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Some Results on Weak Solutions to a Class of Singular Hyperbolic Variational Inequalities.

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SUMMARY - In the natural framework of suitable weigthted spaces, some existence and uniqueness results are proved for weak solutions to a class of singular hyperbolic variational inequalities.

SUNTO - Si dimostrano alcuni risultati di esistenza ed unicità di soluzioni deboli per una classe di disequazioni variazionali iperboliche singolari, nell'ambito naturale di opportuni spazi con peso.

1. Introduction.

The subject of evolution variational inequalities is important in several applications from Mechanics and Physics (see, e.g., the book by Duvant and Lions [4]). The Cauchy problem was investigated for various kinds of such inequalities (in particular, «parabolic» inequalities with unilateral constraints concerning the unknown function or the time derivative of the unknown function; «hyperbolic» inequalities

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with unilateral constraints concerning the time derivative of the unknown function). Several existence and uniqueness results *for strong or weak solutions* to these problems are well known: see, e.g., Lions [6], and in particular, Brezis [3].

During the last years, the inequalities above mentioned were extensively studied also in the presence of singularities or degeneracies with respect to the time variable. Let us refer to: Bernardi and Pozzi [1] (*strong and weak solutions* to singular or degenerate «parabolic» inequalities with unilateral constraints concerning the unknown function); [7] (*strong and weak solutions* to singular or degenerate «parabolic» inequalities with unilateral constraints concerning the time derivative of the unknown function). Moreover, Bernardi, Luterotti and Pozzi [2] obtained, in a framework of suitable weighted spaces, some existence and uniqueness results *for strong solutions* to a class of «hyperbolic» inequalities with unilateral constraints concerning the time derivative of the unknown function. They investigated, precisely, evolution variational of the form

$$u'(t) \in \mathcal{K}, \qquad \text{for a.e. } t \in]0, T[; \tag{1.1}$$

$$(t^2 u''(t) + atu'(t) + t^{2\alpha} Au(t) - f(t), v - u'(t)) \geq 0, \tag{1.2}$$

$$\forall v \in \mathcal{K}, \quad \text{for a.e. } t \in]0, T[,$$

where $0 < T < +\infty$; a and α are given arbitrary real numbers; $V \subseteq H \equiv \equiv H^* \subseteq V^*$ is the standard Hilbert triplet; $(\cdot, \cdot)$ denotes the duality pairing between V^* and V; $\mathcal{K}$ is a closed convek subset of V, with $0 \in \mathcal{K}$; $A \in \in \mathcal{L}(V, V^*)$, and A is a symmetic and V-coercive operator; $f(t)$ is some given «suitably regular» V^*-valued function on $]0, T[$. As we already said, in [2] they dealt with *strong solutions* $u(t)$ of (1.1)-(1-2); i.e. they asked for some V-valued $u(t)$ on $]0, T[$, with a V-valued $u'(t)$, and with a V^*-valued $u''(t)$. I must be noted that (1.2) contains, in particular, the inequalities connected «in a natural way» with the operators of Euler-Poisson-Darboux ($\alpha = 1$), of Euler-Fuchs ($\alpha = 0$), and of Tricomi ($\alpha = 3/2$ and $a = 0$). Thus, the «abstract» results in [2] apply, in particular, to such important special cases.

Now, considering the results in [1] and [7] (*for strong and weak solutions*), a natural problem is (to define and) to study weak solutions to (1.1)-(1.2). In particular, taking in (1.2) some $f(t)$ «less regular» than in [2], what can be said about existence and uniqueness of some $u(t)$ satisfying (1.1)-(1.2) in a suitable weak sense?

The aim of this paper is to study such problem. First of all, (1.1)-(1.2) is reformulated in a suitable weak form (see (3.1)-(3.2) below). Then, we prove existence and uniqueness results for (weak) solutions to (3.1)-(3.2), in natural framework of weighted spaces.

As far as existence is concerned, such results are proved by considering, as a starting point, the existence results for strong solutions given in [2]; then, the main tool of the proof is a procedure of extension by continuity.

As far as the uniqueness is concerned, such results are proved, by suitably modifying a method given by Brezis [3] (chap. 3, Theor. III.7 and Lemma III.3).

After a subsection devoted to notation and assumptions, we recall, in Section 2, the strong formulation of the problem, the functional setting, and the main results of [2]. In Section 3 we start by presenting our weak formulation ((3.1)-(3.2)); then we prove our existence results. In Section 4 we prove our uniqueness result. Various comments and remarks are given throughout the paper.

2. Notation and assumptions. Preliminary results.

2.1. Let T, p, v be given with: $0 < T < +\infty$, $1 \leq p \leq +\infty$, $\nu \in \mathbb{R}$; let X be a Banach space. Define

$$L^p_\nu(X) = \{w(t) \,|\, t^\nu w(t) \in L^p(0, T; X)\}, \tag{2.1}$$

which is a Banach space with respect to its natural norm.

(Clearly, if X is a Hilbert space, $L^p_\nu(X)$ is a Hilbert space too.)

