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A Lattice of Homomorphs.

PAZ JIMÉNEZ SERAL (*)

Preliminary notes.

In this paper all groups are finite and soluble. The homomorph $h(\mathcal{B})$ for a boundary \mathcal{B} consists of all « \mathcal{B} -perfect groups», namely all those groups that have no \mathcal{B} -groups among their epimorphic images. The boundary $b(\mathcal{H})$ for a homomorph \mathcal{H} consists of all groups G such that $G \notin \mathcal{H}$ and if $1 \neq N \leq G$, then $G/N \in \mathcal{H}$. The maps h and b are mutually inverse bijections between the set of non-empty homomorphs and the set of boundaries. Let \mathcal{H} be a homomorph. We recall from [4] that the class $D\mathcal{H}$ of \mathcal{H} comprises all groups G such that $\text{Cov}_{\mathcal{H}}(G) \neq \emptyset$ namely all those groups that have \mathcal{H} -covering subgroups. $D\mathcal{H}$ is also a homomorph. We study in [6] the set

$$\mathbb{H}(\mathcal{U}) = \{\mathcal{H} \mid D\mathcal{H} = \mathcal{U}\}, \text{ where } \mathcal{U} \text{ is a homomorph.}$$

Those homomorphs \mathcal{H} such that $D\mathcal{H} = \mathcal{U}$ behave with regard to \mathcal{U} in a somewhat similar way to the Schunck classes with regard to the whole universe of soluble groups. The class $\mathcal{A}(\mathcal{U})$ (see (2.1) of [6]) is introduced in order to characterize the homomorphs \mathcal{H} of $\mathbb{H}(\mathcal{U})$, when $\mathbb{H}(\mathcal{U}) = \emptyset$ or $|\mathbb{H}(\mathcal{U})| = 1$ and to study the relation of usual containment in $\mathbb{H}(\mathcal{U})$. The class $\mathcal{A}(\mathcal{U})$ consists of those primitive groups G in \mathcal{U} that satisfy:

If $M < X$ and $X/\text{core}_X M \cong G$, we have $M \in \mathcal{U}$ if and only if $X \in \mathcal{U}$.

Let \mathcal{P} denote the class of finite soluble primitive groups.

If $\mathbb{H}(\mathcal{U}) \neq \emptyset$, the minimum in $\mathbb{H}(\mathcal{U})$ with regard to the relation of containment is $\mathcal{H} = h((b(\mathcal{U}) - \mathcal{P}) \cup \mathcal{A}(\mathcal{U}))$ (see [6], (3.3)).

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In this paper we study the relation of strong containment in $\mathbb{H}(\mathcal{U})$ given by

1 DEFINITION. Let \mathcal{U} be a homomorph. Let $\mathcal{X}, \mathcal{Y} \in \mathbb{H}(\mathcal{U})$. We say that \mathcal{X} is strongly contained in \mathcal{Y} , and write $\mathcal{X} \ll \mathcal{Y}$ if, for each $G \in \mathcal{U}$ an \mathcal{X} -covering subgroup of G is contained in some \mathcal{Y} -covering subgroup of G .

For a homomorph \mathcal{C} , we denote $\tilde{\mathcal{C}} := h(b(\mathcal{C}) \cap \mathcal{P})$. For every group $G \in \mathbf{D}\mathcal{C}$ we have: $\text{Cov}_{\mathcal{C}}(G) = \text{Cov}_{\tilde{\mathcal{C}}}(G)$ (see [6], (1.8)).

2 LEMMA. Let \mathcal{C} be a homomorph. We denote $\alpha(\mathcal{C}) := \{G \in \mathbf{D}\mathcal{C} \mid \text{if } H \in \text{Cov}_{\mathcal{C}}(G), H \cap \text{Soc } G = 1\}$.

We have:

- a) $\alpha(\mathcal{C}) = \alpha(\tilde{\mathcal{C}}) \cap \mathbf{D}\mathcal{C}$.
- b) $\tilde{\mathcal{C}} = h(\alpha(\mathcal{C}))$.

