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MARIA JOÁO FERREIRA
MARCO RIGOLI
RENATO TRIBUZY

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Isometric Immersions of Kähler Manifolds.

MARIA JOÁO FERREIRA - MARCO RIGOLI - RENATO TRIBUZY(*)

1. Introduction.

The present article is concerned with obstruction to the existence of (1, 1)-geodesic isometric immersions from a Kähler manifold M.

The concept of a (1, 1)-geodesic map is a natural extension of the notion of a minimal immersion from a Riemann surface; (1, 1)-geodesic maps appear sometimes in the literature with the name of circular or pluriharmonic maps.

It is well known ([D-R]) that the (1, 1)-geodesic isometric immersions from M into $\mathbb{R}^n$ are exactly the minimal isometric immersions. A naive remark allows us to infer that, more generally, the minimal isometric immersions from a Kähler manifold into a locally symmetric Riemannian manifold of non-compact type are (1, 1)-geodesic.

M. Dacjzer and L. Rodrigues have proved in [D-R] that if Q_c is a space form of sectional curvature $c > 0$ (resp. $c < 0$) and $\varphi\colon M^m \to Q_c$ (m = complex dimension of M) is a (1, 1)-geodesic (resp. minimal) isometric immersion, then $m = 1$. We will show that the only (1, 1)-geodesic isometric immersions from M into a 1/4-pinched Riemannian manifold are the minimal isometric immersions from a Riemannian surface. A dual result is obtained for maps into a Riemannian manifold with negative sectional curvature, namely, if N is a Riemannian manifold whose sectional curvatures $K(\sigma)$ satisfy $-1 \leqq K/(\sigma) < -1/4$, the only minimal isometric immersions from a Kähler manifold into N are the minimal immersions from a Riemann surface.

In a different context, space forms can be considered as conformally

(*) Indirizzo degli AA.: M. J. FERREIRA: Departamento de Matemática, Faculdade de Ciencias, Universidade de Lisboa, Rua Ernesto de Vasconcelos, Ed C2, 1700 Lisboa, Portugal; M. RIGOLI: Dipartimento di Matematica, Università degli Studi di Catania, Viale Andrea Doria 6, 95125 Catania, Italy; R. TRIBUZY: Departamento de Matemática - ICE, Universidade do Amazonas, 69000 Manaus, AM, Brazil.

flat Einstein manifolds. In that direction we generalize Theorem 1.2 of [D-R] to isometric immersions into conformally flat Riemannian manifolds with certain bounds on their Ricci curvature.

To a certain extent holomorphic maps between Kähler manifolds are the simplest examples of (1,1)-geodesic maps. In [D-T], M. Dacjzer and Thorbergson have shown that for $m > 1$ the only (1,1)-geodesic isometric immersions from M into a complex space-form with holomorphic sectional curvature $c \neq 0$ are the holomorphic immersions. Regarding $\mathbb{C}P^n$ as the complex Grassmannian of one dimensional complex subspaces of $\mathbb{C}^n$ it is natural to try to extend their results to isometric immersions into a complex Grassmannian. We show that if N is the complex Grassmannian of p-dimensional complex subspaces of $\mathbb{C}^n$ and $m > (p - 1)(n - p - 1) + 1$, the only (1,1)-geodesic isometric immersions from M^m into N are the holomorphic immersions. Furthermore, if N is the corresponding dual symmetric space of non-compact type, for $m > (p - 1)(n - p - 1) + 1$, there are no non-holomorphic minimal isometric immersions from M^m into N.

2. (1,1)-geodesic maps into pinched Riemannian manifolds.

Let M^m be a Kähler manifold with complex dimension m and N be an arbitrary Riemannian manifold.

The complex structure of M^m gives rise to the splitting

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{\mathbb{C}}$ is the complexified tangent bundle and $T^{1,0}M$, the holomorphic tangent bundle, is the eigenbundle of J corresponding to the eigenvalue $+i$.

The second fundamental form of a smooth map $\varphi: M \to N$ is the covariant tensor field $\alpha = \nabla d\varphi \in C(\odot^2 T^* M)$.

