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L^∞-Estimates for Nonlinear Parabolic Equations with Natural Growth Conditions.

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Let Ω be a bounded domain in $\boldsymbol{R}^N$ of boundary $\partial\Omega$ and for $0 < T < \infty$ let Ω_T denote the cylindrical domain $\Omega \times (0,T]$. Let also

$$\Gamma = (\Omega \times \{0\}) \cup (\partial\Omega \times (0, t])$$

be the parabolic boundary of Ω_T. Assume that the boundary $\partial\Omega$ satisfies the property of positive geometric density, i.e.

there exist $c > 0$ and r_0 such that for each $x_0 \in \partial\Omega$, for every ball $B_r(x_0)$ centred in x_0 and with radius $r \leqslant r_0$, the measure of the intersection between Ω and $B_r(x_0)$ is greater than cr^N.

Consider the boundary value problem

$$\text{(1)} \qquad \begin{cases} u \in C(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)), \\ u_t - \operatorname{Div} a(x, t, u, Du) = b(x, t, u, Du) \text{ in } \Omega_T, \\ u|_\Gamma = f \in L^\infty(\Gamma), \end{cases}$$

where p is a number greater than 1 and the p.d.e. satisfies the structure conditions

$$\text{(2)} \qquad a(x, t, u, Du) \cdot Du \geqslant C_0 Du|^p - \phi(x, t),$$

$$\text{(3)} \qquad |b(x, t, u, Du)| \leqslant C_1 |Du|^p + \phi.$$

Here C_i, $i = 0, 1$, are given positive constants and the non-negative function ϕ satisfies

$$\text{(4)} \qquad \phi \in L^{\hat{q}}(\Omega_T)$$

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where

$$\frac{1}{\bar{q}} = (1 - \kappa_0)\,\frac{p}{N+p}\,\kappa_0 \in (0, 1]. \tag{5}$$

The notion of weak solution in the specified classes, is standard (see for instance [8]). The lower order term has the natural or Hadamard growth condition with respect to $|Du|$ (see for instance [8]). Here we stress that if merely require that $|Du| \in L^p(\Omega_T)$, the testing functions must be bounded to account for the growth of the right hand side. The problem we address here is that of finding a sup-bound for a weak solution u. It is known that weak solutions of 1 in general are not bounded, not even in the elliptic case (see counterexample 3.7 of [2]). This is due to the fast growth of the right hand side with respect to $|Du|$. On the other hand the existence theory is based on constructing solutions as limits, in some appropriate topology, of bounded solutions of some sequence of approximating problems. (see [2]-[7]).

Therefore the main problem regarding sup-estimates can be formulated as follows. Assuming that a weak solution u of 1 is *qualitatively* bounded, find a *quantitative* $L^\infty(\Omega_T)$ estimate, depending only upon the data. In such a form the problem was first formulated by Stampacchia in the context of elliptic equations (see [12]-[13]). Sup-estimates for solutions of elliptic equations with natural growth conditions have been recently derived by Boccardo-Murat-Puel. See also for the parabolic case [11]. Related results are due to Maderna and Salsa [10] and Alvino, Lions and Trombetti [1]. Here we propose a different approach. We put it in the context of parabolic equations but the proof for the elliptic counterpart in analogous. We will concerned only with L^∞ estimates. An existence theorem based on these would require more stronger assumptions on the operator and it can be modelled after the argument of [7].

THEOREM 0.1. *Let u be a qualitatively bounded weak solution of* 1 *in Ω_T. There exists a constant C that can be determined quantitatively a priori only in terms of the data, such that*

$$\|u\|_{\infty, \Omega_T} \leqslant \max\{2\|f\|_{\infty, \Gamma}; C\}.$$

PROOF. The proof will be consequence of the following two estimates

$$\|u\|_{\infty, \Omega_T} \leqslant \max\{2\|f\|_{\infty, \Gamma}; C\|u\|_{p, \Omega_T}\}, \tag{6}$$

$$\|u\|_{\infty, \Omega_T} \leqslant C, \tag{7}$$

where C is a constant that can be determined quantitatively a priori only in terms of the data.