It will be useful, for the sequel, to adopt the following notation

$$\begin{cases} z(t) \in {}_0C^0(X) \text{ means that,} \\ z(t) \in C^0([0, T]; X), \text{ and vanishes at } t = 0. \end{cases} \tag{2.2}$$

Let now

$$V \subseteq H \equiv H^* \subseteq V^*, \qquad \text{with } V \text{ separable}, \tag{2.3}$$

be the standard *real* Hilbert triplet; $(.,.)$ denotes both the scalar product in H and the duality pairing between V^* and V. $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ denote respectively the norms in V, H and V^*. Let moreover

$$\mathcal{K} \text{ be a closed convex subset of } V, \text{ with } 0 \in \mathcal{K}. \tag{2.4}$$

Let now A be an operator such that:

$$\begin{cases} A \in \mathcal{L}(V, V^*);\ (Au, v) = (Av, u), & \forall u, v \in V; \\ \exists c > 0 \text{ such that } (Av, v) \geq c\|v\|^2, & \forall v \in V. \end{cases} \tag{2.5}$$

2.2. Let (2.3), (2.4), (2.5) hold. Let T, a, and α be given, with

$$0 < T < +\infty\,, \qquad a \in \mathbb{R}\,, \qquad \alpha \in \mathbb{R}\,. \tag{2.6}$$

The following problem was investigated in [2]. Given some V^*-valued $f(t)$, find a V-valued $u(t)$, with a V-valued $u'(t)$, and a V^*-valued $u''(t)$, such that

$$u'(t) \in \mathcal{K}\,, \qquad \text{for a.e. } t \in]0, T[\,; \tag{2.7}$$

$$(t^2 u''(t) + atu'(t) + t^{2\alpha} Au(t) - f(t), v - u'(t)) \geq 0\,, \qquad \forall v \in \mathcal{K}\,, \quad \text{for a.e. } t \in]0, T[\,. \tag{2.8}$$

To recall the main results of [2], we have firstly to introduce an appropriate functional setting.

First of all, we will consider a given $f(t)$ with the following properties:

$$f(t) = f_1(t) + f_2(t)\,, \qquad \text{where, for some } \mu \in \mathbb{R}: \tag{2.9}$$

$$\begin{cases} f_1(t) \in L^1_\mu(H)\,, \qquad f_1'(t) \in L^1_{\mu+1}(H)\,, \\ \text{(so that (see Lemma 2.1 in [2]):} \\ t^{\mu+1} f_1(t) \in {}_0C^0(H)\,, \qquad f_1(t) \in L^2_{\mu+1/2}(H))\,; \end{cases} \tag{2.10}$$

$$\begin{cases} f_2(t) \in L^1_{\mu-\alpha}(V^*)\,, \qquad f_2'(t) \in L^1_{\mu-\alpha+1}(V^*)\,, \\ f_2''(t) \in L^1_{\mu-\alpha+2}(V^*) \qquad \text{(so that (see Lemma 2.1 in [2]):} \\ t^{\mu-\alpha+1} f_2(t) \in {}_0C^0(V^*)\,, \qquad t^{\mu-\alpha+2} f_2'(t) \in {}_0C^0(V^*)\,, \\ f_2(t) \in L^2_{\mu-\alpha+1/2}(V^*)\,, \qquad f_2'(t) \in L^2_{\mu-\alpha+3/2}(V^*))\,. \end{cases} \tag{2.11}$$

We shall deal with solutions $u(t)$ of (2.7)-(2.8), such that, for some $\mu \in \mathbb{R}$:

$$\begin{cases} u(t) \in L^2_{\mu+\alpha+1/2}(V)\,, \qquad u'(t) \in L^2_{\mu+\alpha+3/2}(V)\,, \\ u'(t) \in L^\infty_{\mu+\alpha+2}(V)\,, \qquad \text{(so that (see Lemma 2.2 in [2]):} \\ t^{\mu+\alpha+1} u(t) \in {}_0C^0(V))\,; \end{cases} \tag{2.12}$$

$$\begin{cases} u'(t) \in L^2_{\mu+3/2}(H)\,, \qquad u''(t) \in L^2_{\mu+5/2}(H)\,, \\ u''(t) \in L^\infty_{\mu+3}(H)\,, \qquad \text{(so that (see Lemma 2.2 in [2]):} \\ t^{\mu+2} u'(t) \in {}_0C^0(H))\,; \end{cases} \tag{2.13}$$

In Section 3 and 4 we shall also deal with a class $\mathcal{W}$ of suitable «test functions». Take some $\mu \in \mathbb{R}$ and cosider the functions $v(t)$ such that:

(2.14) $$v(t) \in L^2_{\mu+\alpha+3/2}(V) \cap L^\infty_{\mu+\alpha+2}(V) \cap L^2_{\mu+3/2}(H) \cap L^\infty_{\mu+2}(H),$$

$$\text{with } v'(t) \in L^2_{\mu+5/2}(H),$$

(2.15) $$v(t) \in \mathcal{K} \quad \text{for a.e. } t \in]0, T[.$$

Then, we define

(2.16) $$\mathcal{W} = \{v(t) \mid v(t) \text{ satisfies (2.14)-(2.15)}\}.$$

2.3. Now, we define

(2.17) $$\mu_0 \equiv \mu_0(a, \alpha) \equiv \min[-\alpha - 1; a - 2; a - 2 - 2\alpha];$$

(2.18) $$\mu_0^* \equiv \mu_0^*(a, \alpha) \equiv \min[-\alpha - 1; a - 2].$$

The following uniqueness result was proved in [2].

