RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

PIERCARLO CRAIGHERO
REMO GATTAZZO

Quintic surfaces of P^3 having a non singular model with $q = p_g = 0, P_2 \neq 0$

Rendiconti del Seminario Matematico della Università di Padova, tome 91 (1994), p. 185-198
<http://www.numdam.org/item?id=RSMUP_1994__91__185_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 91 (1994)

Quintic Surfaces of P^3 Having a Non Singular Model with $q = p_g = 0$, $P_2 \neq 0$.

PIERCARLO CRAIGHERO - REMO GATTAZZO (*)

ABSTRACT - In this paper we construct new examples of quintic surfaces of P^3 having only isolated singularities (r tacnodes or s double points with infinitely near a tacnode, $r + s = 4$) whose non singular model has irregularity $q = 0$ and invariants $p_g = 0$, $P_2 \neq 0$. In particular, an example is found of a quintic with non singular model of general type.

1. Introduction.

Let P^3 be the three-dimensional projective space over an algebraically closed field of characteristic 0, and let $\mathcal{F}$ be an algebraic surface of P^3.

By a Dd-point, or an isolated tacnode, on $\mathcal{F}$ we mean an isolated double point of $\mathcal{F}$ which has just a double straight line (s.line) infinitely near to it in the first neighbourhood.

By a DDd-point on $\mathcal{F}$ we mean an isolated double point of $\mathcal{F}$ which has just a Dd-point infinitely near to it in the first neighbourhood. Both a Dd-point and DDd-point must be uniplanar double points of $\mathcal{F}$.

In this paper we deal with quintic surfaces $\mathcal{F}_5$ of P^3, having on them, as singularities affecting adjointness, that is non rational singularities, only Dd-points or DDd-points, and having a non singular model (n.s.m.) $\tilde{\mathcal{F}}_5$ which is regular and has birational invariants $p_g = 0$, $P_2 \neq 0$.

The first of such examples is due to E. Stagnaro, who in [St.1] con-

(*) Indirizzo degli AA.: P. CRAIGHERO: Istituto di Meccanica Teorica e Applicata, Via Fagagna 208, Udine; R. GATTAZZO: Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Via Belzoni 7, Padova.

structs an $\mathcal{F}_5$ with 4 Dd-points, having as only bicanonical adjoint a reducible quadric. Afterward, in [St.2], a deep analysis of the conditions that Dd-points and DDd-points impose to the canonical and pluricanonical adjoints to a n.s.m. of an algebraic surface $\mathcal{F}$ was carried out, providing methods for computing birational invariants.

The aim of this paper is to complete the picture about quintics of P^3, with only Dd-points or DDd-points (normal quintic surfaces), whose n.s.m. is of the said type, in view of utilizing the surfaces here found in order to construct, by means of convenient procedures, new models of regular sextic threefolds in P^4 with $p_g = 0$, $P_2 \neq 0$, of whom very few examples are known up to now.

We examine the various possible configurations of the considered singularities constructing the following examples:

1) $\mathcal{F}_5$ with 4 Dd-points whose n.s.m. $\tilde{\mathcal{F}}_5$ has $p_g = 0$, $P_2 = 1$, $P_3 = 0$ (Enriques surface), but with an irreducible quadric cone as bicanonical adjoint;

2) $\mathcal{F}_5$ with 3 Dd-points and 1 DDd-points with $\tilde{\mathcal{F}}_5$ again an Enriques surface;

3) $\mathcal{F}_5$ with 2 Dd-points and 2 DDd-points with $\tilde{\mathcal{F}}_5$ having $p_g = 0$, $P_2 = 2$, $P_3 = 1$;

4) $\mathcal{F}_5$ with 1 Dd-points and 3 DDd-points with $\tilde{\mathcal{F}}_5$ having $p_g = 0$, $P_2 = 1$, $P_3 = 2$;

5) $\mathcal{F}_5$ with 4 DDd-points with $\tilde{\mathcal{F}}_5$ of general type, $p_g = 0$, $P_2 = 2$, $P_3 = 4$ (Godeaux surface).

We don't know of any example of surface $\mathcal{F}$ with isolated singularities, and with $\tilde{\mathcal{F}}$ having invariants as in case 4). As for example 5), we remark that the quintic surfaces of general type have been studied in [Y], but no example with $p_g = 0$ is exhibited there. The existence of the quintic 5) proposed by E. Stagnaro was also a problem raised by M. Reid in [R].

On the other hand recently others authors got interested in the study of normal quintic surfaces in P^3 with $p_g = 0$, $P_2 \neq 0$, for example Yonggu Kim, who in [K] proves that every Enriques surface is birationally isomorphic with a quintic surface in P^3.

2. Imposing a DDd-point on a surface.

We want to establish necessary conditions for a surface $\mathcal{F}$ of degree 4 or 5 in order that it has a given point P as DDd-point: presently, as far as the examples given in this paper are concerned, these conditions, toghether with those concerning a Dd-point, turn out to be also sufficient.