PROOF OF 6. By working with u_+ and u_- separately we may assume that u is non-negative. If M is the essential supremum of u in Ω_T, we may assume that $M > 2\|f\|_{\infty,\Gamma}$, otherwise there is nothing to prove. In the weak formulation of 1, we take the testing function $(u-k)_+$, where

$$\|f\|_{\infty,\Gamma} \leqslant k < M.$$

This is admissible, modulo a Steklov averaging process (see for instance [9]), since it is bounded and it vanishes in the sense of the traces on the parabolic boundary of Ω_T. By standard calculations in all analogous to those in [9] one gets

$$\sup_{0<t<T} \int_{\Omega\times\{t\}} (u-k)_+^2\, dx + \frac{C_0}{2} \iint_{\Omega_T} |D(u-k)_+|^p\, dx\, d\tau \leqslant \tag{8}$$

$$\leqslant C_1 \iint_{\Omega_T} |D(u-k)_+|^p (u-k)_+\, dx\, d\tau +$$

$$+\gamma \iint_{\Omega_T} |\{\phi\chi[u>k] + \phi(u-k)_+\}\, dx\, d\tau.$$

Here and in what follows we denote with γ a generic positive constant that can be determined a priori only in terms of the data. Next choose $k = M - 2\varepsilon$ where $\varepsilon \in (0,1)$ is so small that $M - 2\varepsilon \geqslant \|f\|_{\infty,\Gamma}$, and

$$C_1 \iint_{\Omega_T} |D(u-k)_+|^p (u-k)_+\, dx\, d\tau \leqslant 2C_1\varepsilon \iint_{\Omega_T} |D(u-k)_+|^p\, dx\, d\tau \leqslant$$

$$\leqslant \frac{C_0}{4} \iint_{\Omega_T} |D(u-k)_+|^p\, dx\, d\tau.$$

Thus we may take

$$2\varepsilon = \min\left\{\|f\|_{\infty,\Gamma};\ \frac{1}{4}\,\frac{C_0}{C_2}\right\}.$$

Combining these calculations in 8, we arrive at

$$\sup_{0<t<T} \int_{\Omega\times\{t\}} (u-k)_+^2\, dx + \iint_{\Omega_T} |D(u-k)_+|^p\, dx\, d\tau \leqslant$$

$$\leqslant \gamma \iint_{\Omega_T} \{\phi\chi[u>k]\}\, dx\, d\tau .$$

By Holder inequality and 2-5 the last term is majorised by

$$\gamma \|\phi\|_{\hat{q},\,\Omega_T} |A_k|^{(N/(N+p))(1+\kappa)} \qquad \kappa = \kappa_0 \frac{p}{N}$$

where γ is constant depending only upon the data and

$$A_k \equiv \{(x,t) \in \Omega_T \,|\, u(x,t) > k\} .$$

Consider the sequence of increasing levels

$$k_n = M - \varepsilon - \frac{\varepsilon}{2^{n+1}} \qquad n = 0, 1, 2, \dots$$

and the corresponding family of sets

$$A_n \equiv \{(x,t) \in \Omega_T \,|\, u(x,t) > k_n\} .$$

These remarks imply that for all $n \in N$

$$\sup_{0<t<T} \int_{\Omega\times\{t\}} (u-k_n)_+^2\, dx + \iint_{\Omega_T} |D(u-k_n)_+|^p\, dx d\tau \leqslant \tag{9}$$

$$\leqslant \gamma |A_n|^{p/N + p(1+\kappa)} ,$$

for a constant γ depending only upon the data. Let $p > 2$ (the case $1 < p < 2$ will be analyzed later).

From 9 and the Gagliardo-Nirenberg embeding theorem (see formula (3.2), pag. 74 of [9]),

$$\left(\frac{\varepsilon}{2^{n+1}}\right)^{p((N+2)/N)} |A_{n+1}| \leqslant \int\int_{[u>k_{n+1}]} (u-k_n)^{p((N+2)/N)}\, dx d\tau \leqslant$$

$$\leqslant \gamma \left(\sup_{0<t<T} \int_{\Omega\times\{t\}} (u-k)_+^2\, dx \right)^{p/N} \iint_{\Omega_T} |D(u-k_n)_+|^p\, dx\, d\tau \leqslant \gamma |A_n|^{1+\kappa} ,$$

i.e. for all $n = 0, 1, 2, \dots,$

$$|A_{n+1}| \leqslant \gamma b^n \varepsilon^{-p((N+p)/N)} |A_n|^{1+\kappa}, \qquad b = 2^{p((N+2)/N)}.$$

It follows from Lemma 5.6 of [9] that $|A_n| \to 0$ as $n \to \infty$ if

$$|A_0| \leqslant \gamma_* \equiv \left(\frac{\varepsilon^{p((N+p)/N)}}{\gamma} \right)^{1/\kappa} b^{1/\kappa^2}.$$