PROPOSITION 2.1. Let (2.3), (2.4), (2.5), (2.6) hold. Fix any $\mu \leq \mu_0^*$ (see (2.18)), and take any $f(t) \equiv f_1(t) + f_2(t)$, with $f_1(t) \in L^1_\mu(H)$ and $f_2(t) \in L^1_{\mu-\alpha}(V^*)$. Let $u(t) = u_1(t)$ and $u(t) = u_2(t)$ satisfy (2.12), (2.13), (2.7) and (2.8) (with the same previous $f(t)$). Then $u_1(t) = u_2(t)$, $\forall t \in [0, T]$.

A first existence result (see Theorem 2.1 in [2]) is the following.

PROPOSITION 2.2. Let (2.3), (2.4), (2.5), (2.6) hold. Fix any $\mu < \mu_0$ (see (2.17)). Then, for every $f(t)$ satisfying (2.9), (2.10), (2.11) there exists a (unique) $u(t)$ satisfying (2.12), (2.13), solution of (2.7)-(2.8).

The main tool to prove Proposition 2.2 was the penalization method (see e.g., in general, Lions [5]; in particular, for Prop. 2.2, see subsection 3.2 in [2]).

REMARK 2.1. Since $\mu_0 \leq \mu_0^*$ (and precisely $\mu_0 < \mu_0^*$ if and only if $\alpha > \max[0, a - 1]$), and since we have considered a larger setting for $f(t)$, the uniqueness result of Proposition 2.1 holds under more general hypotheses than the ones assumed for the existence result in Proposition 2.2.

REMARK 2.2. The existence result in Proposition 2.2 was improved in some cases (see Proposition 4.1 in [2]). Precisely, by taking the further assumption on the convex $\mathcal{K}$:

$$(2.19) \qquad \exists \text{ a closed convex subset } \mathcal{K}_1 \text{ of } H, \text{ such that } \mathcal{K} = \mathcal{K}_1 \cap V,$$

one has the same conclusions of Proposition 2.2, when $\mu < \tilde{\mu}_0$, where:

$$(2.20) \qquad \tilde{\mu}_0 \equiv \tilde{\mu}_0 \equiv \max_{r \geq 1} \{\min[-\alpha - 1; a - 2; a - 1 - r; -3\alpha - 2 + r]\}.$$

Remark that $\tilde{\mu}_0(a, \alpha) \geq \mu_0(a, \alpha)$, $\forall a, \alpha \in \mathbb{R}$ and, in particular, $\tilde{\mu}_0(a, \alpha) > \mu_0(a, \alpha)$ if and only if $\alpha > \max[0, a - 1]$.

Further details about comparison of the «threshold weights» values, $\mu_0(a, \alpha)$, $\mu_0^*(a, \alpha)$, $\tilde{\mu}_0(a, \alpha)$, are given in Proposition 4.1 and Remark 4.2 in [2].

3. Existence results.

3.1. Under the assumption of proposition 2.2 (and in particular, with $f(t)$ satisfying (2.9), (2.10), (2.11)), we have a unique (strong) solution $u(t)$ of (2.7)-(2.8) satisfying (2.12), (2.13). Such a $u(t)$ also satisfies the following conditions (3.1)-(3.2). Firstly, it results clearly from (2.7) that:

$$(3.1) \qquad -\int_0^T u(t)\phi'(t)\,dt \in \mathcal{K} \qquad \forall \phi(t) \in C_0^\infty(]0, T[),$$

$$\text{s.t. } \phi(t) \geq 0 \text{ in }]0, T[\text{ and } \int_0^T \phi(t)\,dt = 1;$$

(also see: the Remark 2.8 in Lions [6]; Brezis [3], chap. 3; [7], section 5).

Moreover, it also results that:

$$(3.2) \qquad \int_0^T t^{2\mu+1}\{(a - 2\mu - 4)(t^2 u'(t), v(t)) + (\mu - a + 2)t^2 |u'(t)|^2 -$$

$$-t^3(u'(t), v'(t)) + t^{2\alpha+1}(Au(t), v(t)) + (\mu + \alpha + 1)t^{2\alpha}(Au(t), u(t)) -$$

$$-(tf_1(t), v(t) - u'(t)) - (tf_2(t), v(t)) - (2\mu + 2)((f_2(t), u(t)) -$$

$$-(tf_2'(t), u(t))\}\,dt + T^{2\mu+4}(u'(T), v(T)) - \frac{1}{2}T^{2\mu+4}|u'(T)|^2 -$$

$$-\frac{1}{2}T^{2\mu+2\alpha+2}(Au(T), u(T)) +$$

$$+T^{2\mu+2}(f_2(T), u(T)) \geqslant 0, \qquad \forall v(t) \in \mathcal{W} \text{ (see (2.16))}.$$

The inequality (3.2) can be obtained by using as a starting point (2.8), considering $v = v(t) \in \mathcal{W}$, multiplying both sides of the inequality by $t^{2\mu+2}$, and integrating by parts. Since $u(t)$, solution of the problem (2.7)-(2.8), also satisfies (3.1)-(3.2), we are led to consider (3.1)-(3.2) as a «weak» formulation of the problem (2.7)-(2.8).