Within a linear isomorphism we can suppose $P = O$, that the principal tangent plane to $\mathcal{F}$ in O is the plane $\{Z = 0\}$, and that the Dd-point infinitely near to P is in the direction of OX_∞. With this the equation of $\mathcal{F}$ is of the following form:

$$(*) \quad \alpha Z^2 + AX^3 + BX^2Y + CXY^2 + DY^3 + (EX^2 + FXY + GY^2)Z +$$
$$+ (HX + KY)Z^2 + LZ^3 + A_1X^4 + B_1X^3Y + C_1X^2Y^2 + D_1XY^3 + E_1Y^4 +$$
$$+ (F_1X^3 + G_1X^2Y + H_1XY^2 + K_1Y^3)Z +$$
$$+ (L_1X^2 + M_1XY + N_1Y^2)Z^2 + (P_1X + Q_1Y)Z^3 + R_1Z^4 +$$
$$+ A_2X^5 + B_2X^4Y + C_2X^3Y^2 + D_2X^2Y^3 + E_2XY^4 + F_2Y^5 + \phi_4(X, Y)Z +$$
$$+ \phi_3(X, Y)Z^2 + \phi_2(X, Y)Z^3 + \phi_1(X, Y)Z^4 + \phi_0 Z^5 = 0,$$

with $\phi_l(X, Y)$ forms of degree l in (X, Y). Blowing up $\mathcal{F}$ in O, by means of the quadratic transformation of affine equations

$$\begin{cases} X = x \\ Y = xy \\ Z = xz, \end{cases}$$

we have to impose to its proper transform $\mathcal{F}'$ to have a Dd-point in O again, because this point of the exceptional divisor corresponds to the direction of OX_∞ in the blowing up, and this leads to the following systems of 6 equations:

$$(**) \qquad \begin{cases} A = B = C = 0, \\ 4\alpha A_1 = E^2, \\ 2\alpha B_1 = EF, \\ 2\alpha A_2 = EF_1 - 2A_1H. \end{cases}$$

In order for our point to be a DDd-point, we must also have $\alpha \neq 0$ and $D \neq 0$, the conditions being necessary for preventing P from becoming a triple point or a Dd-point. With this the generic quintic having a DDd-point in O, and the infinitely near Dd-point in the said direction, is the following

$$Z^2 + DY^3 + (EX^2 + FXY + GY^2)Z + (HX + KY)Z^2 + LZ^3 +$$
$$+ \frac{E^2}{4}X^4 + \frac{EF}{2}X^3Y + C_1X^2Y^2 + D_1XY^3 + E_1Y^4 +$$

$$+(F_1X^3+G_1X^2Y+H_1XY^2+K_1Y^3)Z+(L_1X^2+M_1XY+N_1Y^2)Z^2+$$

$$+(P_1X+Q_1Y)Z^3+R_1Z^4+\left(-\frac{E^2H}{4}+\frac{EF_1}{2}\right)X^5+B_2X^4Y+C_2X^3Y^2+$$

$$A_1X^2Y^3+E_2XY^4+F_2Y^5+(A_2X^4+B_3X^3Y+B_1X^2Y^2+E_3XY^3+E_4Y^4)Z+$$

$$+(A_3X^3+B_4X^2Y+B_0XY^2+A_0Y^3)Z^2+$$

$$+(A_5X^2+B_5XY+C_0Y^2)Z^3+(A_6X+B_6Y)Z^4+A_7Z^5=0$$

with $D \neq 0$.

REMARK 1. In order to calculate p_g and P_2 for the n.s.m. of the quintic surfaces which are constructed in the sequel one can procede as in [St.2], p. 11, in the case of a Dd-point, and p. 39, in the case of a DDd-point. As for the calculation of P_3, one can again follow [St.2], p. 18, for a Dd-point. On the other hand, for a DDd-point on a quintic surface $\mathcal{F}_5$, one gets the following result: a cubic surface $\mathcal{F}_3$ has a pull-back (via the desingularisation map) that is a threecanonical adjoint to $\mathcal{F}_5$ if and only if, beside verifying the conditions of adjointness imposed by the possible Dd-points of $\mathcal{F}_5$, is has in each of the DDd-points of $\mathcal{F}_5$ or a singular point, or a simple point with tangent plane coincident with the principal tangent plane to $\mathcal{F}_5$ in the considered DDd-point, and moreover, when the equation of $\mathcal{F}_5$ is of the form written above, and the DDd-point is O, we have

$$\mathcal{F}_3\colon\ Z+\frac{E}{2}X^2+aXY+bYZ+cZ^2+\phi_3(X,Y,Z)=0.$$

REMARK 2. In some cases one must prove that the constructed surface has no singularity beside those that have been explicitely imposed to it: for this it is useful to study how much a Dd-point or a DDd-point or a triple point makes the class of the surface decrease.

