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The Existence of Envelopes.

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ABSTRACT - In [7] (pg. 207) a theorem was given guaranteeing the existence of certain kinds of covers. A dual theorem proves the existence of envelopes. These theorems do not prove the existence of injective and pure injective envelopes. However, if we use Maranda's notion of an injective structure [11], we can show that a modification of arguments in [7] prove that envelopes exist for a wide class of injective structures. These envelopes include the injective and pure injective envelopes.

1. Introduction.

DEFINITION 1.1. If R is a ring and $\mathcal{F}$ a class of left R-modules, by an $\mathcal{F}$-preenvelope of a left R-module M we mean a linear map $\phi: M \to F$ where $F \in \mathcal{F}$ and such that for any linear $f: M \to F'$ where $F' \in \mathcal{F}$ there is a $g: F \to F'$ such that $g \circ \phi = f$. If furthermore, when $F' = F$ and $f = \phi$ the only such g are automorphisms of F, then $\phi: M \to F$ is called an $\mathcal{F}$-envelope of M. If envelopes exist they are unique up to isomorphism. Covers are defined dually to envelopes. It is not hard to see that if $\mathcal{F}$ is the class of injective modules then we get the usual injective envelopes. Similarly, we get the pure injective envelopes when $\mathcal{F}$ is the class of pure injective modules. For $\mathcal{F}$ some familiar class of modules, say the class of flat modules, $\mathcal{F}$-envelopes will simply be called flat envelopes.

If R is a noetherian domain, then flat envelopes exist for all modules if and only if R has global dimension at most 2 (see [7], pg. 202). Mar-

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tinez Hernández ([10] and [9]) and Asencio Mayor [10] have considered the question of the existence of flat envelopes in general. We will show that if a ring is right coherent then every left module has a pure injective flat envelope. From this we will deduce that if the ring is also pure injective as a left module, then every finitely presented left module will have a flat envelope. We note that over the group ring $\mathbb{Z}G$ with $G \neq 1$ a finite group, not even $\mathbb{Z}$ with the trivial G-action has a flat envelope although $\mathbb{Z} \to \mathbb{Z}G$ with $1 \to \sum g (g \in G)$ is a flat preenvelope (Akatsa [1]).

It is an open question whether every module has a flat cover. But it is now known that every left module of finite flat dimension over a right coherent ring has a flat cover [3].

We need the following:

PROPOSITION 1.2. If $\mathcal{F}$ is a class of modules closed under products, then a left R-module M has an $\mathcal{F}$-preenvelope if and only if there is a cardinal number $\aleph_\alpha$ such that for any linear map $M \to F$ with $F \in \mathcal{F}$ there is a factorization $M \to G \to F$ with $G \in \mathcal{F}$ and $\operatorname{Card} G \leqslant \aleph_\alpha$.

PROOF. The condition is necessary, for let $\aleph_\alpha = \operatorname{Card} F$ where $M \to F$ is an $\mathcal{F}$-preenvelope. Conversely, given such an $\aleph_\alpha$, it is easy to construct or product $\prod G_i$ with $G_i \in \mathcal{F}$, $\operatorname{Card} G_i \leqslant \aleph_\alpha$ for all i and a map $M \to \prod G_i$ which is an $\mathcal{F}$-preenvelope. ■

DEFINITION 1.3. A left R-module A is said to be absolutely pure it for every $n \geqslant 1$ and every finitely generated submodule $S \subset R^n$, $\operatorname{Hom}(S, A) \to \operatorname{Hom}(R^n, A) \to 0$ is exact.

We now get the known (D. Adams, [2]).

COROLLARY 1.4. Every left R-module M has an absolutely pure preenvelope.

PROOF. We fix the ring R. Let $\aleph_\beta$ be an infinite cardinal. We claim there is another cardinal $\aleph_{\beta'}$ such that if A is any absolutely pure left R-module and $V \subset A$ is a submodule with $\operatorname{Card} V \leqslant \aleph_\beta$, then there is a submodule V' of A with $V \subset V'$ such that $\operatorname{Card} V' \leqslant \aleph_{\beta'}$ and such that if $n \geqslant 1$ and if $S \subset R^n$ is any finitely generated submodule then any linear map $S \to V$ has an extension $R^n \to V'$.

To see this we consider the set of triples (R^n, S, f) with $S \subset R^n$ as above and $f\colon S \to V$ linear. For each such f choose an extension $g\colon R^n \to A$. Let $V' = V + \sum g(R^n)$ (with the sum over all (R^n, S, f)). From the construction of V' it is easy to see how to find $\aleph_{\beta'}$.