φ is said to be (1,1)-geodesic if the (1,1)-part of the complex bilinear extension of α vanishes identically, or equivalently, if for any $X, Y \in C(TM)$

$$\alpha(X; Y) + \alpha(JX; JY) = 0,$$

where J denotes the complex structure of M.

φ is said to be minimal if trace $\alpha = 0$. Clearly (1,1)-geodesic immersions are minimal immersions.

PROPOSITION 1. If N is a non-compact locally symmetric Riemannian manifold and $\varphi: M^m \to N$ is an isometric immersion the following assertions are equivalent:

(i) φ is (1, 1)-geodesic,

(ii) φ is minimal.

PROOF. For each $x \in M$ we consider a local orthonormal frame field $\{e_1, \dots, e_m, Je_1, \dots, Je_m)$ in a neighbourhood of x. We shall use the following notation:

$$\sqrt{2}E_i = e_j + iJe_j \in C(T^{0,1}M)$$

and

$$\sqrt{2}E_{-j} = \sqrt{2}\bar{E}_j \in C(T^{1,0}M), \qquad \text{for each } j \in \{1, \dots, m\}.$$

If R^M and R^N denote respectively the Riemannian curvature tensors of M and N, using the complex multilinear extension of the Gauss equation we can write

$$(1) \quad \langle R^M(E_i, E_j)E_{-i}, E_{-j}\rangle = \langle \alpha(E_i, E_{-i}), \alpha(E_j, E_{-j})\rangle - \\ - \langle \alpha(E_i, E_{-j}), \alpha(E_j, E_{-i})\rangle + \langle R^N(E_i, E_j)E_{-i}, E_{-j}\rangle.$$

Since M^m is Kähler the left-hand side member of (1) vanishes identically. If φ is minimal, summing in i we get

$$(2) \qquad \sum_{i=1}^{m} \langle \alpha(E_i, E_{-j}), \alpha(E_j, E_{-i})\rangle = \sum_{i=1}^{m} \langle R^M(E_i, E_j)E_{-1}, E_{-j}\rangle.$$

On the other hand, the universal covering of N is a Riemannian symmetric space $\tilde{N}$ of the non-compact type. Let $\pi\colon \tilde{N} \to N$ represent the covering map and G the connected component of the identity in the group of isometries of $\tilde{N}$. At some point $y \in \tilde{N}$ such that $\pi(y) = \varphi(x)$ we let K denote the isotropy subgroup of G at y. If S represents the Lie algebra of G and $\mathcal{K}$ the subalgebra corresponding to K, we can identify T_yN with the orthogonal complement $\mathcal{P}$ of $\mathcal{K}$ in $\mathcal{S}$ with respect the Killing-Cartan form of G. Under this identification, for each $l \leq j \leq m$, the lifting of E_j (resp. E_{-j}) corresponds to some vector $\hat{E}_{\underline{j}}$ (resp. $\hat{E}_{-j}$) of $\mathcal{P}^{\mathbb{C}}$ where $\mathcal{P}^{\mathbb{C}}$ denotes the complexification of $\mathcal{P}$. Then if $\tilde{R}$ denotes the Riemannian curvature tensor of $\tilde{N}$ we know that

$$(3) \quad \langle R^N(E_iE_j)E_{-1}E_{-j}\rangle_{\varphi(x)} - \langle \tilde{R}(\hat{E}_i\hat{E}_j)\hat{E}_{-1}\hat{E}_{-j}\rangle_y = \\ = \langle [\hat{E}^i, \hat{E}_j], [\hat{E}_{-i}, \hat{E}_{-j}]\rangle \leqq 0.$$

From (2) we conclude that $\alpha(E_i, E_{-j}) = 0$ for $1 \leqq i, j \leq m$, hence φ is (1, 1)-geodesic.

REMARK. If N is a real symmetric space of rank 1 either of the compact or non-compact type and φ is a $(1,1)$-geodesic isometric immersion, eq. (3) holds, and we recover Theorem 1.2 of [D-R]. Indeed $[\hat{E}_i, \hat{E}_j] = 0$ for all $1 \leqq i, j \leq m$, and it is easily seen that this can happen only when $m = 1$. We now generalize this result to pinched-Riemannian manifolds.