For the case of a Dd-point or a triple point this is done in [En.Ch.], Libro III, vol. 2, p. 612, where it is proved that, by acquiring a Dd-point or a triple point, a surface has its class lowered by at least 12. In the case of a DDd-point one gets the same estimate for the lowering of the class by verifying that the polar surfaces with respect to $\mathcal{F}_5$ of two generic points and $\mathcal{F}_5$ have in the DDd-point a multiplicity of intersection $\geqslant 12$. More precisely, within a linear isomorphism, one can suppose that the point is O, that $\{X+Y+Z=0\}$ is the principal tangent plane to $\mathcal{F}_5=\{F=0\}$ in O, and that the two generic points are X_∞ and Y_∞, whose polars are then $\mathcal{F}_x=\{F_x=0\}$ and $\mathcal{F}_y=\{F_y=0\}$. It turns out that the curve $\mathcal{C} = \mathcal{F}_x \cdot \mathcal{F}_y$ has in O a double point with two linear branches, each of then having with $\mathcal{F}_5$ a multiplicity of intersection $\geqslant 6$, which implies that $\mathcal{F}_x$, $\mathcal{F}_y$ and $\mathcal{F}_5$ intersect in O with multiplicity at least 12, so that the class of $\mathcal{F}_5$ will decrease of at least 12, by the presence of a DDd-point on it. This result, together with that relative to the Dd-point and the triple point, having a local character, it follows that the presence on an $\mathcal{F}_5$ of r Dd-points, t triple points and s DDd-points makes the class of $\mathcal{F}_5$ decrease of at least $(r + s + t)\,12$.

REAMRK 3. As for the irregularity q of the n.s.m. of every surface $\mathcal{F}$ considered in this paper which has four non coplanar double points of the Dd or DDd type, we can say that it is $q = 0$, arguing as in [St.2], p. 25, for Dd-points, and p. 40, for DDd-points.

3. $\mathcal{F}_5$ with 4 Dd-points.

Let us consider the quadric cone $\mathcal{Q}$: $XY + XZ + YZ = 0$; let $\mathcal{F}_3$ be a cubic surface which passes through X_∞, Y_∞, Z_∞, $P(1, 0, 0, 1)$, and having in them the same tangent planes as $\mathcal{Q}$: we can choose

$$\mathcal{F}_3\colon\ X^2(Y+Z) + Y^2(X+Z) + Z^2(X+Y) + XYZ + \\ + (XY + XZ + YZ)\,T + (X+Y)\,T^2 = 0\,.$$

We can then construct the pencil of quintics

$$\mathcal{F}_5\colon \lambda[(X^2(Y+Z) + Y^2(X+Z) + Z^2(X+Y) + XYZ + (XY+XZ+YZ)\,T + \\ + (X+Y)\,T^2](XY + XZ + YZ) + \mu(Z-T)^2(X+Y)\,T^2 = 0$$

whose generic element $\mathcal{F}_5$ is reduced and irreducible, non singular in codimension one, and has only X_∞, Y_∞, Z_∞, P as isolated singularities: this is easily seen by applying Bertini's Theorems (cfr. [Seg.], p. 200).

By construction $\mathcal{F}_5$ has a Dd-point in each of its singularities.

Proceeding as suggested in Remark 1, we can see that $\tilde{\mathcal{F}}_5$ has no canonical adjoint because X_∞, Y_∞, Z_∞, P are non coplanar points; that the only bicanonical adjoint to $\tilde{\mathcal{F}}_5$ is the pull-back of $\mathcal{Q}$; that $\tilde{\mathcal{F}}_5$ has no threecanonical adjoint because it would be the pull-back $\tilde{\mathcal{F}}$ of a cubic surface $\mathcal{F}$ with double biplanar points in the four Dd-points of $\mathcal{F}_5$, with one of the two principal tangent planes in each of them coincident with the tacnodal plane of $\mathcal{F}_5$ in it: this is not possible, because $\mathcal{F}_\infty = \mathcal{F} \cap \{T = 0\}$ would split in the three straight lines joining X_∞, Y_∞, Z_∞, hence it could not have as principal tangents in these points the tangents to $\mathcal{Q}_\infty = \mathcal{Q} \cap \{T = 0\}$, as it should be, by what has been said above (see Remark 1). With this, we can say that $\widetilde{\mathcal{F}}_5$ has the following plurigenera

$$p_g = 0\,, \qquad P_2 = 1\,, \qquad P_3 = 0$$

so that, having also irregularity $q = 0$, it is an Enriques surface.

The previous examples of quintics of Enriques type in [Ca.1], [St.1], [St.2] (whose n.s.m.) has a bicanonical adjoint who is the pull-back of a degenerate quadric $\mathcal{Q}$, whereas for this example the quadric $\mathcal{Q}$ is non degenerate.