PROOF. a) It is evident by the definition.

b) Since $b(\tilde{\mathcal{C}}) = b(\mathcal{C}) \cap \mathcal{P}$, we have $b(\tilde{\mathcal{C}}) \subseteq \alpha(\tilde{\mathcal{C}}) \cap \mathbf{D}\mathcal{C} = \alpha(\mathcal{C})$ and therefore $h(\alpha(\mathcal{C})) \subseteq h(b(\tilde{\mathcal{C}})) = \tilde{\mathcal{C}}$. Since $\tilde{\mathcal{C}} = h(\alpha(\tilde{\mathcal{C}}))$ (see [2], VI (1.4)) and $\alpha(\mathcal{C}) \subseteq \alpha(\tilde{\mathcal{C}})$, we have $\tilde{\mathcal{C}} = h(\alpha(\tilde{\mathcal{C}})) \subseteq h(\alpha(\mathcal{C}))$.

Let us recall now the following

3 DEFINITION ([5] and [3] (8.2)). Let $\mathcal{B} \subset \mathcal{P}$. We define $\mathcal{B}_0 = \mathcal{B}$, and if \mathcal{B}_i has already been defined, let

$$\mathcal{B}_{i+1} = \left\{ (X/C_X(V)) [V] \mid H \leq X \leq K < G = KF(G) \in \mathcal{B}_i, \right. \\ \left. H \in \text{Cov}_{h(\mathcal{B}_i)}(K), \text{ is } X\text{-composition factor of } F(G) \right\}.$$

We denote by \mathcal{B}^∞ the union of all class \mathcal{B}_i previously defined.

In a similar way to (8.3) from [3] we have

4 PROPOSITION. Let \mathcal{C} be a homomorph and $\mathcal{B} \subset \mathcal{P}$ such that $\mathcal{B} \subseteq \alpha(\mathcal{C})$. We have that $\mathcal{B}^\infty \subseteq \alpha(\mathcal{C})$ (in particular $\alpha(\mathcal{C})^\infty = \alpha(\mathcal{C})$).

PROOF. Let us prove that $\mathcal{B}_i \subseteq \alpha(\mathcal{C})$ for every $i \in \mathbb{N}$. We proceed by induction on i . We have that $\mathcal{B} = \mathcal{B}_0 \subseteq \alpha(\mathcal{C})$. Suppose $\mathcal{B}_i \subseteq \alpha(\mathcal{C})$. Let $B \in \mathcal{B}_{i+1}$. There exists $G \in \mathcal{B}_i \subseteq \alpha(\mathcal{C})$, $Y \leq X \leq K$, K complement of $F(G)$, $H \in \text{Cov}_{h(\mathcal{B}_i)}(K)$, V, W, X -subgroups of $F(G)$, $V/W, X$ -composition

of $F(G)$ such that $B = X/C_X(V/W)[(V/W)]$. Since $\mathcal{B}_i \subseteq a(\mathcal{D}) \subseteq a(\tilde{\mathcal{D}})$, by [1] (2.2), we have $\tilde{\mathcal{D}} \ll h(\mathcal{B}_i)$, hence there exists $H \in \text{Cov}_{\tilde{\mathcal{D}}}(K)$ such that $H \leq Y$. As $G \in a(\mathcal{D}) \subseteq \mathbf{D}\mathcal{D}$, we have $H \in \text{Cov}_{\mathcal{D}}(K) \subseteq \text{Cov}_{\mathcal{D}}(G)$. Besides, it can be confirmed that

$$B = X/C_X(V/W)[(V/W)] \cong XV/C_X(V/W)W.$$

By the properties of covering subgroups $H \in \text{Cov}_{\mathcal{D}}(XV)$ and

$$HC_X(V/W)W/C_X(V/W)W \in \text{Cov}_{\mathcal{D}}(XV/C_X(V/W)W),$$

therefore $B \in \mathbf{D}\mathcal{D}$. We know from [3] (8.3), that $B \in a(\tilde{\mathcal{D}})$, so we can deduce that $B \in a(\tilde{\mathcal{D}}) \cap \mathbf{D}\mathcal{D} = a(\mathcal{D})$.

Below we study the relation « \ll » in $\mathbb{H}(\mathcal{U})$.

5 PROPOSITION. Let $\mathcal{X}, \mathcal{Y} \in \mathbb{H}(\mathcal{U})$. We have $\mathcal{X} \ll \mathcal{Y}$ if and only if $\tilde{\mathcal{X}} \ll \tilde{\mathcal{Y}}$.

PROOF. \Leftarrow) It is evident from that comment before Lemma 2.