Repeating this procedure with $\aleph_{\beta'}$ playing the role of $\aleph_\beta$ we can find an ordinal $\aleph_{\beta''}$ and then a submodule $U'' \subset A$, $U' \subset U''$ with $\aleph_{\beta''}$ having the obvious properties. Hence we see that we have a sequence of cardinals $\aleph_\beta, \aleph_{\beta'}, \aleph_{\beta''}, \ldots$ and then a sequence $V \subset V' \subset V'' \subset \ldots$ of submodules of A with the obvious properties.

Letting $\aleph_\alpha = \sup \aleph_\beta^{(n)}$ (over $n \geqslant 0$) we have that if $B = UV^{(n)}$ then $\mathrm{Card}\, B \leqslant \aleph_\alpha$. From the construction we see that B is absolutely pure.

Hence if M is a left R-module, suppose $\mathrm{Card}\, M \leqslant \aleph_\beta$ for some $\aleph_\beta$. Let $\aleph_\alpha$ be as above. Then if $\sigma\colon M \to A$ is linear where A is absolutely pure, let $V = \sigma(M)$. Then $\mathrm{Card}\, V \leqslant \aleph_\beta$. By the above there is an absolutely pure submodule $B \subset A$ with $V \subset B \subset A$ and $\mathrm{Card}\, B \leqslant \aleph_\alpha$. We then have the factorization $M \to B \to A$. The result now follows from the Proposition. ■

If G is a left or right R-module, let $G^+ = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$. Then G^+ is a pure injective R-module. G is pure injective if and only if the canonical injection $G \to G^{++}$ admits a retraction ([8] or [12]).

If G is a right and M a left R-module then $(G \otimes_R M)^+ \cong \cong \mathrm{Hom}_R(M, G^+)$. A sequence $0 \to M' \to M \to M'' \to 0$ is exact if and only if $0 \to (M'') \to M^+ \to (M') \to 0$ is exact. If R is right coherent and G is an absolutely pure right R-module then G^+ is a flat R-module. This is seen using the isomorphism

$$P \otimes_R (G^+) \cong \mathrm{Hom}_R(P, G)^+$$

for any finitely presented right R-module P.

2. The main result.

For a ring R, Maranda [11] calls an injective structure on the category of left R-modules a pair $(\mathcal{A}, \mathcal{F})$ where $\mathcal{A}$ is a class of linear maps between left R-modules and where $\mathcal{F}$ is a class of left R-module such that $F \in \mathcal{F}$ if and only if $\mathrm{Hom}(N, F) \to \mathrm{Hom}(M, F) \to 0$ is exact for all $M \to \to N \in \mathcal{A}$, and $M \to N \in \mathcal{A}$ if and only if $\mathrm{Hom}(N, F) \to \mathrm{Hom}(M, F) \to 0$ is exact for all $F \in \mathcal{F}$, and such that every left R-module M has an $\mathcal{F}$-preenvelopes $M \to F$ (so $M \to F \in \mathcal{A}$).

We recall that Eilenberg and Moore have essentially the same notion in [4] (but with different terminology).

Let $\mathcal{G}$ be a class of right R-modules. We say that an injective structure $(\mathcal{A}, \mathcal{F})$ is determined by $\mathcal{G}$ if and only if $M \to N \in \mathcal{A}$ if and only if $0 \to \to G \otimes M \to G \otimes N$ is exact for all $G \in \mathcal{G}$.

Then we have

THEOREM 2.1. If an injective structure $(\mathcal{A}, \mathcal{F})$ on the category of left R-modules is determined by a class G of right R-modules then every left R-module has an $\mathcal{F}$-envelope. If $\mathcal{G}$ is any set of right R-modules then there is a unique injective structure $(\mathcal{A}, \mathcal{F})$ determined by $\mathcal{G}$. In this case, $\mathcal{F}$ will consist of all $\mathcal{F}$ which are isomorphic to a direct summand of products of copies of the modules G^+ for $G \in \mathcal{G}$.

PROOF. We first need a sort of Zorn's lemma, namely:

LEMMA 2.2. Let F be any class of left R-modules and let M be some left R-module. Suppose $\mathcal{F}$ is closed under taking summands and that M has an $\mathcal{F}$-preenvelope, say $M \to F$. If for any well ordered system of $\mathcal{F}$-preenvelopes $((M \to F_i), (f_{ji}))$ there exists a map of $\varinjlim (M \to F_i) = M \to \to \varinjlim F_i$ into $M \to F$ (i.e. there is a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & \varinjlim F_i \arrow[d] \\ & F \end{tikzcd}$$

then M has an $\mathcal{F}$-envelope.

