RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

M. KUZUCUOĞLU

A note on barely transitive permutation groups satisfying $min - 2$

Rendiconti del Seminario Matematico della Università di Padova, tome 90 (1993), p. 9-15

<http://www.numdam.org/item?id=RSMUP_1993__90__9_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 90 (1993)

A Note on Barely Transitive Permutation Groups Satisfying min-2.

M. KUZUCUOĞLU (*)

We recall that a group of permutations G of an infinite set Ω is called a barely transitive group if G acts transitively on Ω and every orbit of every proper subgroup is finite. An abstract group is called barely transitive, if it is isomorphic to some barely transitive permutation group. Recall also that [2] an infinite group G can be represented faithfully as a barely transitive permutation group if and only if G possesses a subgroup H such that $\bigcap_{x \in G} H^x = 1$ and $|K: K \cap H| < \infty$ for every proper subgroup $K < G$. The subgroup H is a point stabilizer of a barely transitive permutation group. Locally finite barely transitive groups are studied and the following theorem is proved in [5].

THEOREM [5] (1.2). *A locally finite barely transitive permutation group containing an element of order p and satisfying* min-p *is isomorphic to* C_{p^∞}.

In the proof of the above theorem we invoke the classification of finite simple groups. In this paper we will prove the same result for the prime 2 without using the classification of finite simple groups and extend the above theorem by reducing the min-p condition on H.

By assuming some restrictions on point stabilizer H one might expect to obtain some results about the structure of a locally finite barely transitive group. On the lines of this idea we have three propositions which might be of interest. Proposition 4 might have independent interest.

PROPOSITION 1. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If H satisfies* min-p, *then G satisfies* min-p.

(*) Indirizzo dell'A.: Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey.

PROOF. Let Q be a Sylow p-subgroup of H. Then by [6] Q is a Černikov group. Since H is a proper subgroup of G the group Q is a proper subgroup hence residually finite. But a residually finite Černikov group is finite. Hence Q is finite. Let P be a Sylow p-subgroup of G. If G is a p-group, then finiteness of $|K:K \cap H|$ for each proper subgroup $K < G$ implies that each proper subgroup of G is finite hence G satisfies min-p.

Assume that P is a proper subgroup of G. Since $P \cap H$ is a p-subgroup of H it is contained in a Sylow p-subgroup of H which is finite. Barely transitivity implies $|P:P \cap H| < \infty$ hence P is finite i.e. G satisfies min-p.

COROLLARY. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If G contains an element of order p and H satisfies* min-p, *then* $G \cong C_{p^\infty}$.

PROOF. Use Proposition 1 and the above Theorem.

THEOREM. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If G contains an element of order* 2 *and H satisfies* min-2, *then* $G \cong C_{2^\infty}$.

PROOF. By Proposition 1 G satisfies min-2. Let S be a Sylow 2-subgroup of G. Then S is Černikov [6] and so S has a divisible abelian normal subgroup of finite index. Residual finiteness of each proper subgroup of G [5] Lemma (2.13) and non residual finiteness of C_{2^∞} implies that either S is isomorphic to C_{2^∞} and so $G = S$ or S is proper and hence finite. In the first case we are done. We show that the second case is impossible.

Assume that G is a locally finite barely transitive group with finite Sylow 2-subgroups.

a) Each proper subgroups K of G satisfies $|K: O_{2'}(K)| < \infty$.

We prove this by induction on the order of Sylow 2-subgroups of proper subgroups of G.

Let $K < G$. If Sylow 2-subgroup of K is a trivial group, then $K = O_{2'}(K)$. By the Feit-Thompson theorem K is locally solvable.