4. $\mathcal{F}_5$ with 3 Dd-points and 1 DDd-point.

On the quadric $\mathcal{Q}$: $XY + XZ + YZ + uXT + YT + (-1-u)ZT = 0$ let us choose the four points X_∞, Y_∞, Z_∞, $O(1,0,0,0)$; we want to construct an $\mathcal{F}_5$ with Dd-point in X_∞, Y_∞, Z_∞, and tacnodal planes in them coincident with the tangent planes to $\mathcal{Q}$, and moreover having a DDd-point in O with the infinitely near Dd-point in the direction of the straight line $\{X = Y = Z\}$ which is tangent to $\mathcal{Q}$ in O. Let the principal tangent plane to $\mathcal{F}_5$ in O be $\{Z - 2X + Y = 0\}$. First we write down the equation of the generic quintic having the prescribed Dd-point in Z_∞, and uniplanar double points in X_∞, Y_∞, with as principal tangent plane in them the tangent plane there to $\mathcal{Q}$: it has the following equation

$$\begin{aligned}
&Z^3[X + Y - (1+u)T]^2 + \\
&+Z^2[X + Y - (1+u)T](A_1X^2 + BXY + CY^2 + DXT + AYT + T^2) + \\
&+Z[2A_1X^3Y + B_1X^2Y^2 + 2CXY^3 + 2A_1uX^3T + S_1X^2YT + F_1XY^2T + \\
&+2CY^3T + H_1X^2T^2 + K_1XYT^2 + L_1Y^2T^2 + 2(u+1)(2X - Y)T^3] + \\
&+A_1X^3Y^2 + CX^2Y^3 + 2uA_1X^3YT + B_2X^2Y^2T + 2CXY^3T + A_1u^2X^3T^2 + \\
&\qquad\qquad + S_2X^2YT^2 + F_2XY^2T^2 + CY^3T^2 - (u+1)(2X - Y)^2T^3 = 0\,.
\end{aligned}$$

By imposing in X_∞ and Y_∞ the required Dd-points one gets first the condition $u^2 + u + 1 = 0$, which makes the quadric $\mathcal{Q}$ a irreducible cone with vertex $V(1, \theta, 1, -1-\theta)$, which, in order to simplify the calculations, we can choose as new origin. The remaining conditions for being X_∞ and Y_∞ Dd-points, and for being O a DDd-point, allow to express, as functions of A and B, the remaining parameters, and thus we get the following linear systems Σ of dimension two of quintics

$$\mathscr{F}_5\colon\ \lambda\{2\theta(Y+X)^2Z^3+[(4\theta+3)Y^3+[(4\theta+3)X+(8\theta+12)T]Y^2+$$
$$+[(-5\theta-9)X^2+(4\theta+18)XT+(\theta+3)YT]Y+$$
$$+(-5\theta-9)X^3+(6-4\theta)X^2T+(\theta+3)XT^2]Z^2+$$
$$+[(8\theta+6)XY^3+[(-3\theta-6)X^2+(10\theta+24)XT+\theta T^2]Y^2+$$
$$[(-10\theta-18)X^3+(18-2\theta)X^2T+24(\theta+1)XT^2-12T^3]Y+$$
$$+(4\theta+3)X^2T^2+12(\theta+1)XT^3-6T^4]Z+$$
$$+(4\theta+3)X^2Y^3+[(-5\theta-9)X^2+(2\theta+12)XT+\theta T^2]XY^2+$$
$$+[(4\theta+3)X^2-12\theta XT+6(\theta+1)T^2]YT^2-6\theta XT^4\}-$$
$$-2\mu\{(YZ+XZ+XY)^2[(\theta+1)(Y-X)+(\theta+2)T]\}+$$
$$+\nu\{(YZ+XZ+XY)[Y^2Z+XY^2+2\theta XYZ-(\theta+1)X^2Z-(\theta+1)X^2T+$$
$$+4XYT+2(\theta+1)XZT-2\theta YZT-(\theta+1)ZT^2+\theta YT^2+XT^2]\}=0\,.$$

Always using Bertini's Theorems we can see that the generic quintic $\mathscr{F}_5$ of Σ has only the 4 imposed singularities.

$\tilde{\mathscr{F}}_5$ is regular and has $p_g=0$, $P_2=1$ (the pull-back of $\mathcal{Q}$ is the only bicanonical adjoint), $P_3=0$ (because the only cubic whose pull-back is threecanonical adjoint splits in $\mathcal{Q}$ and in the plane $\{T=0\}$, but this cubic doesn't satisfy the condition of threeadjointness in O (see Remark 1), not having the same tangent plane in O as $\mathscr{F}_5$.

5. $\mathscr{F}_5$ with 2 Dd-points and 2 DDd-points.