Let S be a positive real number. A Riemannian manifold N is said to be positively (negatively) S-pinched at a point $y \in N$ if there exists a positive real number L such that $LS < K_y(\sigma) \leqq L(-L \leqq K_y(\sigma) < -LS)$ for any 2-dimensional subspace σ of $T_y N$. N is said to be positively (negatively) S-pinched if it is positively (negatively) S-pinched at each point $y \in N$.

THEOREM 1. Let N be a positively $1/4$-pinched Riemannian manifold. If $\varphi\colon M^m \to N$ is a $(1,1)$-geodesic isometric immersion, then $m = 1$.

THEOREM 2. Let N be a negatively $1/4$-pinched Riemannian manifold. If $\varphi\colon M^m \to N$ is a minimal isometric immersion, then $m = 1$.

To prove Theorems 1 and 2 we need the following lemma:

LEMMA 1. Let N be a Riemannian manifold whose sectional curvatures satisfy one of the following inequalities:

i) $-1 \leqq K(\sigma), -1/4,$

ii) $(1/4) < K(\sigma) \leqq 1.$

Then if X, Y, Z, W is a local orthonormal frame field the following inequality holds:

$$|\langle R(X, Y)Z, W\rangle| \leq \frac{1}{2}\,.$$

PROOF OF LEMMA 1. Assume (i) holds. By polarization we get

$$\begin{aligned}(4)\qquad -16 \leqq{}& \langle R(X+Z, Y+W)(X+Z), Y+W\rangle + \\ &+\langle R(X-Z, Y-W)(X-Z), Y-W\rangle + \\ &+\langle R(-X+Y, W+Z)(-X+Y), W+Z\rangle +\end{aligned}$$

$$+\langle R(X+Y, W-Z)(X+Y), W-Z\rangle =$$
$$= 4\langle R(X, W)X, W\rangle + 4\langle R(Y, Z)Y, Z\rangle + 2\langle R(X, Y)X, Y\rangle +$$
$$+2\langle R(Z, W)Z, W\rangle + 2\langle R(X, Z)X, Z\rangle + 2\langle R(Y, W)Y, W\rangle +$$
$$+12\langle R(X, W)Z, Y\rangle < -4 .$$

Therefore from the left-hand side inequality we have

$$(5)\quad 8 + 2\langle R(X, W)X, W\rangle + 2\langle R(Y, Z)Y, Z\rangle + \langle R(X, Y)X, Y\rangle +$$
$$+\langle R(Z, W)Z, W\rangle + \langle R(X, Z)X, Z\rangle + \langle R(Y, W)Y, W\rangle +$$
$$+6\langle R(X, W)Z, Y\rangle \geqq 0 .$$

Replacing X by $-X$ in the right-hand side of inequality (4) we get

$$(6)\quad -2 - 2\langle R(X, W)X, W\rangle - 2\langle R(Y, Z)Y, Z\rangle - \langle R(X, Y)X, Y\rangle -$$
$$-\langle R(Z, W)Z, W\rangle - \langle R(X, Z)X, Z\rangle - \langle R(Y, W)Y, W\rangle +$$
$$+6\langle R(X, W)Z, Y\rangle > 0 .$$

(5) and (6) lead to

$$6 + 12\langle R(X, W)Z, Y\rangle > 0, \qquad \text{or} \qquad \langle R(X, W)Z, Y\rangle > -\frac{1}{2} .$$

Now a similar procedure with Z replaced by $-Z$ in all inequalities gives

$$6 - 12\langle R(X, W)Z, Y\rangle > 0, \qquad \text{or} \qquad \langle R(X, W)Z, Y\rangle > -\frac{1}{2} .$$

PROOF OF THEOREMS 1 AND 2. If necessary normalizing the metric we can assume, without loss of generality, that $L = 1$.