PROOF. The proof is a fairly straightforward modification of the proofs of Lemmas 2.1, 2.2 and 2.3 in [7]. There injective precovers $E \to \to M$ over a left noetherian ring were considered. Given an inductive system $((E_i \to M), (f_{ji}))$ of such precovers, $\varinjlim E_i$ is injective, so there is a commutative diagram

$$\begin{tikzcd} \varinjlim E_i \arrow[r] \arrow[d] & M \\ E \arrow[ur] & \end{tikzcd}$$

where $E \to M$ is an injective precover. The last condition in our hypothesis is seen to be exactly what is needed to carry through the rest of the argument.

We now immediately get the first part of the theorem, for if $((M \to F_i), (f_{ji}))$ is an inductive system of $\mathcal{F}$-precovers of M, then $M \to F_i \in \mathcal{A}$ for each i, so $0 \to G \otimes M \to G \otimes F_i$ is exact for all $G \in \mathcal{G}$.

Hence $0 \to G \otimes M \to G \otimes \varinjlim F_i$ is exact. So $M \to \varinjlim F_i \in \mathcal{A}$. But then if $M \to F$ is an $\mathcal{F}$-preenvelope, we have a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & \varinjlim F_i \arrow[d] \\ & F \end{tikzcd}$$

Then by the lemma, M has an $\mathcal{F}$-envelope. ∎

Now suppose $\mathcal{G}$ is a set of right R-modules. We let $\mathcal{A}$ be the class of all $M \to N$ such that $0 \to G \otimes M \to G \otimes N$ is exact for all $G \in \mathcal{G}$ and let $\mathcal{F}$ consist of al modules F which are isomorphic to direct summands of products of copies of the G^+ for $G \in \mathcal{G}$. If $(\mathcal{A}, \mathcal{F})$ is an injective structure, it is clearly the only one determined by $\mathcal{G}$.

Let $G \in \mathcal{G}$. Since $0 \to G \otimes M \to G \otimes N$ is exact for $M \to N \in A$ we get $\mathrm{Hom}(N, G^+) \to \mathrm{Hom}(M, G^+) \to 0$ exact. Hence by our choice of $\mathcal{F}$, we see that $\mathrm{Hom}(N, F) \to \mathrm{Hom}(M, F) \to 0$ is exact for all $F \in \mathcal{F}$. Since for $G \in \mathcal{G}$, $0 \to G \otimes M \to G \otimes N$ is exact if and only if $\mathrm{Hom}(N, G^+) \to$ $\to \mathrm{Hom}(M, G^+) \to 0$ is exact, we see that $M \to N \in \mathcal{A}$ if and only if $\mathrm{Hom}(N, F) \to \mathrm{Hom}(M, F) \to 0$ is exact for all $F \in \mathcal{F}$.

Now let M be any left R-module. Since $\mathcal{G}$ is a set, it is easy to construct an $F \in \mathcal{F}$ and a linear map $M \to F$ such that for any $G \in \mathcal{G}$ and any linear $M \to G^+$,

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & F \arrow[d, dashed] \\ & G^+ \end{tikzcd}$$

can be completed to a commutative diagram. But then for any $F' \in \mathcal{F}$, $\mathrm{Hom}(F, F') \to \mathrm{Hom}(M, F') \to 0$ is exact, so since $F \in \mathcal{F}$, $M \to F$ is an $\mathcal{F}$-preenvelope.

Now let L be such that $\mathrm{Hom}(N, L) \to \mathrm{Hom}(M, L) \to 0$ is exact for all $M \to N \in \mathcal{A}$. Letting $M = L$ in the above we get an $\mathcal{F}$-preenvelope $L \to F$. As noted, then $L \to F \in \mathcal{A}$, so by hypothesis $\mathrm{Hom}(F, L) \to$ $\to \mathrm{Hom}(L, L) \to 0$ is exact. But this means that L is a direct summand of F so $L \in \mathcal{F}$. This shows that $(\mathcal{A}, \mathcal{F})$ is an injective structure and so completes the proof.

Examples. If $\mathcal{G} = \{R\}$ then $\mathcal{A}$ is just the class of all injections and we get the usual injective envelopes.

If $\mathcal{G} = \{R/I \mid I$ is a finitely generated right ideal then $\mathcal{A}$ consistes of all pure injections and we get the pure injective envelopes.

PROPOSITION 2.3. If R is right coherent then every left R-module has a pure injective flat envelope.

PROOF. We claim there is a set $\mathcal{G}$ of absolutely pure right R-modules such that every absolutely pure right R-module A, $A \cong \varinjlim G_i$ with $G_i \in \mathcal{G}$ for each i. If suffices to use a modification of the proof of Corollary 1.3 to find a cardinal $\aleph_\alpha$ such that if $S \subset A$ with S finitely generates and A absolutely pure there is an absolutely pure submodule $B \subset A$ with $S \subset B$ and $\operatorname{Card} B \leq \aleph_\alpha$. Then choosing a set X with $\operatorname{Card} X = \aleph_\alpha$, let $\mathcal{G}$ be all absolutely pure right R-modules G with $G \subset X$ (as a set).