3.2. Now, considering in (3.2) sone $f(t)$ «less regular» than $f(t)$ in (2.9), (2.10), (2.11), we want to see if there exists a corresponding $u(t)$, solution of (3.1)-(3.2) (possibly less regular than $u(t)$, solution of (2.7)-(2.8) obtained in Proposition 2.2).
The main tool to obtain the existence result is the following

THEOREM 3.1. Let (2.3), (2.4), (2.5), (2.6) hold. Let $\mu < \mu_0$. Let $f(t)$ satisfy (2.9), (2.10), (2.11). We condider the mapping $\mathfrak{J}$: $f \to u$, where $u(t)$ is the unique (strong) solution of (2.7)-(2.8) corresponding to $f(t)$ and satisfying (2.12), (2.13) (thanks to Proposition 2.2). Then, the mapping $\mathfrak{J}$ is the Lipschitz continuous from $\{f(t) \mid f(t)$ satisfying (2.9), (2.10), (2.11)$\}$, endowed with the natural norm of $\{f(t) \mid f(t) = f_1(t) + f_2(t);\ f_1(t) \in L^1_\mu(H),\ f_2(t) \in L^1_{\mu-\alpha}(V^*),\ f_2'(t) \in L^1_{\mu-\alpha+1}(V^*)\}$, to $\{u(t) \mid u(t)$ satisfying (2.12), (2.13)$\}$, endowed with the natural norm of $\{u(t) \mid u(t) \in L^2_{\mu+\alpha+1/2}(V),\ u(t) \in L^\infty_{\mu+\alpha+1}(V),\ u'(t) \in L^2_{\mu+3/2}(H),\ u'(t) \in L^\infty_{\mu+2}(H)\}$.

PROOF. Let $f(t)$, $h(t)$ satisfy (2.9), (2.10), (2.11). Let $u(t)$ (respectively $w(t)$) be the strong solution corresponding to $f(t)$ (respectively to $h(t)$). By taking $v = w'(t)$ in the inequality (2.8) relative to u, and $v = u'(t)$ in the inequality relative to w, and by adding the resulting inequalities, we obtain

$$(3.3) \qquad (t^2(u''(t) - w''(t)) + at(u'(t) - w'(t)) + t^{2\alpha}A(u(t) - w(t)) -$$

$$-(f(t) - h(t), u'(t) - w'(t)) \leqslant 0, \qquad \text{for a.e. } t \in [0, T].$$

Now, multiplying both sides of (3.3) by $t^{2\mu+2}$, integranting from 0 to t ($t \leqslant T$) and denoting by ε, δ, η, ϑ some positive numbers, thanks also to

(2.5), we obtain

$$(3.4)\qquad \frac{1}{2}t^{2\mu+4}\,|u'(t)-w'(t)|^2-(\mu+2-a)\int_0^t \tau^{2\mu+3}\,|u'(t)-w'(t)|^2\,d\tau+$$

$$+\frac{c}{2}t^{2\mu+2\alpha+2}\|u(t)-w(t)\|^2-c(\mu+\alpha+1)\int_0^t \tau^{2\mu+2\alpha+1}\|u(\tau)-w(\tau)\|^2\,d\tau\leqslant$$

$$\leqslant\frac{1}{\varepsilon}\left(\int_0^T \tau^{\mu}\,|f_1(\tau)-h_1(\tau)|\,d\tau\right)^2+\frac{1}{\delta}\sup_{0\leqslant\tau\leqslant t}\,[\tau^{2\mu-2\alpha+2}\|f_2(\tau)-h_2(\tau)\|_*^2]+$$

$$+\frac{|2\mu+2|}{\eta}\left(\int_0^T \tau^{\mu-\alpha}\|f_2(\tau)-h_2(\tau)\|_*\,d\tau\right)^2+$$

$$+\frac{1}{\vartheta}\left(\int_0^T \tau^{\mu-\alpha+1}\|f_2'(\tau)-h_2'(\tau)\|_*\,d\tau\right)^2+\varepsilon\sup_{0\leqslant\tau\leqslant t}\,[\tau^{2\mu+4}\,|u'(\tau)-w'(\tau)|^2]+$$

$$+[\delta+\eta|2\mu+2|+\vartheta]\sup_{0\leqslant\tau\leqslant t}\,[\tau^{2\mu+2\alpha+2}\|u(\tau)-w(\tau)\|^2],\qquad 0\leqslant t\leqslant T.$$

Taking ε, δ, η, ϑ sufficiently small, (3.4) implies that there exists a constant $c(a, \alpha, \mu)$ such that:

$$(3.5)\qquad \|u'(t)-w'(t)\|_{L^\infty_{\mu+2}(H)}+\|u'(t)-w'(t)\|_{L^2_{\mu+3/2}(H)}+$$

$$+\|u(t)-w(t)\|_{L^\infty_{\mu+\alpha+1}(V)}+\|u(t)-w(t)\|_{L^2_{\mu+\alpha+1/2}(V)}\leqslant$$

$$\leqslant c(a,\alpha,\mu)[\|f_1(t)-h_1(t)\|_{L^1_\mu(H)}+\|f_2(t)-h_2(t)\|_{L^1_{\mu-\alpha}(V^*)}+$$

$$+\|f_2'(t)-h_2'(t)\|_{L^1_{\mu-\alpha+1}(V^*)}].$$

Thanks to Theorem 3.1, we can extend by continuity the mapping $\mathfrak{J}$: $f\to u$ to functions $f(t)\in\{f(t)\,|\,f(t)=f_1(t)+f_2(t);\,f_1(t)\in L^1_\mu(H),\,f_2(t)\in L^1_{\mu-\alpha}(V^*),\ f_2'(t)\in L^1_{\mu-\alpha+1}(V^*)\}$. The corresponding «solution» $u(t)$ satisfies: $t^{\mu+\alpha+1}u(t)\in{}_0C^0(V)$, $u(t)\in L^2_{\mu+\alpha+1/2}(V)$, $u'(t)\in L^2_{\mu+3/2}(H)$, $t^{\mu+2}u'(t)\in{}_0C^0(H)$.

