_1 = \{1, 2, 3\}$, $\tau_2 = \{2, 3, 4\}$, $\tau_3 = \{1, 5, 6\}$ and $\tau_4 = \{4, 5, 6\}$ in $T = 2^{\bar{6}}$ the power set of $\bar{6}$. It is easy to see that $\tau =$ $= (\tau_1, \tau_2, \tau_3, \tau_4)$ is regular and strongly indecomposable. However

$$\tau_{\{2,3\}} \vee \tau_{\{1,4\}} = (\{2, 3, 4\}) \cap \{1, 5, 6\}) \cup (\{1, 2, 3\} \cap \{4, 5, 6\}) = \emptyset$$

while $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ is the column of an admissible mod 2 matrix $E \in \mathcal{F}$. For example, $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. But $\tau E_1 = \sigma$ cannot be a representation type of τ since $\sigma_2 \leqslant \sigma_i$ for all i so σ cannot be strongly indecomposable. In this case, there are less than $\prod_{i=0}^{2}(2^3 - 2^i)/24 = 7$ representation types of σ.

Three are 7 pertinent matrices from $\mathcal{F}$: $E_0 =$ identity,

$$E_1, E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

These are the matrices of concern because no complementing and/or interchanging of columns will transform one into the other. Set $\tau_5 = \{1, 4\}$ and $\tau_6 = \{2, 3, 5, 6\}$. Of the vectors $\tau E_i, i = 0, \dots, 6$, only $\tau E_0 = \tau$, $\sigma = \tau E_2 = (\tau_6, \tau_5, \tau_1, \tau_4)$ and $\gamma = \tau E_4 = (\tau_5, \tau_6, \tau_2, \tau_3)$ are representation types of τ. One easily checks that σ and γ are strongly indecomposable and regular, and that

$$\sigma \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \tau = \gamma \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

so that there are 3 representation types of τ. ■

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