We now let $(\mathcal{A}, \mathcal{F})$ be the injective structure determined by $\mathcal{G}$. Then by the above, $M \to N \in \mathcal{A}$ if and only if $0 \to A \otimes M \to A \otimes N$ is exact for all absolutely pure right R-modules $\mathcal{A}$. Hence by Theorem 2.1, every left R-module has an $\mathcal{F}$-envelope. We only need argue that $\mathcal{F}$ consists of all pure injective flat modules. But since R is right coherent, A^+ is flat for every absolutely pure right R-module A (so in particular for $A = G \in \mathcal{G}$). Hence each $F \in \mathcal{F}$ is flat. Also each G^+ for $G \in \mathcal{G}$ is pure injective so each $F \in \mathcal{F}$ is pure injective and flat.

Conversely let F be flat and pure injective. We want to show that $F \in \mathcal{F}$. But F^+ is injective, so absolutely pure, hence $0 \to F^+ \otimes M \to F^+ \otimes N$ is exact for all $M \to N \in \mathcal{A}$. So

$$\operatorname{Hom}(N, F^{++}) \to \operatorname{Hom}(M, F^{++}) \to 0$$

is exact for $M \to N \in \mathcal{A}$. Since F is pure injective, F is a direct summand of F^{++}, so $\operatorname{Hom}(N, F) \to \operatorname{Hom}(M, F) \to 0$ is also exact for all $M \to N \in \mathcal{A}$. Hence $F \in \mathcal{F}$ is flat since R is right coherent. ■

COROLLARY 2.4. If R is right coherent and pure injective as a module on the left, then every finitely presented left R-module has a flat envelope.

PROOF. Let M be a finitely presented left R-module. If $\phi: M \to F$ is a pure injective flat envelope, then since F is flat and M finitely presented there is a factorization $M \xrightarrow{f} R^n \xrightarrow{g} F$. But R^n is flat and pure injective so $M \to R^n$ can be factored $M \xrightarrow{f} F \xrightarrow{h} R^n$. Then since $g \circ h \circ \phi = \phi$ and since $\phi: M \to F$ is an envelope, $g \circ h$ is automorphism of F, F is a direct summand of R^n, i.e. F is finitely generated and projective. Then if

$M \to G$ is linear with G flat, there is a factorization $M \to R^m \to G$. R^m is both flat and pure injective so

$$\begin{tikzcd} M \arrow[r] \arrow[dr] & F \arrow[d, dashed] \\ & R^m \end{tikzcd}$$

can be completed to a commutative diagram. This shows that $M \to G$ can be factored $M \xrightarrow{\phi} F \to G$ and so $M \to F$ is flat preenvelope and so an envelope. ■

REFERENCES

[1] V. K. A. AKATSA, *Flat envelopes and negative torsion functors*, Thesis, University of Kentucky (1991).

[2] D. D. ADAMS, *Absolutely pure modules*, Thesis, University of Kentucky (1978).

[3] R. BELSHOFF - E. ENOCHS - XU JINZHONG, *The existence of flat covers* (submitted).

[4] S. EILENBERG - J. C. MOORE, *Foundations of Relative Homological Algebra*, A.M.S. Memoirs, **55**, A.M.S., Providence (1965).

[5] E. ENOCHS, *Flat covers and flat cotorsion modules*, Proc. Amer. Math. Soc., **92** (1984), pp. 179-184.

[6] E. ENOCHS, *Minimal pure injective resolutions of flat modules*, J. Alg., **105** (1987), pp. 351-364.

[7] E. ENOCHS, *Injective and flat covers, envelopes and resolvents*, Israel J. Math., **39** (1981), pp. 189-209.

[8] L. GRUSON - M. RAYNAUD, *Critères de platitude et de projectivité*, Invent. Math., **13** (1971), pp. 1-89.

[9] J. MARTÍNEZ HERNÁNDEZ, *Relatively flat envelopes*, Comm. Alg., **14** (1986), pp. 867-884.

[10] J. MARTÍNEZ HERNÁNDEZ - J. ASENCIO MAYOR, *Flat envelopes in commutative rings*, Israel J. Math., **62** (1978), pp. 123-128.

[11] J. M. MARANDA, *Injective structures*, Tams, **110** (1964), pp. 98-135.

[12] R. B. WARFIELD, *Purity and algebraic compactness for modules*, Pacific J. Math., **28** (1969), pp. 699-719.

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