Assume that $|L: O_{2'}(L)| < \infty$ if the order of Sylow 2-subgroup of L is less than the order of a Sylow 2-subgroup of K. Let x be an involution in K. Since K is residually finite then there exist a normal subgroup N_x of K such that $x \notin N_x$ and $|K: N_x|$ is finite. So the order of Sylow 2-subgroup of N_x is less than the order of Sylow 2-subgroup of K. By induction assumption $|N_x: O_{2'}(N_x)| < \infty$ and so $|K: O_{2'}(N_x)| < < |K: N_x|\,|N_x: O_{2'}(N_x)| < \infty$. As $O_{2'}(N_x) \operatorname{char} N_x \lhd K$, we have $O_{2'}(N_x) \lhd K$ hence $O_{2'}(K) \geq O_{2'}(N_x)$ and so $|K: O_{2'}(K)| < \infty$.

b) G is not simple.

Assume that G is simple with finite Sylow 2-subgroup S. For each involution x in G, the subgroup $C_G(x)$ is a proper subgroup and by the previous paragraph, $C_G(x)$ is almost locally solvable. The group G contains an elementary abelian 2-subgroup of order four. Otherwise there is a unique involution i in the centre of the Sylow 2-subgroup S of G. Since Sylow 2-subgroups are conjugate every Sylow 2-subgroup contains at most one conjugate of i, then by [4] Theorem (1.I.4) G is not simple. Hence we may assume that G contains an elementary abelian 2-subgroup V of order four. Let x_1, x_2, x_3 be the nontrivial involutions in V. Then

$$|C_G(x_i): O_{2'}(C_G(x_i))| < \infty, \qquad i = 1, 2, 3.$$

Since S is finite, the 2-rank of G is finite. Then by [1] Theorem 9

$$|G: \langle O_{2'}(C_G(x_i)): i = 1, 2, 3 \rangle| < \infty.$$

Since our group does not have a subgroup of finite index

$$G = \langle C_G(x_i): i = 1, 2, 3 \rangle.$$

But again $C_G(x_i)$ is proper subgroup of G for all $i = 1, 2, 3$. But by [5] Lemma 2.10 G cannot be generated by two proper subgroups. Hence $G = C_G(x_i)$ for some $i = 1, 2, 3$ which is impossible since $x_i \notin Z(G) = 1$. So G is not simple.

Since we have non-trivial normal subgroups either G has a maximal normal subgroup or G is a union of an ascending series of proper normal subgroups N_i. In the latter case there exists i such that $S \subseteq N_i \lhd G$ and by a Frattini argument

$$G = N_i N_G(S).$$

But G cannot be generated by two proper subgroups, and N_i is a proper subgroup so $G = N_G(S)$. Hence S is a normal subgroup of G. The group S is finite and normal whence [5] Lemma 2.2 implies $S \leq Z(G)$. Since S is finite abelian and a maximal 2-subgroup G/S is a $2'$-group. Let Σ be a local system consisting of finite subgroups and containing S. We can find such a local system since G is countable by [5] Lemma 2.14 and S is finite. Any element K_i in the local system is a finite subgroup of G containing S and $(|K_i/S|, |S|) = 1$. Then by the Schur-Zassenhaus theorem $K_i = S \times L_i$ as $S \leqslant Z(G)$. The group L_i is a $2'$-group. But this is true for all $K_i \in \Sigma$. Since the complements L_i of S are unique by embedding for each i, $L_i < L_{i+1}$ we get

$$G = S \times O_{2'}(G).$$

Since S is finite and G does not have a subgroup of finite index $G = O_{2'}(G)$ which is impossible since there exists nontrivial $x \in G$ such that $2 \mid o(x)$.

It remains to show the first possibility, that G contains a maximal normal subgroup is impossible. If there exists a maximal normal subgroup N, then G/N is a simple group satisfying min-2. By [5] Lemma 2.4 G/N is barely transitive and by the previous paragraph a barely transitive locally finite group satisfying min-2 cannot be simple.