If φ is $(1, 1)$-geodesic in the case of Theorem 1, or if φ is minimal in the case of Theorem 2, we know from (2) that, for $1 \leqq i, j \leq m$

$$0 = \sum_{i=1}^{m} \langle R^N(E_i, E_j)E_{-i}, E_{-j}\rangle =$$
$$= \sum_{i=1}^{m} \{\langle R(e_i, e_j)e_i, e_j\rangle + \langle R(Je_i, Je_j)(Je_i), Je_j\rangle + \langle R(Je_i, e_j)(Je_i), e_j\rangle +$$
$$+\langle R(e_i, Je_j)e_i, Je_j\rangle - 2\langle R(Je_i, Je_j)e_i, e_j\rangle + 2\langle R(Je_i, e_j)e_i, Je_j\rangle\} =$$

$$= \sum_{i=1}^{m} \{\langle R(e_i, e_j) e_i, e_j\rangle + \langle R(Je_i, Je_j)(Je_i) Je_j\rangle + \langle R(Je_i, e_j)(Je_i), e_j\rangle +$$

$$+\langle R(e_i, Je_j) e_i, Je_j\rangle + 2\langle R(Je_i, e_i) e_j, Je_j\rangle\},$$

where we have used the Bianchi identity.

Now the hypothesis of Theorem 1 (2) implies that

$$\sum_{i=1}^{m} \langle R^N(E_i, E_j) E_{-i} E_{-j}\rangle > 0 \quad (< 0)$$

which cannot happen. Then m must be 1.

3. Minimal isometric immersions into conformally flat Riemannian manifolds.

Riemannian manifolds with constant sectional curvature are very special examples of conformally flat Riemannian manifolds.

A Riemannian manifold (N, h) is said to be conformally flat if there exists a smooth function $f: N \to \mathbb{R}$ such that $(N, e^{2f} h)$ is flat.

The main invariant under conformal changes of the metric is the Weyl curvature tensor W. The vanishing of W completely characterizes the conformally flat Riemannian manifolds.

For each $x \in N$ we denote by $\mathcal{C}_x(n)$ the subspace of $(\Lambda^2 T^x N)$ consisting of «curvature like» tensors: that means those tensors satisfying the Bianchi identity. The action of $O(n)$ $(n = \dim N)$ on $\mathcal{C}_x(N)$ gives rise to the following decomposition into irreducible subspaces

$$\mathcal{C}_x(N) = \mathcal{U}_x(N) \oplus R_x(N) \oplus \mathcal{W}_x(N),$$

where $\mathcal{U}_x(N) = \mathbb{R} Id_{\Lambda^2 T^* N}$ and $R_x(N)$ is formed by the «Ricci traceless» tensors, that is, those tensors θ whose Ricci contraction $c(\theta)$ $(c(\theta)(x, y) = \operatorname{trace} \theta(x, \cdot, y, \cdot))$ vanishes. The orthogonal complement $\mathcal{W}_x(N)$ of $\mathcal{U}_x(N) \oplus R_x(N)$ is called the space of Weyl tensors. The Weyl tensor of a Riemannian manifold is the Weyl part of its curvature tensor.

It is an easy matter to verify that the Riemannian curvature tensor of a conformally flat Riemannian manifold N with Ricci curvature Ricci^N and normalized scalar curvature $S(N) = 1/n$ trace Ricci^N is given by

$$(7) \quad \langle R^N(X, Y)Z, W\rangle = \frac{1}{n-2}\{\langle X, Z\rangle \mathrm{Ricci}^N(Y, W) +$$

$$\langle Y, W\rangle \mathrm{Ricci}^N(X, Z) - \langle X, W\rangle \mathrm{Ricci}^N(Y, Z) - \langle Y, Z\rangle \mathrm{Ricci}^N(N, W)\} -$$

$$- \frac{nS(N)}{(n-1)(n-2)}\{\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle\}.$$

In this section we analyse the existence of minimal isometric immersions into certain conformally flat Riemannian manifolds.

As above M^m will represent a Kähler manifold with complex dimension m. We shall use the following notation

$$r = \inf_{\substack{x\in M\\ \|v\|_x=1}} \mathrm{Ricci}_x^N(v, v), \qquad R = \sup_{\substack{x\in M\\ \|v\|_x=1}} \mathrm{Ricci}_x^N(v, v),$$

$$s = \inf_{x\in M} S(N)_x \quad \text{and} \quad S = \sup_{x\in M} S(N)_x .$$

THEOREM 3. Let N be a conformally flat Riemannian manifold with positive scalar curvature such that $r/S > n/2(n-1)$.