We want to construct an $\mathscr{F}_5$ with 2 Dd-points and with 2 DDd-points. We start from a quartic $\mathscr{F}_4$ which has

1) a DDd-point in O with the infinitely near Dd-point in the direction of OX_∞;

2) a DDd-point in Z_∞ with the infinitely near Dd-point in the direction of $Z_\infty X_\infty$;

In order to construct such an $\mathscr{F}_4$, we can take it as invariant under the homography

$$\sigma\colon (T,X,Y,Z)\to(Z,X,Y,T)$$

that interchanges conditions 1) and 2), and impose to it the conditions required for the DDd-point just in O. This leads to the following

$$\mathcal{F}_4\colon (Z+Y)^2(T+Y)^2 + dY^3(T+Y) + dY^3(Z+Y) + \\ +(eX^2+fXY+gY^2)(Z+Y)(T+Y) + \\ +\frac{e^2}{4}X^4 + \frac{ef}{2}X^3Y + c_1X^2Y^2 + d_1XY^3 + e_1Y^4 = 0\,,$$

with $d \neq 0$.

Now we can impose to $\mathcal{F}_4$ to be tangent in Y_∞ and in another point P_∞ ($\neq Y_\infty$) of the s.line $\{Z = T = 0\}$ to the plane $\{Z + T = 0\}$. This implies the conditions

$$1 + 2d + g + e_1 = 0\,, \qquad f + d_1 = 0\,, \qquad c_1 = \frac{f^2}{4} - e\,.$$

so that P_∞ is the point $(0, f, -e, 0)$, with $ef \neq 0$. Now we consider the pencil Φ of quintics generated by $\mathcal{F}_4 \cup \{Z + T = 0\}$ and $\mathcal{Q} \cup \{YZT = 0\}$, where $\mathcal{Q}$ is the quadric cone with vertex in Y_∞ and tangent to $\{Z = T = 0\}$. The generic quintic of Φ is

$$\mathcal{F}_5\colon\ \lambda(Z+T)[(Z+Y)^2(T+Y)^2 + dY^3(T+2Y+Z) + (eX^2+fXY+gY^2)\cdot \\ \cdot(Z+Y)(T+Y) + \frac{e^2}{4}X^4 + \frac{ef}{2}X^3Y + \left(\frac{f^2}{4} - e\right)X^2Y^2 - fXY^3 + \\ +(-1-2d-g)Y^4] + \mu[(Y+Z)T + YZ]YZT = 0\,.$$

(For $d = -(g+1)/2$, $\mathcal{F}_5$ acquires a triple point in Y_∞). $\mathcal{F}_5$ has only the four imposed singularities (by Bertini's Theorems). $\widetilde{\mathcal{F}}_5$ has $q = 0$, $p_g = 0$, $P_2 = 2$ (indeed the pull-back of the quadrics of the pencil $\alpha[TY + YZ] + \beta TZ = 0$ are the bicanonical adjoints to it), $P_3 = 1$ (the only three-canonical adjoint to it is the pull-back of the cubic $(Z + T)(X^2 + XY + YZ + YT + ZT) = 0$).

6. $\mathcal{F}_5$ with 1 Dd-point and 3 DDd-points.

We start from the generic quintic having a Dd-point in O, with tacnodal plane $\{X + Y + Z = 0\}$, and which is invariant under the three elements of the cyclic group $\mathcal{G}$ of order 3 generated by the homography

$$\sigma\colon (T, X, Y, Z) \to (T, Z, X, Y)\,.$$

This quintic has equation of the following form

$$Z^3(X + pY + T)^2 + X^3(Y + pZ + T)^2 + Y^3(Z + pX + T)^2 +$$
$$+ rXYZ(YZ + XZ + XY) + [f(X^2Y^2 + X^2Z^2 + Y^2Z^2) +$$
$$+ gXYZ(X + Y + Z)]T - 3XYZT^2 + u(X + Y + Z)^2T^3 = 0\,.$$

It will be enough then to impose to such a quintic to have a DDd-point in X_∞ and it will have a DDd-point in Y_∞ and in Z_∞ too, because these two points are in the $\mathcal{G}$-orbit of X_∞.

Let us choose as direction of the Dd-point infinitely near to X_∞ that of the s.line $\{Y + pZ + T = Y = 0\}$.

Imposing the conditions for a DDd-point we get the relations

$$p = i + 1\,, \quad u = -1\,, \quad f = 2i + 1\,, \quad g = 5i + 4\,, \quad r = 5i \quad (i^2 = -1)\,,$$

so that we get the required quintic

$$\mathcal{F}_5\colon\ X^3[(i + 1)Z + Y + T]^2 + Y^3[Z + (i + 1)X + T]^2 +$$
$$+ [(i + 1)Y + X + T]^2Z^3 + 5iXYZ(YZ + XZ + XY) +$$
$$+ T[(2i + 1)(Y^2Z^2 + X^2Z^2 + Y^2Z^2) + (5i + 4)XYZ(X + Y + Z)] -$$
$$-3T^2XYZ + T^3(Z + Y + X)^2 = 0\,.$$

We can see that $\mathcal{F}_5 = \{F = 0\}$ is non singular in codimension one in the following way.