This proof also says that in a locally finite barely transitive group $G \neq C_{2^\infty}$ all maximal 2-subgroups are infinite and indeed not *Černikov.*

PROPOSITION 2. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If for a fixed prime p every p-subgroup of H is solvable, then G is a union of proper normal subgroups. In particular G is not simple.*

PROOF. Assume if possible that, G is a locally finite barely transitive simple group. Let P be a maximal p-subgroup of G. The subgroup $P \cap H$ is a p-subgroup of H so it is solvable. By bare transitivity we have $|P : P \cap H| < \infty$ which implies that P is solvable. Therefore every p-subgroup of G is solvable. Every locally finite simple group is either linear or non-linear. But a non-linear locally finite simple group contains isomorphic copies of alternating groups A_n for all natural number n and hence contains finite p-subgroups of arbitrary derived length. Hence G cannot be a non-linear group. Then G is a linear group, but we show in [5] Lemma 2.11 that a locally finite barely transitive group cannot be a group of Lie type.

Let N be a proper normal subgroup of G. If N is a maximal normal subgroup of G, then G/N is a simple barely transitive group with HN/N its point stabilizer moreover every p-subgroup of HN/N is solvable. Hence there exists no maximal normal subgroup and G is a union of its proper normal subgroups.

PROPOSITION 3. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If the order of every simple section of H is bounded, then*

1) *G is not simple,*

2) *G can be written as a union of proper normal subgroups.*

PROOF. Assume if possible that such a simple barely transitive group exists. By [3] a non-linear locally finite simple group contains subgroups $C \triangleright D$ such that C/D is a direct product of finite alternating groups of unbounded orders. So C has normal subgroups D_i such that $C/D_i \cong \mathrm{Alt}(n_i)$ where $n_i \to \infty$. As G is barely transitive and C is a proper subgroup of G we have $|C : C \cap H| < \infty$. Let $K = \bigcap_{x \in C} (C \cap H)^x$. Then K is a normal subgroup of C and $|C:K|$ is finite.

Consider $KD_i/D_i \triangleleft C/D_i$. Since C/D_i is simple either *a*) $KD_i = C$ for infinitely many i or *b*) $KD_i = D_i$ for infinitely many i.

If $KD_i = C$ for some i where $|\mathrm{Alt}(n_i)|$ is greater than the order of the simple sections of H, then $K/(K \cap D_i) \cong KD_i/D_i = C/D_i \cong \mathrm{Alt}(n_i)$. Since $K \leq H$ then H involves a simple subgroup isomorphic to $\mathrm{Alt}(n_i)$. and this is impossible.

Assume that (*b*) holds. Then $KD_i = D_i$ for infinitely many i, i.e. $K \leq D_i$. But then $|C/D_i| \leq |C/K|$. Since K is fixed and D_i's are variable there exists j such that $|\mathrm{Alt}(n_j)| = |C/D_j| \nleq |C/K|$. Hence this case is also impossible. Hence if such a simple group exists, then it must be linear. But by [5] Lemma 2.11 a locally finite barely transitive group can not be isomorphic to a simple linear group. So G cannot be simple.

One can show easily as in Proposition 2 that there exists no maximal normal subgroup of G. Hence G can be written as a union of its proper normal subgroups.

In particular if H is locally soluble, then G satisfies (1) and (2) of the theorem.

PROPOSITION 4. *Let G be a locally finite barely transitive group and H be a point stabilizer of G. If a proper subgroup X of G involves an infinite simple group, such that $Y \triangleleft X$ and X/Y isomorphic to an infinite simple group, then*