If $\varphi\colon M^m \to N$ is a $(1,1)$-geodesic isometric immersion, then $m = 1$.

THEOREM 4. Let N be a conformally flat Riemannian manifold with negative scalar curvature such that $R/s > n/2(n-1)$.

If $\varphi\colon M^m \to N$ is a minimal isometric immersion, then $m = 1$.

COROLLARY 1. Let N be a conformally flat Riemannian manifold such that $nA/2(n-1) < \mathrm{Ricci}^N \leqq A$ for some positive real number A.

If $\varphi\colon M^m \to N$ is a $(1,1)$-geodesic isometric immersion, then $m = 1$.

COROLLARY 2. Let N be a conformally flat Riemannian manifold such that $-A \leqq \mathrm{Ricci}^N < -(n/2(n-2))A$ for some positive real number A.

If $m > 1$ there does not exist minimal isometric immersions from M^m into N.

REMARK. If N has non-zero constant sectional curvature, $r/S = R/s = 1$.

PROOF OF THEOREM 3. Assuming that $\varphi\colon M^m \to N$ is $(1,1)$-geodesic, we obtain from eq. (2)

$$\langle R^N(E_i, E_j)E_{-i}, E_{-j}\rangle = 0 .$$

On the other hand, if $m > 1$, taking $i \neq j$ we conclude from (7) that

$$\langle R^N(E_i, E_j)E_{-i}, E_{-j}\rangle =$$

$$= \frac{1}{n-2}\{\mathrm{Ricci}^N(E_i, E_{-i}) + \mathrm{Ricci}^N(E_j, E_{-j})\} - \frac{nS(N)}{(n-1)(n-2)} =$$

$$= \frac{1}{2(n-2)}\{\mathrm{Ricci}^N(e_i, e_j) + \mathrm{Ricci}^N(Je_i, Je_j) + \mathrm{Ricci}^N(e_j, e_j) +$$

$$+ \mathrm{Ricci}^N(Je_j, Je_j)\} - \frac{S(N)}{n-2} \geqq \frac{1}{n-2}\left\{2r - \frac{S(n)}{n-1}\right\} > 0,$$

which is a contradiction.

Proof of Theorem 4. If φ is minimal, eq. (2) establishes that

$$\sum_{i=1}^{m} \langle R^N(E_i, E_j)E_{-i}, E_{-j}\rangle \geq 0.$$

But from (7)

$$\sum_{i=1}^{m} \langle R^N(E_i, E_j)E_{-i}, E_{-j}\rangle =$$

$$= \frac{1}{n-2}\left\{(m-2)\,\mathrm{Ricci}^N(E_j, E_j) + \sum_{i=1}^{m} \mathrm{Ricci}^N(E_i, E_i)\right\} -$$

$$- \frac{m-1}{n-2}\,\frac{nS(N)}{(n-1)(n-2)} \leqq \frac{m-1}{n-2}\left\{2r - \frac{nS}{n-1}\right\},$$

which can only happen if $m = 1$.

4. Holomorphicity of minimal isometric immersion into a complex Grassmannian.

Dacjzer and Thorbergson have studied in [D-R] minimal isometric immersions into a complex space form $\mathbb{C}Q_c$ with non-zero constant holomorphic sectional curvature c. Their result states that if $m > 1$, $c > 0$ (< 0) and $\varphi: M^m \to \mathbb{C}Q_c$ is a (1,1)-geodesic (minimal) isometric immersion, then φ is $\pm$ holomorphic.

Regarding $\mathbb{C}P^n$ as the complex Grassmannian manifold of complex 1-planes we extend these results to isometric immersions into a complex Grassmannian (respectively to its dual symmetric manifold of non-compact type).