\Rightarrow) We have $b(\tilde{\mathcal{Y}}) = b(\mathcal{Y}) \cap \mathcal{P}$. By definition of \ll and $a(\mathcal{X})$, we have that $b(\mathcal{Y}) \cap \mathcal{P} = b(\mathcal{Y}) \cap \mathbf{D}\mathcal{Y} \subseteq a(\mathcal{X})$. Moreover, $a(\mathcal{X}) \subseteq a(\tilde{\mathcal{X}})$, hence $b(\tilde{\mathcal{Y}}) \subseteq b(\tilde{\mathcal{X}})$ and by [1] (2.2), $\tilde{\mathcal{X}} \ll \tilde{\mathcal{Y}}$.

Since the mapping $\mathcal{X} \rightarrow \tilde{\mathcal{X}}$ from $\mathbb{H}(\mathcal{U})$ to the set of Schunck classes is injective (see [6], 3.1), $\mathbb{H}(\mathcal{U})$ can be considered a subset of the Schunck classes ordered by \ll .

In the examples described in [6] (1.9), (3.8), (3.9), $(\mathbb{H}(\mathcal{U}), \ll)$ has a lattice structure. In these examples we have $\mathcal{A}(\mathcal{U}) = a(\mathcal{N})$. In this respect, we can say:

6 PROPOSITION. Let \mathcal{U} be a homomorph and \mathcal{N} the minimum for \subseteq in $\mathbb{H}(\mathcal{U})$. The following statements are equivalent:

- a) $\mathcal{A}(\mathcal{U}) = a(\mathcal{N})$;
- b) $\mathcal{A}(\mathcal{U})^\infty = \mathcal{A}(\mathcal{U})$.

PROOF. $a) \Rightarrow b)$ It follows immediately from Proposition 4.

$b) \Rightarrow a)$ By $b)$ we obviously have $\mathcal{A}(\mathcal{U})^\infty \cap h(\mathcal{A}(\mathcal{U})) = \emptyset$. By [3] (8.4), we have $\mathcal{A}(\mathcal{U}) \subseteq a(h(\mathcal{A}(\mathcal{U})))$. By [6] (3.3), $h(\mathcal{A}(\mathcal{U})) = \tilde{\mathcal{N}}$ and therefore $\mathcal{A}(\mathcal{U}) \subseteq a(\tilde{\mathcal{N}})$.

Besides, $\mathfrak{a}(\mathfrak{u}) \subseteq \mathfrak{u} = \mathbf{D}\mathfrak{M}$ implies $\mathfrak{a}(\mathfrak{u}) \subseteq \mathfrak{a}(\mathfrak{M}) \cap \mathbf{D}\mathfrak{M} = \mathfrak{a}(\mathfrak{M})$. By [6] (1.7), we have $\mathfrak{a}(\mathfrak{M}) \subseteq \mathfrak{a}(\mathfrak{u})$ and therefore the equality.

7 THEOREM. Let \mathfrak{u} be a homomorph such that $b(\mathfrak{u}) \cap \mathcal{P} = \emptyset$. (These homomorphs are known as totally unsaturated).

$(\mathbb{H}(\mathfrak{u}), \ll)$ is a lattice if and only if $\mathfrak{a}(\mathfrak{u})^\infty = \mathfrak{a}(\mathfrak{u})$.

PROOF. \Rightarrow By the proposition above and [6] (1.7), it suffices to prove that $\mathfrak{a}(\mathfrak{u}) \subseteq \mathfrak{a}(\mathfrak{M})$. Let $G \in \mathfrak{a}(\mathfrak{u})$. Let $\mathfrak{X} = h(b(\mathfrak{u}) \cup \{G\})$. By [6] (2.3), $\mathfrak{X} \in \mathbb{H}(\mathfrak{u})$. Since \ll implies \subseteq , the infimum of $\{\mathfrak{X}, \mathfrak{M}\}$ must be \mathfrak{M} . Thus $\mathfrak{M} \ll \mathfrak{X}$, therefore $\mathfrak{M} \ll \mathfrak{X}$ and consequently $b(\mathfrak{X}) \subseteq \mathfrak{a}(\mathfrak{M})$. As $\{G\} = b(\mathfrak{X})$, we have that

$$G \in \mathfrak{a}(\mathfrak{M}) \cap \mathfrak{u} = \mathfrak{a}(\mathfrak{M}) \cap \mathbf{D}\mathfrak{M} = \mathfrak{a}(\mathfrak{M}).$$