Let π_0 be the tacnodal plane O to $\mathcal{F}_5$. We have $\mathcal{C}_0 = \pi_0 \cap \mathcal{F}_5 = r_1 + r_2 + \mathcal{C}$, where r_1, r_2 are distincts s.lines, principal tangents to the node of the plane irredicible cubic $\mathcal{C}$ in O, so that O is the only singular point of $\mathcal{C}_0$. Hence O is the only singular point of $\mathcal{F}_5$ belonging to π_0. Now, if $\mathcal{C}'$ were the singular locus of $\mathcal{F}_5$, we should have $\pi_o \cap \mathcal{C}' = \{O\}$, hence $O \in \mathcal{C}'$. On the other hand $\mathcal{C}' \subset \{F = 0\} \cap \{F_x = 0\}$, and, as one can verify, the curve $\{F = 0\} \cap \{F_x = 0\}$ has 2 linear branches centered in O, each of them having a multiplicity of intersection 4 in O with $\{F_y = 0\}$. This means that $\mathcal{C}' \not\subset \{F_y = 0\}$, as it should be.

We can also verify that the only singularities of $\mathcal{F}_5$ affecting adjointness are just the four we have imposed to it. Indeed the fixed points of σ are those of the s.line $\{X = Y = Z\}$, together with the two points

$$(0,\ -1 + i\sqrt{3},\ 2,\ -1 - i\sqrt{3}) \quad \text{and} \quad (0,\ -1 - i\sqrt{3},\ 2,\ -1 + i\sqrt{3})$$

(cyclic points of the plane $\{X + Y + Z = 0\}$): with the only exception of O, no one of these fixed points is a singular point of $\mathcal{F}_5$. A singular point of $\mathcal{F}_5$, different from O, X_∞, Y_∞, Z_∞, has then a $\mathcal{G}$-orbit of 3 points. Now a singular point P affecting; adjointness for $\mathcal{F}_5$ would be a Dd-point, DDd-point, or a triple point, but each of these points makes the class of $\mathcal{F}_5$ decrease of at least 12. On the other hand the $\mathcal{G}$-orbit of P consists of 3 points, and each of them is a singular point of the same kind of P: this would imply that the class of $\mathcal{F}_5$ is $\leqslant 80-(3+4)\,12<0$: absurd.

As for the previous examples $\widetilde{\mathcal{F}}_5$ has $q=p_g=0$. The only bicanonical adjoint to $\widetilde{\mathcal{F}}_5$ is the pull-back of the quadric

$$\mathcal{Q}\colon\ XY+XZ+YZ+XT+YT+ZT=0\,,$$

so that $P_2=1$.

As for P_3, we have to construct the linear system of the threecanonical adjoints, which is the pull-back of the linear system

$$\alpha[T(X+Y+Z)^2+X^2Y+X^2Z+XY^2+Y^2Z+XZ^2+$$
$$+YZ^2+i(XY^2+YZ^2+X^2Z)]+\beta XYZ=0\,.$$

For $\widetilde{\mathcal{F}}_5$ then we have $P_3=2$.

REMARK 4. We don't know of any surface, with only isolated singularities, having the same invariants as this $\widetilde{\mathcal{F}}_5$:

$$q=p_g=0\,,\qquad P_2=1\,,\qquad P_3=2\,.$$

7. $\mathcal{F}_5$ with 4 DDd-points.

A DDd-point imposes 15 conditions to a surface, 12 linear and 3 quadratic ones. So, an $\mathcal{F}_5$ with 4 DDd-points is subjected to 60 conditions, but depends only on 56 parameters: thus means that, for a generic choice of the 4 DDd-points, the problem has no solution. But by carefully choosing the configuration of the singularities, it is still possible to construct this example, which is undoubtedly the most significant of those exhibited in this paper.

First we construct the generic quintic surface having a uniplanar double point in O and which is invariant under the action of the 4 elements of the cyclic group $\mathcal{G}$ of order 4 generated by the homography

$$\sigma\colon (T,X,Y,Z)\to(X,Y,Z,T)\,.$$

This quintic $\mathscr{F}$ has equation of the form

$$(X + mY + aZ)^2 T^3 + [a^2 X^3 + XY(bX + cY) + m^2 Y^3 +$$
$$+(eX^2 + fXY + cY^2) Z + (bX + eY) Z^2 + Z^3] T^2 +$$
$$+[2aX^3 Y + eX^2 Y^2 + 2amXY^3 + (2amX^3 + fX^2 Y + fXY^2 + 2mY^3) Z +$$
$$+(cX^2 + fXY + bY^2) Z^2 + 2(mX + aY) Z^3] T +$$
$$+X^3 Y^2 + a^2 X^2 Y^3 + XY(2mX^2 + bXY + 2aY^2) Z +$$
$$+(m^2 X^3 + cX^2 Y + eXY^2 + Y^3) Z^2 + (mX + aY)^2 Z^3 = 0$$