We let $G_p(\mathbb{C}^n)$ denote the Grassmannian manifold of p-dimensional complex subspaces of $\mathbb{C}^n$. The action of the unitary group $U(n)$ on $G_p(\mathbb{C}^n)$ endows $G_p(\mathbb{C}^n)$ with the structure of a Hermitian symmetric space isometric to $U(n)/U((p) \times (n-p))$. In particular, $G_p(\mathbb{C}^n)$ is a Kähler manifold. We represent by $H_p(\mathbb{C}^n)$ its dual symmetric space of non-compact type $U(p, n-p)/U((p) \times U(n-p))$, where $U(p, n-p)$ is the group of matrices in $Gl(\mathbb{C}^n)$ which leave invariant the Hermitian form $-Z_1, \overline{Z}_1 - \ldots - Z_p\overline{Z}_p + Z_{p+1}\overline{Z}_{p+1} + \ldots + Z_n\overline{Z}_n$.

THEOREM 5. Let $\varphi\colon M^m \to G_p(\mathbb{C}^n)$ be a $(1,1)$-geodesic isometric immersion. If $m > (p-1)(n-p-1)+1$, then φ is $\pm$ holomorphic.

THEOREM 6. Let $\varphi\colon M^m \to H_p(\mathbb{C}^n)$ be a minimal isometric immersion. If $m > (p-1)(n-p-1)+1$, then φ is $\pm$ holomorphic.

REMARKS. 1) When $p = 1$ we get Theorems A and B of [D-T].

2) As an easy consequence of Theorems 5 and 6 we also get Theorem 1.2 of [D-R] which asserts that for Riemannian manifolds with constant sectional curvature $c > 0$ (< 0) the only $(1,1)$-geodesic (minimal) isometric immersions are the minimal immersions from a Riemann surface.

In fact, let Q_c be a Riemannian manifold with constant sectional curvature c. Assume, for instance, that $c > 0$ and $m > 1$. Without loss of generality we can assume Q_c is simply connected, hence isometric to S^n. Therefore, since there exists a totally geodesic immersion from S^n to $\mathbb{C}P^n$, a $(1,1)$-geodesic immersion $\varphi\colon M^m \to Q_c$ would originate a $(1,1)$-geodesic immersion $\tilde{\varphi}\colon M^m \to \mathbb{C}P^n$. This cannot happen, since $\tilde{\varphi}$ would be simultaneously holomorphic and totally real. The case $c < 0$ is analogous.

PROOF OF THEOREM 5 AND 6. Let $\mathcal{U}(k)$ represent the Lie algebra of $U(k)$, $N = G_p(\mathbb{C}^n)$ and $q = n - p$.

At some point $y = \varphi(x)$ we identify T_yN with the orthogonal complement $\mathcal{P}$ of $\mathcal{U}(p) \times \mathcal{U}(q)$ in $\mathcal{U}(n)$ with respect to the Killing-Cartan form of $U(n)$. Let $\mathcal{P}^{\mathbb{C}}$ denote the complexification of $\mathcal{P}$.

Under this identification we obtain from (2)

$$\langle R^N(E_i, E_j)E_{-i}, E_{-j}\rangle_y = \langle [E_i, E_j], [E_{-i}, E_{-j}]\rangle = \langle [E_i, E_j], \overline{[E_i, E_j]}\rangle = 0\,.$$

Therefore, we conclude that $d\varphi(x)(T^{1,0}M) = W$ is an Abelian isotropic subspace of $\mathcal{P}^{\mathbb{C}}$. We remark that

$$\mathcal{P}^{\mathbb{C}} = \left\{ \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} : \quad \begin{matrix} A \text{ and } B \text{ are respectively } p \times q \\ \text{and } q \times p \text{ complex matrices} \end{matrix} \right\}$$

N is a Kähler manifold. It is easily seen that, under the above identification, the type decomposition of $T_y N$ gives rise to the splitting

$$\mathcal{P}^{\mathbb{C}} = \mathcal{P}^+ \oplus \mathcal{P}^- \cong T_y^{\mathbb{C}} N ,$$

where

$$\mathcal{P}^+ = \left\{ \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \in \mathcal{P}^{\mathbb{C}} : B = 0 \right\} \cong T_y^{1,0} N$$

and

$$\mathcal{P}^- = \left\{ \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \in \mathcal{P}^{\mathbb{C}} : A = 0 \right\} \cong T_y^{0,1} N .$$

Clearly, if φ is holomorphic (respectively -holomorphic) $W \subset \mathcal{P}^+$ (respectively $W \subset \mathcal{P}^-$). It is also a well-known fact that $\mathcal{P}^+$ and $\mathcal{P}^-$ are Abelian subspaces of $\mathcal{P}^{\mathbb{C}}$.