Now, it can be obtained, by using some standard arguments, that such $u(t)$ satisfies (3.1)-(3.2).

Hence, we have proved the following

THEOREM 3.2. Let (2.3), (2.4), (2.5), (2.6) hold. Let $\mu < \mu_0$. Let $f(t)$ satisfy (2.9), with $f_1(t) \in L^1_\mu(H)$, $f_2(t) \in L^1_{\mu-\alpha}(V^*)$, $f_2'(t) \in L^1_{\mu-\alpha+1}(V^*)$. Then, there exists $u(t)$ with $u(t) \in L^2_{\mu+\alpha+1/2}(V)$, $t^{\mu+\alpha+1}u(t) \in_0 C^0(V)$, $u'(t) \in L^2_{\mu+3/2}(H)$, $t^{\mu+2}u'(t) \in_0 C^0(H)$, solution of (2.7)-(2.8) in a «weak sense», i.e. satisfying (3.1)-(3.2).

REMARK 3.1. Clearly, the main reasons for choosing (3.1)-(3.2) as a weak formulation of (2.7)-(2.8) are: $u'(t)$ is, in general, no longer a V-valued function; moreover, in general, no information on $u''(t)$ can be obtained.

REMARK 3.2. (3.2) in only *a* possible weak formulation of (2.8); other weak formulations, in an integral form, can be consider.

REMARK 3.3. We used in the proof, as a starting point, the «strong» existence result of Proposition 2.2. When (2.19) also holds, we could easily obtain an improvement (in the sense of Remark 2.2) of the «weak» existence result of Theorem 3.2, by using the «strong» existence result, recalled in Remark 2.2.

4. The uniqueness result.

4.1. To prove our uniqueness result we need the following auxiliary Lemma.

LEMMA 4.1. Let (2.3), (2.4), (2.5), and (2.6) hold. Let some $\mu \in \mathbb{R}$ be given. Let $f(t)$ satisfy (2.9), with $f_1(t) \in L^1_\mu(H)$, $f_2(t) \in L^1_{\mu-\alpha}(V^*)$, $f_2'(t) \in L^1_{\mu-\alpha+1}(V^*)$. Let $u(t)$ satisfy (3.1), with $u(t) \in L^2_{\mu+\alpha+1/2}(V)$, $t^{\mu+\alpha+1}u(t) \in_0 C^0(V)$, $u'(t) \in L^2_{\mu+3/2}(H)$, $t^{\mu+2}u'(t) \in_0 C^0(H)$. Then, recalling (2.16), we have that

$$(4.1) \quad \operatorname{Inf}\Bigg\{(a - 2\mu - 4)\int_0^T t^{2\mu+3}(u'(t), v(t))\,dt -$$

$$-\int_0^T t^{2\mu+4}(u'(t), v'(t))\,dt + \int_0^T {}^{2\mu+2\alpha+2}(Au(t), v(t))\,dt -$$

$$-\int_0^T t^{2\mu+2}(f_1(t), v(t) - u'(t))\,dt - \int_0^T t^{2\mu+2}(f_2(t), v(t))\,dt +$$

$$+T^{2\mu+4}(u'(T), v(T))\,|\,v(t)\in \mathcal{W}\Big\} + (\mu - a + 2)\int_0^T t^{2\mu+3}\,|u'(t)|^2\,dt +$$

$$+(\mu+\alpha-1)\int_0^T t^{2\mu+2\alpha+2}(Au(t), u(t))\,dt - (2\mu+2)\int_0^T t^{2\mu+1}(f_2(t), u(t))\,dt -$$

$$-\int_0^T t^{2\mu+1}(f_2'(t), u(t))\,dt \leqslant \frac{1}{2}T^{2\mu+4}\,|u'(T)|^2 +$$

$$+\frac{1}{2}T^{2\mu+2\alpha+2}(Au(T), u(T)) - T^{2\mu+2}(f_2(T), u(T))\,.$$

Proof. We adapt here a method by Brezis (see the proof of Lemma III.3, chap. 3 in [3]), with several changes due to: the fact that the problem is singular or degenerate, the presence (more generally than in [3]) of a V^*-valued $f(t)$, and the weighted spaces framework.

Firstly, take $\varepsilon \in \mathbb{R}^+$ and $m \in \mathbb{N}$ with $m > 2/T$; then, define:

$$v_{\varepsilon m}(t) = \sigma_m(t)[\check{\rho}_\varepsilon * \rho_\varepsilon * (\sigma_m u')](t), \qquad \forall t \in [0, T], \tag{4.2}$$

where $\sigma_m(t) \in C_0^\infty(]0, T])$, with $\sigma_m(t) = 1$, $t \in [1/m, T - 1/m]$; $\rho_\varepsilon(t) \in C_0^\infty(\mathbb{R})$, with $\rho_\varepsilon(t) \geqslant 0$, $\operatorname{supp}(\rho_\varepsilon(t)) \subset]-\varepsilon, \varepsilon[$, $\int_{-\infty}^{+\infty} \rho_\varepsilon(t)\,dt = 1$; $\check{\rho}_\varepsilon(t) \equiv \rho_\varepsilon(-t)$.