The fixed points for the action of σ are

$$P_0 = (1,1,1,1), P_1 = (1, i, -1, -i), P_2 = (1, -1, 1-1), P_3 = (1, -i, -1, i)$$

($i^2 = -1$); whereas those that are fixed for the action of σ^2 are the points of the two s.lines

$$r' : \{Z - X = Y - T = 0\} = P_0 P_2 \quad \text{and} \quad r'' : \{X + Z = Y + T = 0\} = P_1 P_3 .$$

P_1, P_2, P_3 are simple points of $\mathscr{F}$, and in each of them $\mathscr{F}$ has the same tangent plane α: $X + Y + Z + T = 0$; moreover $r'' \subset \alpha \cap \mathscr{F}$, $P_0 \notin \mathscr{F}$.

On the other hand one can verify that every point $P \notin r' \cup r''$ has a $\mathcal{G}$-orbit of 4 distinct points

$$\{P, \sigma(P), \sigma^2(P), \sigma^3(P)\} .$$

Being $\mathscr{F}$ invariant under σ, it will be enough to impose to it to have a DDd-point in O, and it will have a similar singularity in $Z_\infty Y_\infty, X_\infty$, which are in the $\mathcal{G}$-orbit of O. By this we find the following 6 equations in the 6 parameters a, b, c, e, f, m:

$$-ab + a^2 e + 1 - a^5 = 0 ,$$
$$-mb + a^2 b + 2ame + e - af - 3a^4 m = 0 ,$$
$$2amb + c - ac + m^2 e - mf - 3a^3 m^2 = 0 ,$$
$$-6a^4 b + 4a^2 c + 12a^5 e - b^2 + 4abe - 4a^2 e^2 - 8am - 8a^4 m - 9a^8 = 0 ,$$
$$mb + 3a^3 mb - abc + a^2 ce - 4a^4 me - a^2 m^2 + a^3 m^2 - 3a^4 m + 3a^7 m = 0 ,$$
$$-6a^3 mb + 6a^5 b + 4amc + 18a^4 me - 2af + 2a^2 f - 3a^4 f + 2ab^2 + 2mbe -$$
$$-4a^2 be - bf + 2aef - 4ame^2 + 4a - 12a^3 m^2 - 18a^7 m - 4a^4 - 4m^2 = 0 .$$

By solving this system we find, among other possible solutions, the following three

$$a = r^2\,, \qquad e = -\frac{3r^2+6r-8}{3r^2+r-2}\,, \qquad b = \frac{7r^2+4r-6}{3r-2}\,,$$

$$m = -\frac{r^2-r+1}{2r^2-4r+1}\,, \quad f = -\frac{34r^2-18r-7}{29r^2+22r-33}\,, \quad c = -\frac{225r^2-156r-10}{5r^2+212r-163}$$

where r is a root of

$$r^3 + r^2 - 1 = 0\,.$$

We have now to verify that the obtained $\mathcal{F}_5$ has no singularity affecting adjointness beside $O, X_\infty, Y_\infty, Z_\infty$.

First we ascertain that $\mathcal{F}_5$ is non singular in codimension one in the following way.

r'', which belongs to $\mathcal{F}_5$, doesn't contain any singularity of $\mathcal{F}_5$: this can be seen by examining the two sections of $\mathcal{F}_5$ with the two distinct planes through r'' α and $\beta\colon X + Z = 0$. Moreover $\mathcal{F}_5$ has non singularity on r'.

Now, $\alpha \cap \mathcal{F}_5$ splits in r'' and in a plane quartic $\mathcal{C}_4$ having a double point in P_2. If then $\mathcal{F}_5$ were singular in codimension 1, and $\mathcal{C}$ were its singular locus, $\mathcal{C} \cap \alpha$ would contain a point $P \in \mathcal{C}_4 - (r' \cup r'')$. But $\mathcal{C}_4 \subset \mathcal{F}_5 \cap \alpha$ is σ-invariant, and the 4 points of the $\mathcal{G}$-orbit of P would be singular for $\mathcal{F}_5$ and for $\mathcal{C}_4$. $\mathcal{C}_4$ then would have at least 5 singular points, P_2 and the above 4 points. This can happen only if $\mathcal{C}_4$ splits in two s.lines and in a conic (possibly reducible). Through P_2 would then pass a s.line, component of $\mathcal{C}_4$: but this can be excluded by direct verification. So $\mathcal{F}_5$ is non singular in codimension 1.