Theorem 5 is now a direct consequence of the next lemma. By duality we obtain Theorem 6 in the same way.

LEMMA 1. If $\dim_{\mathbb{C}} W > (p-1)(q-1)+1$, then $W \subset \mathcal{P}^+$ or $W \subset \mathcal{P}^-$.

PROOF. We assume that $W \not\subset \mathcal{P}^+$ and $W \not\subset \mathcal{P}^-$ and prove that $\dim_{\mathbb{C}} W \geqq (p-1)(q-1)+1$.

We shall consider two cases:

Case 1. – $W \cap \mathcal{P}^- \neq \emptyset$ or $W \cap \mathcal{P}^+ \neq \emptyset$.

Since the procedure is similar we only consider $W \cap \mathcal{P}^- \neq \emptyset$.

There exists, at least, one matrix $0 \neq X = \begin{bmatrix} 0 & 0 \\ B_X & o \end{bmatrix} \in W$ where B_X is a $q \times p$ complex matrix and $Y = \begin{bmatrix} 0 & A_Y \\ B_Y & o \end{bmatrix}$ where A_Y is a non-zero $p \times q$ complex matrix.

Now $[X, Y] = 0$ implies that

$$\begin{cases} A_Y B_X = 0\,, \\ B_X A_Y = 0\,. \end{cases}$$

We can assume without loss of generality that $p \leqq q$. From $B_X A_Y = 0$ we see that the rank of B_X is strictly smaller than p, otherwise A_y would be identically zero.

Let

$$k = \max\left\{\operatorname{rank} B_X \colon X = \begin{bmatrix} 0 & 0 \\ B_X & 0 \end{bmatrix} \in W\right\}$$

and take

$$X_0 = \begin{bmatrix} 0 & 0 \\ B_{X_0} & 0 \end{bmatrix} \in W \quad \text{with rank } B_{X_0} = k\,.$$

Since the metric of $\mathscr{P}$ is invariant by the action of $U(p) \times U(q)$, without loss of generality we can asume that $B_{X_0} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, where I is a diagonal non-singular matrix.

We consider the subspaces

$$W_1 \left\{X = \begin{bmatrix} 0 & A_X \\ B_X & 0 \end{bmatrix} \in W \colon A_X = 0\right\}$$

and

$$W_2 = \left\{X = \begin{bmatrix} 0 & A_X \\ B_X & 0 \end{bmatrix} \in W \colon A_X \neq 0\right\}$$

As we can see from the equations $A_X B_{X_0} = B_{X_0} A_X = 0$, for each $X \in W_2$, we must have $A_X = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A}_X \end{bmatrix}$, when $\tilde{A}_X$ is a $l \times l$ matrix with $l \leqq p = k$.

Now let

$$r = \max\left\{\operatorname{rank} \tilde{A}_X \colon X = \begin{bmatrix} 0 & A \\ B_X & 0 \end{bmatrix} \in W_2\right\}$$

and choose

$$X_1 = \begin{bmatrix} 0 & A_{X_1} \\ B_{X_1} & 0 \end{bmatrix} \in W_2 \qquad \text{such that rank } \tilde{A}_{X_1} = r .$$

If necessary changing W_1 and W_2 we can assume, without loss of generality, that this particular $\tilde{A}_{X_1}$ is a diagonal nonsingular matrix. Again the equations $A_{X_1} B_X = B_X A_{X_1} = 0$ allow us to conclude that, for each $X = W_2$ we must have $B_X = \begin{bmatrix} \tilde{B}_X & 0 \\ 0 & 0 \end{bmatrix}$ where $\tilde{B}_X$ is a $(p-r) \times (q-r)$ matrix. Thus

$$\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} W_1 + \dim_{\mathbb{C}} W_2 \leqq$$

$$\leqq (p-r) \times (q-r) + r^2 \leqq (p-1)(q-1) + 1 ,$$

since $1 \leqq r \leqq p-1$. The equality $(p-r)(q-r) + r^2 = (p-1)(q-1) + 1$ is attained when $r = 1$.