We have that the functions $v_{\varepsilon m}(t)$ belong to $\mathcal{W}$. In fact, $v_{\varepsilon m}(t) \in C_0^\infty(]0, T[, V)$, thanks to the properties of $\rho_\varepsilon(t)$ and $\sigma_m(t)$; hence $v_{\varepsilon m}(t)$ satisfies (2.14). The proof of (2.15) for $v_{\varepsilon m}(t)$ can be carried out (thanks to the fact that $0 \in \mathcal{K}$), by following the analogous Brezis' proof in Lemma III.3, chap. 3, in [3], concerning $w_\varepsilon(t)$.

Now, we denote the left hand side of (4.1) by I; since the functions $v_{\varepsilon m}(t)$ belong to $\mathcal{W}$, we have

$$I \leqslant (a - 2\mu - 4)\int_0^T t^{2\mu+3}(u'(t), v_{\varepsilon m}(t))\,dt + (\mu - a + 2)\cdot \tag{4.3}$$

$$\cdot\int_0^T t^{2\mu+3}\,|u'(t)|^2\,dt - \int_0^T t^{2\mu+4}(u'(t), v_{\varepsilon m}'(t))\,dt + (\mu + \alpha + 1)\cdot$$

$$\cdot \int_0^T t^{2\mu+2\alpha+2}(Au(t), u(t))\,dt + \int_0^T t^{2\mu+2\alpha+2}(Au(t), v_{\varepsilon m}(t))\,dt -$$

$$- \int_0^T t^{2\mu+2}(f_1(t), v_{\varepsilon m}(t) - u'(t))\,dt - \int_0^T t^{2\mu+2}(f_2(t), v_{\varepsilon m}(t))\,dt -$$

$$-(2\mu+2)\int_0^T t^{2\mu+1}(f_2(t), u(t))\,dt - \int_0^T t^{2\mu+1}(f_2'(t), u(t))\,dt,$$

$$\forall v_{\varepsilon m}(t) \text{ as in } (4.2).$$

We set:

$$I_{1\varepsilon m} = \int_0^T t^{2\mu+3}(u'(t), v_{\varepsilon m}(t))\,dt; \tag{4.4}$$

$$I_{2\varepsilon m} = \int_0^T t^{2\mu+4}(u'(t), v_{\varepsilon m}'(t))\,dt; \tag{4.5}$$

$$I_{3\varepsilon m} = \int_0^T t^{2\mu+2\alpha+2}(Au(t), v_{\varepsilon m}(t))\,dt; \tag{4.6}$$

$$I_{4\varepsilon m} = -\int_0^T t^{2\mu+2}(f_2(t), v_{\varepsilon m}(t))\,dt; \tag{4.7}$$

$$I_{5\varepsilon m} = -\int_0^T t^{2\mu+2}(f_1(t), v_{\varepsilon m}(t))\,dt. \tag{4.8}$$

Since we have that

$$I_{1\varepsilon m} = \int_0^T t^{2\mu+3}(\rho_\varepsilon * \sigma_m u'(t), \rho_\varepsilon * \sigma_m u'(t))\,dt, \tag{4.9}$$

we obtain

$$\lim_{\varepsilon\to 0} I_{1\varepsilon m} = \int_0^T t^{2\mu+3}\,|\sigma_m(t)\,u'(t)|^2\,dt. \tag{4.10}$$

Since we have that

$$(4.11)\quad v'_{\varepsilon m}(t) = \sigma'_m \check{\rho}_\varepsilon * \rho_\varepsilon * \sigma_m u'(t) + \sigma_m \check{\rho}_\varepsilon * \frac{d}{dt}(\rho_\varepsilon * \sigma_m u'(t)),$$

we can write

$$(4.12)\qquad I_{2\varepsilon m} = -\int_0^T t^{2\mu+4}(\rho_\varepsilon * \sigma'_m u'(t), \rho_\varepsilon * \sigma_m u'(t))\,dt -$$

$$-\int_0^T t^{2\mu+4}\left(\rho_\varepsilon * \sigma_m u'(t), \frac{d}{dt}[\rho_\varepsilon * \sigma_m u'(t)]\right)dt\,;$$

hence we obtain

$$(4.13)\qquad \lim_{\varepsilon\to 0} I_{2\varepsilon m} = -\int_0^T t^{2\mu+4}\,|u'(t)|^2 \sigma'_m(t)\,\sigma_m(t)\,dt +$$

$$+(\mu+2)\int_0^T t^{2\mu+3}\,|\sigma_m(t)\,u'(t)|^2\,dt\,.$$

Taking into account that

$$(4.14)\qquad \int_0^T t^{2\mu+2\alpha+2}(A(\rho_\varepsilon * \sigma_m u(t)), \rho_\varepsilon * \sigma_m u'(t))\,dt =$$

$$= \int_0^T t^{2\mu+2\alpha+2}\,\frac{1}{2}\,\frac{d}{dt}(A(\rho_\varepsilon * \sigma_m u(t)), \rho_\varepsilon * \sigma_m u(t))\,dt -$$

$$-\int_0^T t^{2\mu+2\alpha+2}(A(\rho_\varepsilon * \sigma_m u(t)), \rho_\varepsilon * \sigma'_m u(t))\,dt\,,$$

we have

$$(4.15)\qquad \lim_{\varepsilon\to 0} I_{3\varepsilon m} = -(\mu+\alpha+1)\int_0^T t^{2\mu+2\alpha+1}(Au(t), u(t))\,|\sigma_m(t)|^2\,dt -$$

