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with natural growth conditions**

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## **$L^\infty$ -Estimates for Nonlinear Parabolic Equations with Natural Growth Conditions.**

VINCENZO VESPRI (\*)

Let  $\Omega$  be a bounded domain in  $\mathbf{R}^N$  of boundary  $\partial\Omega$  and for  $0 < T < \infty$  let  $\Omega_T$  denote the cylindrical domain  $\Omega \times (0, T]$ . Let also

$$\Gamma = (\Omega \times \{0\}) \cup (\partial\Omega \times (0, t])$$

be the parabolic boundary of  $\Omega_T$ . Assume that the boundary  $\partial\Omega$  satisfies the property of positive geometric density, i.e.

there exist  $c > 0$  and  $r_0$  such that for each  $x_0 \in \partial\Omega$ , for every ball  $B_r(x_0)$  centred in  $x_0$  and with radius  $r \leq r_0$ , the measure of the intersection between  $\Omega$  and  $B_r(x_0)$  is greater than  $cr^N$ .

Consider the boundary value problem

$$(1) \quad \begin{cases} u \in C(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)), \\ u_t - \text{Div } a(x, t, u, Du) = b(x, t, u, Du) \text{ in } \Omega_T, \\ u|_\Gamma = f \in L^\infty(\Gamma), \end{cases}$$

where  $p$  is a number greater than 1 and the p.d.e. satisfies the structure conditions

$$(2) \quad a(x, t, u, Du) \cdot Du \geq C_0 |Du|^p - \phi(x, t),$$

$$(3) \quad |b(x, t, u, Du)| \leq C_1 |Du|^p + \phi.$$

Here  $C_i, i = 0, 1$ , are given positive constants and the non-negative function  $\phi$  satisfies

$$(4) \quad \phi \in L^{\hat{q}}(\Omega_T)$$

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where

$$(5) \quad \frac{1}{\bar{q}} = (1 - \kappa_0) \frac{p}{N + p} \kappa_0 \in (0, 1].$$

The notion of weak solution in the specified classes, is standard (see for instance[8]). The lower order term has the natural or Hadamard growth condition with respect to  $|Du|$  (see for instance[8]). Here we stress that if merely require that  $|Du| \in L^p(\Omega_T)$ , the testing functions must be bounded to account for the growth of the right hand side. The problem we address here is that of finding a sup-bound for a weak solution  $u$ . It is known that weak solutions of 1 in general are not bounded, not even in the elliptic case (see counterexample 3.7 of[2]). This is due to the fast growth of the right hand side with respect to  $|Du|$ . On the other hand the existence theory is based on constructing solutions as limits, in some appropriate topology, of bounded solutions of some sequence of approximating problems. (see [2]-[7]).

Therefore the main problem regarding sup-estimates can be formulated as follows. Assuming that a weak solution  $u$  of 1 is *qualitatively* bounded, find a *quantitative*  $L^\infty(\Omega_T)$  estimate, depending only upon the data. In such a form the problem was first formulated by Stampacchia in the context of elliptic equations (see [12]-[13]). Sup-estimates for solutions of elliptic equations with natural growth conditions have been recently derived by Boccardo-Murat-Puel. See also for the parabolic case[11]. Related results are due to Maderna and Salsa[10] and Alvino, Lions and Trombetti[1]. Here we propose a different approach. We put it in the context of parabolic equations but the proof for the elliptic counterpart is analogous. We will concerned only with  $L^\infty$  estimates. An existence theorem based on these would require more stronger assumptions on the operator and it can be modelled after the argument of [7].

**THEOREM 0.1.** *Let  $u$  be a qualitatively bounded weak solution of 1 in  $\Omega_T$ . There exists a constant  $C$  that can be determined quantitatively a priori only in terms of the data, such that*

$$\|u\|_{\infty, \Omega_T} \leq \max \{ 2\|f\|_{\infty, \Gamma}; C \}.$$

**PROOF.** The proof will be consequence of the following two estimates

$$(6) \quad \|u\|_{\infty, \Omega_T} \leq \max \{ 2\|f\|_{\infty, \Gamma}; C\|u\|_{p, \Omega_T} \},$$

$$(7) \quad \|u\|_{\infty, \Omega_T} \leq C,$$

where  $C$  is a constant that can be determined quantitatively a priori only in terms of the data.