\Leftarrow) Let $\mathfrak{X}, \mathfrak{Y} \in \mathbb{H}(\mathfrak{u})$. Recall from [5] Theorem A that

$$\tilde{\mathfrak{X}} \wedge \tilde{\mathfrak{Y}} = h((b(\tilde{\mathfrak{X}}) \cup b(\tilde{\mathfrak{Y}}))^\infty).$$

By Proposition 6 we have $\mathfrak{a}(\mathfrak{u})^\infty = \mathfrak{a}(\mathfrak{u}) = \mathfrak{a}(\mathfrak{M})$. Since $b(\tilde{\mathfrak{X}}) \cup b(\tilde{\mathfrak{Y}}) \subseteq \mathfrak{a}(\mathfrak{M})$, by Proposition 4, we have that $(b(\tilde{\mathfrak{X}}) \cup b(\tilde{\mathfrak{Y}}))^\infty \subseteq \mathfrak{a}(\mathfrak{M})$ and therefore $b(\tilde{\mathfrak{X}} \wedge \tilde{\mathfrak{Y}}) \subseteq \mathfrak{a}(\mathfrak{u})$. By [6] (2.3), we have that $\mathfrak{X} = h(b(\mathfrak{u}) \cup b(\tilde{\mathfrak{X}} \wedge \tilde{\mathfrak{Y}})) \in \mathbb{H}(\mathfrak{u})$ and it can easily be confirmed that $\mathfrak{X} = \mathfrak{X} \wedge \mathfrak{Y}$.

Now let, $\mathfrak{Y} = h(\mathfrak{a}(\mathfrak{X}) \cap \mathfrak{a}(\mathfrak{Y}))$. Again by the characterization in [6] (2.3) and (3.1), of the homomorphs in $\mathbb{H}(\mathfrak{u})$ we have that $\mathfrak{Z} = \mathfrak{Y} \cap \mathfrak{u} \in \mathbb{H}(\mathfrak{u})$, and $\mathfrak{Y} = \mathfrak{Z}$. It can be confirmed that $\mathfrak{Z} = \mathfrak{X} \vee \mathfrak{Y}$.

8 PROPOSITION. Let \mathfrak{u} be a totally unsaturated homomorph such that $(\mathbb{H}(\mathfrak{u}), \ll)$ is a lattice. For every $\mathfrak{X}, \mathfrak{Y} \in \mathbb{H}(\mathfrak{u})$ we have:

a) $\widetilde{\mathfrak{X} \wedge \mathfrak{Y}} = \tilde{\mathfrak{X}} \wedge \tilde{\mathfrak{Y}}$.

b) $\mathfrak{X} \ll \mathfrak{Z} \neq \mathfrak{u}$ implies $\mathfrak{X} = \mathfrak{Z}$ if and only if $|b(\mathfrak{X}) \cap \mathcal{P}| = 1$.

PROOF. a) It is clear from the previous proof that

$$b(\mathfrak{X} \wedge \mathfrak{Y}) \cap \mathcal{P} = b(\tilde{\mathfrak{X}} \wedge \tilde{\mathfrak{Y}}).$$

b) \Rightarrow) If $|b(\mathfrak{X}) \cap \mathcal{P}| \neq 1$, we can have $\emptyset \neq \mathfrak{B} \subset b(\mathfrak{X}) \cap \mathcal{P} \subseteq \mathfrak{a}(\mathfrak{u})$. Now $\mathfrak{Z} = h(b(\mathfrak{u}) \cup \mathfrak{B}) \in \mathbb{H}(\mathfrak{u})$, $\mathfrak{Z} \neq \mathfrak{X}$ and $\mathfrak{X} \ll \mathfrak{Z} \neq \mathfrak{u}$ in contradiction with the hypothesis.

\Leftarrow) As $\tilde{\mathfrak{X}} = h(b(\mathfrak{X}) \cap \mathcal{P})$, $\tilde{\mathfrak{X}}$ is maximal, hence $\tilde{\mathfrak{X}} \ll \tilde{\mathfrak{Z}} \neq \mathfrak{S}$ implies $\tilde{\mathfrak{X}} = \tilde{\mathfrak{Z}}$ and by Proposition 5 we have the thesis.

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