$$-\int_0^T t^{2\mu+2\alpha+2}(Au(t), u(t))\,\sigma'_m(t)\,\sigma_m(t)\,dt\,.$$

Taking into account that

$$(4.16)\qquad -\int_0^T t^{2\mu+2}(\rho_\varepsilon * \sigma_m f_2(t), \rho_\varepsilon * \sigma_m u'(t))\,dt =$$

$$= -\int_0^T t^{2\mu+2}\left(\rho_\varepsilon * \sigma_m f_2(t), \frac{d}{dt}[\rho_\varepsilon * \sigma_m u(t)]\right) dt +$$

$$+ \int_0^T t^{2\mu+2}(\rho_\varepsilon * \sigma_m f_2(t), \rho_\varepsilon * \sigma'_m u(t))\,dt\,,$$

we obtain

$$(4.17)\qquad \lim_{\varepsilon\to 0} I_{4\varepsilon m} = (2\mu + 2)\int_0^T t^{2\mu+1}(f_2(t), u(t))\,|\sigma_m(t)|^2\,dt +$$

$$+2\int_0^T {}^{2\mu+2}(f_2(t), u(t))\,\sigma'_m(t)\,\sigma_m(t)\,dt + \int_0^T t^{2\mu+2}(f'_2(t), u(t))\,|\sigma_m(t)|^2\,dt\,.$$

We also have that

$$(4.18)\qquad \lim_{\varepsilon\to 0} I_{5\varepsilon m} = \int_0^T t^{2\mu+2}(f_1(t), u'(t))\,|\sigma_m(t)|^2\,dt\,.$$

Now, thanks to (4.10), (4.13), (4.15), (4.17), (4.18), passing to the limit with ε in (4.3), we obtain

$$(4.19)\qquad I \leqslant \Bigg\{(a - 2\mu - 4)\int_0^T t^{2\mu+3}\,|u'(t)|^2\,|\sigma_m(t)|^2\,dt +$$

$$+(\mu - a + 2)\int_0^T t^{2\mu+3}\,|u'(t)|^2\,dt + (\mu + 2)\int_0^T t^{2\mu+3}\,|u'(t)|^2\,|\sigma_m(t)|^2\,dt -$$

$$-\int_0^T t^{2\mu+4}\,|u'(t)|^2\,\sigma'_m(t)\,\sigma_m(t)\,dt -$$

$$-(\mu+\alpha+1)\int_0^T t^{2\mu+2\alpha+1}(Au(t),\,u(t))\,(|\sigma_m(t)|^2-1)\,dt-$$

$$-\int_0^T t^{2\mu+2\alpha+2}(Au(t),\,u(t))\,\sigma_m'(t)\,\sigma_m(t)\,dt-$$

$$-\int_0^T t^{2\mu+2}(f_1(t),\,u'(t))\,(|\sigma_m(t)|^2-1)\,dt+$$

$$+2\int_0^T t^{2\mu+2}(f_2(t),\,u(t))\,\sigma_m'(t)\,\sigma_m(t)\,dt+$$

$$+(2\mu+2)\int_0^T t^{2\mu+1}(f_2(t),\,u(t))\,(|\sigma_m(t)|^2-1)\,dt+$$

$$+\int_0^T t^{2\mu+2}(f_2'(t),\,u(t))\,(|\sigma_m(t)|^2-1)\,dt\,.$$

We also remark that:

$$(4.20)\qquad -\int_0^T t^{2\mu+4}\,|u'(t)|^2\,\sigma_m'(t)\,\sigma_m(t)\,dt=$$

$$=\frac{1}{2}\int_0^T \frac{d}{dt}[t^{2\mu+4}\,|u'(t)|^2]\,|\sigma_m(t)|^2\,dt\,,$$

$$(4.21)\qquad -\int_0^T t^{2\mu+2\alpha+2}(Au(t),\,u(t))\,\sigma_m'(t)\,\sigma_m(t)\,dt=$$

$$=\frac{1}{2}\int_0^T \frac{d}{dt}[t^{2\mu+2\alpha+2}(Au(t),\,u(t))]\,|\sigma_m(t)|^2\,dt\,,$$

$$(4.22)\qquad 2\int_0^T t^{2\mu+2}(f_2(t),\,u(t))\,\sigma_m'(t)\,\sigma_m(t)\,dt=$$

$$=-\int_0^T \frac{d}{dt}[t^{2\mu+2}(f_2(t),\,u(t))]\,|\sigma_m(t)|^2\,dt\,.$$

Now, we pass to the limit with m in (4.19); thanks also to the properties of $u(t)$, $u'(t)$, and $f_2(t)$, we obtain (4.1).

4.2. Now, we state and prove our uniqueness result.

THEOREM 4.1. Let (2.3), (2.4), (2.5), (2.6) hold. Let $\mu \leq \mu_0^*$ with also $\mu \neq -\alpha - 1$. Let $f(t)$ satisfy (2.9), with $f_1(t) \in L^1_\mu(H)$, $f_2(t) \in L^1_{\mu-\alpha}(V^*)$, $f_2'(t) \in L^1_{\mu-\alpha+1}(V^*)$. Let $u(t) = u_1(t)$ and $u(t) = u_2(t)$ satisfy (3.1)-(3.2) with $u_i(t) \in L^2_{\mu+\alpha+1/2}(V)$, $t^{\mu+\alpha+1}u_i(t) \in {}_0C^0(V)$, $u_i'(t) \in L^2_{\mu+3/2}(H)$, $t^{\mu+2}u_i'(t) \in {}_0C^0(H)$, $i = 1, 2$. Then $u_1(t) = u_2(t)$ for every $t \in [0, T]$.