PROOF OF 6. By working with  $u_+$  and  $u_-$  separately we may assume that  $u$  is non-negative. If  $M$  is the essential supremum of  $u$  in  $\Omega_T$ , we may assume that  $M > 2\|f\|_{\infty, \Gamma}$ , otherwise there is nothing to prove. In the weak formulation of 1, we take the testing function  $(u - k)_+$ , where

$$\|f\|_{\infty, \Gamma} \leq k < M.$$

This is admissible, modulo a Steklov averaging process (see for instance [9]), since it is bounded and it vanishes in the sense of the traces on the parabolic boundary of  $\Omega_T$ . By standard calculations in all analogous to those in [9] one gets

$$\begin{aligned} (8) \quad \sup_{0 < t < T} \int_{\Omega \times \{t\}} (u - k)_+^2 dx + \frac{C_0}{2} \int \int_{\Omega_T} |D(u - k)_+|^p dx d\tau &\leq \\ &\leq C_1 \int \int_{\Omega_T} |D(u - k)_+|^p (u - k)_+ dx d\tau + \\ &+ \gamma \int \int_{\Omega_T} |\{\phi\chi[u > k] + \phi(u - k)_+\}| dx d\tau. \end{aligned}$$

Here and in what follows we denote with  $\gamma$  a generic positive constant that can be determined a priori only in terms of the data. Next choose  $k = M - 2\varepsilon$  where  $\varepsilon \in (0, 1)$  is so small that  $M - 2\varepsilon \geq \|f\|_{\infty, \Gamma}$ , and

$$\begin{aligned} C_1 \int \int_{\Omega_T} |D(u - k)_+|^p (u - k)_+ dx d\tau &\leq 2C_1 \varepsilon \int \int_{\Omega_T} |D(u - k)_+|^p dx d\tau \leq \\ &\leq \frac{C_0}{4} \int \int_{\Omega_T} |D(u - k)_+|^p dx d\tau. \end{aligned}$$

Thus we may take

$$2\varepsilon = \min \left\{ \|f\|_{\infty, \Gamma}; \frac{1}{4} \frac{C_0}{C_2} \right\}.$$

Combining these calculations in 8, we arrive at

$$\begin{aligned} \sup_{0 < t < T} \int_{\Omega \times \{t\}} (u - k)_+^2 dx + \int_{\Omega_T} |D(u - k)_+|^p dx d\tau &\leq \\ &\leq \gamma \int_{\Omega_T} \{\phi \chi[u > k]\} dx d\tau. \end{aligned}$$

By Holder inequality and 2-5 the last term is majorised by

$$\gamma \|\phi\|_{\bar{q}, \Omega_T} |A_k|^{(N/(N+p))(1+\kappa)} \quad \kappa = \kappa_0 \frac{p}{N}$$

where  $\gamma$  is constant depending only upon the data and

$$A_k \equiv \{(x, t) \in \Omega_T \mid u(x, t) > k\}.$$

Consider the sequence of increasing levels

$$k_n = M - \varepsilon - \frac{\varepsilon}{2^{n+1}} \quad n = 0, 1, 2, \dots$$

and the corresponding family of sets

$$A_n \equiv \{(x, t) \in \Omega_T \mid u(x, t) > k_n\}.$$

These remarks imply that for all  $n \in N$

$$\begin{aligned} (9) \quad \sup_{0 < t < T} \int_{\Omega \times \{t\}} (u - k_n)_+^2 dx + \int_{\Omega_T} |D(u - k_n)_+|^p dx d\tau &\leq \\ &\leq \gamma |A_n|^{p/N + p(1+\kappa)}, \end{aligned}$$

for a constant  $\gamma$  depending only upon the data. Let  $p > 2$  (the case  $1 < p < 2$  will be analyzed later).