a) *Y cannot be locally solvable.*

b) *Y cannot be finite.*

c) *H involves an infinite simple group isomorphic to X/Y.*

PROOF. *a*) Assume if possible that Y is locally solvable and X/Y is infinite simple. Since each proper subgroup of G is residually finite X is residually finite. Then for all $1 \neq x \in X$ we have $N_x \triangleleft X$ such that $x \notin N_x$ and $|X : N_x| < \infty$. But then $N_x Y/Y \trianglelefteq X/Y$. Since X/Y is infinite simple we have either $N_x Y = Y$ or $N_x Y = X$. Assume if possible that there exists $1 \neq x \in X$ such that $N_x Y = Y$. Then $N_x \leqslant Y$. But then $|X: Y| < |X: N_x| < \infty$ which is impossible. Hence we have $N_x Y = X$ for all $1 \neq x \in X$. Then $Y/(Y \cap N_x) \cong (YN_x)/N_x = X/N_x$. Finiteness of $|X/N_x|$ and locally solvableness of Y implies that, there exist $n_x \in N$ satisfying $X^{(n_x)} \leqslant N_x$ for all $x \in X$. If there exists an upper bound m for the set $I = \{n_x \mid 1 \neq x \in X\}$, then $X^{(m)} \leqslant N_x$ for all $1 \neq x \in X$. Hence $X^{(m)} \leqslant \bigcap_{x \in X} N_x = 1$ i.e. X is solvable which is not the case. Hence we may assume that there exists no upper bound for the set I. But then $X^{(n_x)} \leqslant N_x$ hence $\bigcap_{n_x \in I} X^{(n_x)} \subset \bigcap_{x \in X} N_x = 1$. But this implies X is locally solvable which is impossible. Indeed let $A = \langle x_1, x_2, \ldots, x_t \rangle$ be a finite subgroup of X. Then consider $A^{(1)}, A^{(2)}, \ldots$. If A is not solvable, then there exists $k \in N$ such that $1 \neq A^{(k)} = A^{(k+1)} = \ldots$. But then $A^{(k)} \leqslant \bigcap_{x_x \in I} X^{(n_x)} = 1$. Hence A is solvable. This proves (a).

b) If Y is finite, then by residual finiteness of X, there exists a normal subgroup N_Y of X such that $N_Y \cap Y = 1$ and X/N_Y has finite order. Then

$$N_Y Y/Y \trianglelefteq X/Y.$$

But X/Y is infinite simple. Hence $N_Y Y = X$, so $N_Y \cong N_Y/N_Y \cap Y \cong N_Y Y/Y = X/Y$. The group N_Y is residually finite hence finiteness of Y is impossible.

c) By bare transitivity for each proper subgroup X of G we have $|X: X \cap H| < \infty$, so there exists $K \leqslant X \cap H$ such that $K \triangleleft X$ and $|X: K| < \infty$. Then $KY/Y \triangleleft X/Y$. Since X/K is finite and X/Y infinite simple, then $KY = X$. But $K/K \cap Y \cong KY/Y = X/Y$. Hence $K \leqslant H \cap X$ and involves the infinite simple group $K/(K \cap Y)$.

So in case of H is locally solvable, G does not have a proper subgroup X which involves an infinite simple group.

Acknowledgment. – I am indebted to Prof. B. Hartley for valuable suggestions. I would also like to thank the TUBITAK for enabling us to have Prof. Hartley at METU.

REFERENCES

[1] V. V. Belyaev, *Locally finite groups with Černikov Sylow p-subgroups*, Alg. Logic, **20** (1981), pp. 393-402.

[2] B. Hartley, *On the normalizer condition and barely transitive permutation groups*, Alg. Logic, **13** (1974), pp. 334-340.

[3] B. HARTLEY, *Centralizing properties in simple locally finite groups and large finite classical groups*, J. Austral. Math. Soc., **49** (1990), pp. 502-513.

[4] O. H. KEGEL - B. WEHRFRITZ, *Locally Finite Groups*, North Holland, Amsterdam (1973).

[5] M. KUZUCUOĞLU, *Barely transitive permutation groups*, Arch. Math., **55** (1990), pp. 521-532.

[6] V. P. ŠHUNKOV, *On the minimality problem for locally finite groups*, Alg. Logic, **9** (1970), pp. 137-151.

Manoscritto pervenuto in redazione l'8 ottobre 1991.