PROOF. We put $u_1(t)$ (respectively $u_2(t)$) in the inequality (3.2). We add the two inequalities obtained, and we multiply by $1/2$; defining

$$(4.23)\qquad u(t) = [u_1(t) + u_2(t)]/2 \quad \text{and} \quad a(u(t)) = (Au(t), u(t)),$$

we get

$$(4.24)\qquad (a - 2\mu - 4)\int_0^T t^{2\mu+3}(u'(t), v(t))\,dt +$$

$$+\frac{1}{2}(\mu - a + 2)\int_0^T t^{2\mu+3}[\,|u_1'(t)|^2 + |u_2'(t)|^2]\,dt -$$

$$-\int_0^T t^{2\mu+4}(u'(t), v'(t))\,dt + \int_0^T t^{2\mu+2\alpha+2}(Au(t), v(t))\,dt +$$

$$+\frac{1}{2}(\mu + \alpha + 1)\int_0^T t^{2\mu+2\alpha+1}[a(u_1(t)) + a(u_2(t))]\,dt -$$

$$-\int_0^T t^{2\mu+2}(f_1(t), v(t) - u'(t))\,dt - \int_0^T t^{2\mu+2}(f_2(t), v(t))\,dt -$$

$$-(2\mu + 2)\int_0^T t^{2\mu+1}(f_2(t), u(t))\,dt - \int_0^T t^{2\mu+2}(f_2'(t), u(t))\,dt +$$

$$+t^{2\mu+4}(u'(T), v(T)) - \frac{1}{4}T^{2\mu+4}[|u_1'(T)|^2 + |u_2'(T)|^2 -$$

$$-\frac{1}{4}T^{2\mu+2\alpha+2}[a(u_1(T)) + a(u_2(T))] + T^{2\mu+2}(f_2(T), u(T)) \geqslant 0 \quad \forall v(t) \in \mathcal{W}.$$

Now, it results clearly that:

$$(4.25) \qquad \begin{cases} \dfrac{1}{2}[|u_1'(t)|^2 + |u_2'(t)|^2] = \dfrac{1}{4}|u_1'(t) - u_2'(t)|^2 + |u'(t)|^2, \\ \dfrac{1}{2}[a(u_1(t)) + a(u_2(t))] = \dfrac{1}{4}a(u_1(t) - u_2(t)) + a(u(t)), \\ \hfill \forall t \in [0, T]. \end{cases}$$

Considering (4.25) and rewriting $a(u(t))$ as $(Au(t), u(t))$, we obtain that:

$$(4.26) \qquad -\frac{1}{4}(\mu - a + 2)\int_0^T t^{2\mu+3}|u_1'(t) - u_2'(t)|^2 dt -$$

$$-\frac{1}{4}(\mu + \alpha + 1)\int_0^T t^{2\mu+2\alpha+1}a(u_1(t) - u_2(t))\,dt +$$

$$+\frac{1}{8}T^{2\mu+4}|u_1'(T) - u_2'(T)|^2 + \frac{1}{8}T^{2\mu+2\alpha+2}a(u_1(T) - u_2(T)) \leqslant$$

$$\leqslant (a - 2\mu - 4)\int_0^T t^{2\mu+3}(u'(t), v(t))\,dt + (\mu - a + 2)\int_0^T t^{2\mu+3}|u'(t)|^2 dt -$$

$$-\int_0^T t^{2\mu+4}(u'(t), v'(t))\,dt + \int_0^T t^{2\mu+2\alpha+2}(Au(t), v(t))\,dt +$$

$$+(\mu + \alpha + 1)\int_0^T t^{2\mu+2\alpha+1}(Au(t), u(t))\,dt - \int_0^T t^{2\mu+2}(f_1(t), v(t) - u'(t))\,dt -$$

$$-\int_0^T t^{2\mu+2}(f_2(t), v(t))\,dt - (2\mu + 2)\int_0^T t^{2\mu+1}(f_2(t), u(t))\,dt -$$

$$
-\int_0^T t^{2\mu+2}(f_2'(t), u(t))\,dt + T^{2\mu+4}(u'(T), v(T)) -
$$

$$
-\frac{1}{2}T^{2\mu+4}|u'(T)|^2 - \frac{1}{2}T^{2\mu+2\alpha+2}(Au(T), u(T)) +
$$

$$
+T^{2\mu+2}(f_2(T), u(T)), \qquad \forall v(t) \in \mathcal{W}.
$$

Now, thanks to Lemma 4.1 (see (4.1)), we have that the right hand side of (4.26) is $\leqslant 0$. Since (2.5) holds and since $\mu \leqslant \mu_0^*$ and $\mu \neq -\alpha - 1$ mean $\mu \leqslant a - 2$ and $\mu < \alpha - 1$, we deduce that $u_1(t) = u_2(t)$, for every $t \in [0, T]$.

REMARK 4.1. The paper [7] deals with a class of degenerate parabolic variational inequalities, where the unilateral constraints concern the time derivative of the unknown function. In [7], existence and uniqueness results for strong solutions and existence results for weak solutions were proved. A uniqueness result for weak solutions can also be obtained there, by suitably modifying our proofs of Lemma 4.1 and Theorem 4.1 above. We have to observe that such modifications make those proofs simpler than the ones in the present paper.

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