From 9 and the Gagliardo-Nirenberg embedding theorem (see formula (3.2), pag. 74 of [9]),

$$\begin{aligned} \left(\frac{\varepsilon}{2^{n+1}}\right)^{p(N+2)/N} |A_{n+1}| &\leq \int_{[u > k_{n+1}]} (u - k_n)^{p(N+2)/N} dx d\tau \leq \\ &\leq \gamma \left( \sup_{0 < t < T} \int_{\Omega \times \{t\}} (u - k)_+^2 dx \right)^{p/N} \int_{\Omega_T} |D(u - k_n)_+|^p dx d\tau \leq \gamma |A_n|^{1+\kappa}, \end{aligned}$$

i.e. for all  $n = 0, 1, 2, \dots$ ,

$$|A_{n+1}| \leq \gamma b^n \varepsilon^{-p((N+p)/N)} |A_n|^{1+\kappa}, \quad b = 2^{p((N+2)/N)}.$$

It follows from Lemma 5.6 of [9] that  $|A_n| \rightarrow 0$  as  $n \rightarrow \infty$  if

$$|A_0| \leq \gamma^* \equiv \left( \frac{\varepsilon^{p((N+p)/N)}}{\gamma} \right)^{1/\kappa} b^{1/\kappa^2}.$$

In this case we would have

$$u \leq M - \varepsilon \text{ a.e. } \Omega_T$$

which contradicts the definition of  $M$ . Now

$$\left( \frac{M}{2} \right)^p |A_{M/2}| \leq \left( \frac{M}{2} \right)^p |A_0| \leq \int \int_{\Omega_T} |u|^p dx d\tau,$$

i.e.

$$|A_0| \leq \left( \frac{2}{M} \right)^p \int_{\Omega} |u|^p dx d\tau.$$

If the right hand side is less than  $\gamma^*$  we have a contradiction. Thus

$$\sup_{\Omega_T} u \leq 2\gamma_*^{1/p} \|u\|_{p, \Omega_T}.$$

Consider, now, the case  $1 < p < 2$

Let

$$s_n = M - \varepsilon - \frac{3\varepsilon}{2^{n+3}}, \quad n = 0, 1, 2, \dots$$

By repeating the previous argument

$$\begin{aligned} \left( \frac{\varepsilon}{2^{n+3}} \right)^{p((N+2)/p)} |A_{n+1}| &\leq \int \int_{[u > k_{n+1}]} (u - s_n)_+^{p((N+2)/N)} dx d\tau \leq \\ &\leq \gamma \left( \sup_{0 < t < T} \int_{\Omega \times \{t\}} (u - s_n)_+^p dx \right)^{p/(N+p)}. \\ &\cdot \left( \int \int_{\Omega_T} |D(u - k_n)_+|^p dx d\tau \right)^{N/(N+p)} |A_n|^{p/(N+p)}. \end{aligned}$$

Therefore

$$\begin{aligned}
\left(\frac{\varepsilon}{2^{n+3}}\right)^{p(N+2)/p} |A_{n+1}| &\leq \int \int_{[u > k_{n+1}]} (u - s_n)_+^{p(N+2)/N} dx d\tau \leq \\
&\leq \gamma \left( \sup_{0 < t < T} \int_{\Omega \times \{t\}} (u - k_n)_+^2 dx \right)^{p/(N+p)} \cdot \\
&\cdot \left( \int_{\Omega_T} |D(u - k_n)_+|^p dx d\tau \right)^{N/(N+p)} |A_n|^{p/(N+p)}, \\
|A_n|^{p/(N+p)} \left(\frac{2^{n+3}}{\varepsilon}\right)^{(p/(N+p))(2-p)} &\leq \left(\frac{2^{n+3}}{\varepsilon}\right)^{(p/(N+p))(2-p)} |A_n|^{1+p/(N+p)}.
\end{aligned}$$

Hence for all  $n = 0, 1, 2, \dots$ ,

$$|A_{n+1}| \leq \gamma b^n \varepsilon^{-p(N+2)/p + (p/(N+p))(2-p)} |A_n|^{1+\kappa},$$

and this inequality implies 6.