Next we verify that $\mathcal{F}_5$ has no other isolated singularity affecting adjointness beside $O, X_\infty, Y_\infty, Z_\infty$. Indeed, if P were such a singular point, it would have a $\mathcal{G}$-orbit of 4 points, being $P \in r' \cup r''$, and every one of them would be a singularity of the same kind of P, that is a Dd-point, or DDd-point, or a triple point or a more complicated singular point, e.g. an oscnode, a 4-fold point.... $\mathcal{F}_5$ would have then at least 8 points, each of them making the class decrease of at least 12, which is impossible, being the class of the generic quintic surface $80 < 8.12$.

$\tilde{\mathcal{F}}_5$ which has again irregularity $q = 0$, has the following invariants:

$$p_g = 0\,, \qquad P_2 = 2\,, \qquad P_3 = 4\,.$$

$p_g = 0$ by the same reason as the previous examples. The bicanonical adjoints are the pull-backs of the pencil of quadrics $\{\lambda YT + \mu XZ = 0\}$

which are tangent to the four s.lines

$$\{Y=X+r^2Z=0\},\{Z=r^2T+Y=0\},\{T=r^2X+Z=0\},\{X=r^2Y+T=0\},$$

hence $P_2 = 2$. The threecanonical adjoints are the pull-backs of the cubics of P^3 satisfying the conditions of Remark 1. They form the following 3-dimensional linear system

$$\lambda\phi_0 + \mu\phi_1 + \nu\phi_2 + \rho\phi_3 = 0$$

where

$$\phi_0 = T\{(3r-2)[(X+mY+r^2Z)T+(r+1)XY+ \\ +(-6r^2+2r+2)XZ+(-2r^2-5r+5)YZ\},$$

$$\phi_1 = X\{(3r-2)[(Y+mZ+r^2T)X+(r+1)YZ]+ \\ +(-6r^2+2r+2)YT+(-2r^2-5r+5)ZT\},$$

$$\phi_2 = Y\{(3r-2)[(Z+mT+r^2X)Y+(r+1)ZT]+ \\ +(-6r^2+2r+2)ZX+(-2r^2-5r+5)TX\},$$

$$\phi_3 = Z\{(3r-2)[(T+mX+r^2Y)Z+(r+1)TX]+ \\ +(-6r^2+2r+2)TY+(-2r^2-5r+5)XY\},$$

hence $P_3 = 4$. Now we can calculate the Kodaira dimension of $\tilde{\mathscr{F}}_5$ by studying the threecanonical map

$$\phi_{3K}\colon \tilde{\mathscr{F}}_5 \to P_1^3,$$

where $\phi_{3K} = \phi \circ \pi$ with

$$\pi\colon \tilde{\mathscr{F}}_5 \to \mathscr{F}_5,$$

is the desingularisation map, and $\phi = (\phi_0, \phi_1, \phi_2, \phi_3)$. The irreducible variety $\phi_{3K}(\tilde{\mathscr{F}}_5) \subset P_1^3$ contains the s.line

$$s\colon \phi_{3K}(r'') = \{T_1 + Y_1 = X_1 + Z_1 = 0\}.$$

On the other hand $\phi_{3K}((1, -1, 1, -1)) = (1, -1, 1-1) \notin s$. It follows that $\phi_{3K}(\tilde{\mathscr{F}}_5)$ cannot be of dimension 1, and we have $\kappa(\tilde{\mathscr{F}}_5) = 2$: $\tilde{\mathscr{F}}_5$ is then of general type.

REFERENCES

[Ca.1] G. CASTELNUOVO, *Sulle superficie di genere zero*, Rend. Accad. Naz. Sci. dei XL, Mem. Mat., **10** (3) (1896), pp. 103-123.

[En.Ch.] F. ENRIQUES - O. CHISINI, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, Zanichelli, Bologna (ristampa 1985).

[K] YONGGU KIM, *On Normal Quintic Enriques Surfaces*, Ph.D. Thesis, University of Michigan (1991).

[R] M. REID, *Campedelli, versus Godeaux*, Symposia Mathematica Ist. Naz. Alta Matematica, Vol. XXXII (1989), pp. 309-365.

[Seg.] B. SEGRE, *Prodromi di geometria algebrica*, Cremonese, Roma (1972).

[St.1] E. STAGNARO, *Constructing Enriques Surfaces from Quintics in P_k^3*, *Algebraic Geometry. - Open Problems, Lectures Notes in Math.*, Springer Notes in Math. Springer-Verlag (1982), pp. 400-403.

[St.2] E. STAGNARO, *Canonical and pluricanonical adjoints to an algebraic surfaces, I*, R.T. n. 21 Ottobre 1991 (Dip. di Metodi e Modelli Matematici per le Sc. Appl. dell'Università di Padova).

[Y] JIN-GEN YANG, *On quintic surfaces of general type*, Trans. A.M.S., **295** (2) (1986), pp. 431-472.

Manoscritto pervenuto in redazione il 3 settembre 1992.