Case 2. – Assume now that $W \cap \mathcal{P}^+ = \phi$ and $W \cap \mathcal{P}^- = \phi$. For each $l \times s$ complex matrix $C = (c_{ij})$ we let $C_1, \ldots, C_l$ denote the lines of C and $C^1, \ldots, C^s$ its columns.

First notice that if there exist two linearly independent elements

$$X = \begin{bmatrix} 0 & A_X \\ B_X & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & A_Y \\ B_Y & 0 \end{bmatrix} \quad \text{in } W$$

with $(A_X)_1 = \ldots = (A_X)_{p-1} = 0$ and $(A_Y)_1 = \ldots = (A_Y)_{p-2} = 0$ we shall have $B_X^p = B_Y^p = 0$. Indeed, from $[X, Y] = 0$ we get that for i, $j \in \{1, \ldots, q\}$,

$$b(X)_{ip}\, a(Y)_{pj} = b(Y)_{ip}\, a(X)_{pj} ,$$

so that if $B_X^p \neq 0$, there exists $1 \leq i \leq q$ such that $X = (b(Y)_{ip} / b(X)_{ip})\, Y \in \mathcal{P}^-$ which cannot happen.

Using an inductive argument we conclude that we can only have two alternative situations:

A) There exists one and only one element $X = \begin{bmatrix} 0 & A_X \\ B_X & 0 \end{bmatrix} \in W$ such that $(A_X)_1 = \ldots = (A_X)_{p-1} = 0$.

B) There exists $1 \leqq j \leq p-1$ such that in W there is no element $X = \begin{bmatrix} 0 & A_X \\ B_X & 0 \end{bmatrix}$ with $(A_X)_2 = \ldots = (A_X)_j = 0$ and $(A_X)_{j+1} \neq 0$.

Suppose A holds. Then there exist at most $(q - 1)$ linearly independent elements $Y = \begin{bmatrix} 0 & A_Y \\ B_Y & 0 \end{bmatrix}$ in W with $(A_Y)_1 = \ldots = (A_Y)_{p-2} = 0$. In fact, for such Y_1, $(A_Y)_{p-1}$ is a solution of the equation $\langle (A_Y)_{p-1}, (B_X^j)^T \rangle = 0$ $(1 \leq k \leq p)$. Moreover any other $Z = \begin{bmatrix} 0 & A_Z \\ B_Z & 0 \end{bmatrix} \in W$ is such that for any $1 \leq k \leq p - 1$ $\langle (A_Z)_k, (B_X^j)^T \rangle = 0$. Therefore there exist at most $(q-1)(p-1)+1$ linearly independent elements in W.

If B holds with a similar reasoning we easily obtain that $W \leqq$ $\leqq (p-1)(q-1)+1$ as well.

REMARKS. In the same way other bounds on the dimension of M^m, can be obtained preventing the existence of non-holomorphic $(1,1)$-geodesic (respectively minimal for the non-compact case) isometric immersion into other classical irreducible Hermitian symmetric manifolds. For instance if N is the complex quadric $Q_c \subset \mathbb{C}P^{n+1}$ isometric to $SO(n+2)/(SO(2) \times SO(n))$ (respectively $SO(2,n)/(SO(2) \times SO(n))$) we can prove analogously that if $m > 2$ and $\varphi: M^m \to N$ is a $(1,1)$-geodesic (respectively minimal) isometric immersion, the φ is $\pm$ holomorphic.

The authors were informed that Ohnita and Udagawa [O-U] have obtained this result using different methods.

In a forthcoming paper we analyse the minimal isometric immersions from a Kähler manifold into a real Grassmannian manifold.

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