**PROOF OF 7.** To prove that  $\|u\|_{p, \Omega_T}$  is bounded above only in terms of the data, we may assume, modulo a shift that involves the supremum of the boundary data, that  $u$  is a bounded non-negative weak solution of 1 vanishing on  $\Gamma$  in the sense of the traces. In the weak formulation of 1, take the testing function

$$\psi = ue^{\alpha u}, \quad \alpha > 1 \text{ to be chosen.}$$

Setting also

$$w \equiv (e^{\alpha u} - e^\alpha)_+,$$

we obtain by standard calculations

$$\begin{aligned}
\sup_{0 < t < T} \int_{\Omega} w^p(x, t) dx + C_0 \int \int_{\Omega_T} |Du|^p (1 + \alpha pu) e^{\alpha pu} dx d\tau &\leq \\
&\leq C_1 \int \int_{\Omega_T} |Du|^p u e^{\alpha u} dx d\tau + \gamma M \int \int_{\Omega_T} (1 + \phi)(1 + w^p) dx d\tau.
\end{aligned}$$

We choose  $\alpha = 2C_1/C_0$  and derive

$$\sup_{0 < t < T} \int_{\Omega} w^p(x, t) dx + \int \int_{\Omega_T} |Dw|^p dx d\tau \leq \gamma M \int \int_{\Omega_T} (1 + \phi)(1 + w^p) dx d\tau,$$

for a constant  $\gamma$  depending only upon the data. The proof is concluded by applying again the Gagliardo-Nirenberg embedding theorem.

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#### REFERENCES

- [1] A. ALVINO - P. L. LIONS - G. TROMBETTI, *Comparison results for elliptic and parabolic equations via Schwartz symmetrisation*, Ann. Inst. Henri Poincaré, Analyse non lineaire, to appear.
- [2] L. BOCCARDO - F. MURAT - J. P. PUEL, *Existence de solutions faible pour des equations elliptiques quasi lineaires a croissance quadratique*, Research Notes in Mathematics, 84, Pitman, London (1983), pp. 19-73.
- [3] L. BOCCARDO - F. MURAT - J. P. PUEL, *Resultats d'existence pour certains problemes quasi lineaires*, Ann. Sc. Norm. Sup. Pisa, 11 (1984), pp. 213-235.
- [4] L. BOCCARDO - F. MURAT - J. P. PUEL, *Existence of bounded solutions for nonlinear elliptic unilateral problems*, Ann. Mat. Pura Appl., 152 (1988), pp. 183-196.
- [5] L. BOCCARDO - F. MURAT - J. P. PUEL, *Quelques proprietes des operateurs elliptiques quasi lineaires*, C.R.A.S., 307, Serie I (1988), pp. 749-752.
- [6] L. BOCCARDO - F. MURAT - J. P. PUEL, *Existence results for some quasilinear parabolic equations*, NonLinear Anal., 13 (1989), pp. 373-392.
- [7] L. BOCCARDO - F. MURAT - J. P. PUEL,  *$L^\infty$  estimate for some nonlinear elliptic partial differential equation and application to an existence result*, SIAM J. Math. Anal., to appear.
- [8] M. GIAQUINTA, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Mathematics Studies, Princeton, N.Y. (1983).
- [9] A. LADYZHENSKAJA - V. A. SOLONNIKOV - N. N. URAL'CEVA, *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc. Transl., Providence, R. I. (1968).
- [10] C. MADERNA - S. SALSZA, *Dirichlet problem for elliptic equations with nonlinear first order term: a comparison result*, Ann. Mat. Pura Appl., 148 (1987), pp. 277-288.
- [11] L. ORSINA - M. M. PORZIO,  *$L^\infty(\Omega)$  estimate and existence of solutions for some nonlinear parabolic equations*, B.U.M.I., to appear.



- [12] G. STAMPACCHIA, *Le probleme de Dirichlet pour les equations elliptiques du second ordre a coefficients discontinus*, Ann. Inst. Fourier, **15** (1965), pp. 189-258.
- [13] G. STAMPACCHIA, *Equation elliptiques du second ordre a coefficients discontinus*, Séminaires de Mathématiques Supérieures, **16**, Les presses de l'Université de Montreal, Montreal (1966).

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