In this case we would have

$$u \leqslant M - \varepsilon \text{ a.e } \Omega_T$$

which contradicts the definition of M. Now

$$\left(\frac{M}{2}\right)^p |A_{M/2}| \leqslant \left(\frac{M}{2}\right)^p |A_0| \leqslant \int\int_{\Omega_T} |u|^p \, dx \, d\tau,$$

i.e.

$$|A_0| \leqslant \left(\frac{2}{M}\right)^p \int_{\Omega} |u|^p \, dx \, d\tau.$$

If the right hand side is less than γ^* we have a contradiction. Thus

$$\sup_{\Omega_T} u \leqslant 2\gamma_*^{1/p} \|u\|_{p, \Omega_T}.$$

Consider, now, the case $1 < p < 2$

Let

$$s_n = M - \varepsilon - \frac{3\varepsilon}{2^{n+3}}, \qquad n = 0, 1, 2, \dots.$$

By repeating the previous argument

$$\left(\frac{\varepsilon}{2^{n+3}}\right)^{p((N+2)/p)} |A_{n+1}| \leqslant \int\int_{[u > k_{n+1}]} (u - s_n)_+^{p((N+2)/N)} \, dx \, d\tau \leqslant$$

$$\leqslant \gamma \left(\sup_{0 < t < T} \int_{\Omega \times \{t\}} (u - s_n)_+^p \, dx \right)^{p/(N+p)} \cdot$$

$$\cdot \left(\int\int_{\Omega_T} |D(u - k_n)_+|^p \, dx \, d\tau \right)^{N/(N+P)} |A_n|^{p/(N+p)}.$$

Therefore

$$\left(\frac{\varepsilon}{2^{n+3}}\right)^{p((N+2)/p)} |A_{n+1}| \leqslant \int\int_{[u>k_{n+1}]} (u-s_n)_+^{p((N+2)/N)}\,dx\,d\tau \leqslant$$

$$\leqslant \gamma\left(\sup_{0<t<T}\int_{\Omega\times\{t\}}(u-k_n)_+^2\,dx\right)^{p/(N+p)}\cdot$$

$$\cdot\left(\int\int_{\Omega_T}|D(u-k_n)_+|^p\,dx\,d\tau\right)^{N/(N+P)} |A_n|^{p/(N+p)},$$

$$|A_n|^{p/(N+p)}\left(\frac{2^{n+3}}{\varepsilon}\right)^{(p/(N+p))(2-p)} \leqslant \left(\frac{2^{n+3}}{\varepsilon}\right)^{(p/(N+p))(2-p)} |A_n|^{1+p/(N+p)}.$$

Hence for all $n=0,1,2,\dots$,

$$|A_{n+1}| \leqslant \gamma b^n \varepsilon^{-p((N+2)/p+(p/(N+p))(2-p))} |A_n|^{1+\kappa},$$

and this inequality implies 6.

Proof of 7. To prove that $\|u\|_{p,\Omega_T}$ is bounded above only in terms of the data, we may assume, modulo a shift that involves the supremum of the boundary data, that u is a bounded non-negative weak solution of 1 vanishing on Γ in the sense of the traces. In the weak formulation of 1, take the testing function

$$\psi = ue^{\alpha pu}, \qquad \alpha > 1 \text{ to be chosen}.$$

Setting also

$$w \equiv (e^{\alpha u} - e^{\alpha})_+,$$

we obtain by standard calculations

$$\sup_{0<t<T}\int_{\Omega} w^p(x,t)\,dx + C_0\int\int_{\Omega_T}|Du|^p(1+\alpha pu)e^{\alpha pu}\,dx\,d\tau \leqslant$$

$$\leqslant C_1\int\int_{\Omega_T}|Du|^p ue^{\alpha u}\,dx\,d\tau + \gamma M\int\int_{\Omega_T}(1+\phi)(1+w^p)\,dx\,d\tau.$$

We choose $\alpha = 2C_1/C_0$ and derive

$$\sup_{0<t<T} \int_\Omega w^p(x,t)dx + \int\int_{\Omega_T} |Dw|^p dx\, d\tau \leq \gamma M \int\int_{\Omega_T} (1+\phi)(1+w^p)dx\, d\tau,$$

for a constant γ depending only upon the data. The proof is concluded by applying again the Gagliardo-Nirenberg embeding theorem.

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