Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 92 (1994), p. 221-230
<http://www.numdam.org/item?id=RSMUP_1994__92__221_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 92 (1994)

Meromorphic Starlike Functions of Order α with Alternating Coefficients.

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ABSTRACT - Coefficient inequalities, distortion theorems and class preserving integral operators are obtained for meromorphic functions with alternating coefficients that are starlike of order α, $0 \leq \alpha < 1$.

1. Introduction.

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{1.1}$$

which are regular in the punctured disc $U^* = \{z: 0 < |z| < 1\}$. The Hadamard product or convolution of two functions $f, g \in \Sigma$ will be denoted by $f * g$. Let

$$\begin{cases} D^n f(z) = \dfrac{1}{z(1-z)^{n+1}} * f(z), \qquad n \geq 0, \\ D^n f(z) = \dfrac{1}{z}\left(\dfrac{z^{n+1} f(z)}{n!}\right)^{(n)}, \\ D^n f(z) = \dfrac{1}{z} + \displaystyle\sum_{k=1}^{\infty} \delta(n, k+1) a_k z^k, \end{cases} \tag{1.2}$$

where

$$\delta(n, k+1) = \binom{n+k+1}{n}. \tag{1.3}$$

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In [1] the authors obtained a new criterion for meromorphic starlike functions of order $\alpha (0 \leq \alpha < 1)$ via the basic inclution relationship $M_{n+1}(\alpha) \subset M_n(\alpha)$, $0 \leq \alpha < 1$, $n \in N_0 = N \cup \{0\}$, $N = \{1, 2, \ldots\}$, where $M_n(\alpha)$ is the class consisting of functions in Σ satisfying

$$\text{(1.4)} \quad \mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - 2\right\} < -\frac{n+\alpha}{n+1}, \quad |z| < 1, \quad 0 \leq \alpha < 1, \quad n \in N_0 .$$

The condition (1.4) is equivalent to

$$\text{(1.5)} \quad \text{(i)} \qquad \frac{D^{n+1}f(z)}{D^n f(z)} = \frac{(n+1) + (n+3-2\alpha)w(z)}{1+w(z)},$$

$w(z) \in H = \{w$ regular, $w(0) = 0$ and $|w(z)| < 1$, $z \in U = \{z: |z| < 1\}\}$, or equivalently,

$$\text{(1.6)} \quad \text{(ii)} \qquad \left| \frac{(n+1)\left(\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right)}{(n+1)\left(\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right) - 2(1-\alpha)} \right| < 1 .$$

We note that $M_0(\alpha) = \Sigma^*(\alpha)$, is the class of meromorphically starlike functions of order $\alpha (0 \leq \alpha < 1)$ and $M_0(0) = \Sigma^*$, is the class of meromorphically starlike functions. The class $M_n(0) = M_n$ was introduced by Ganigi and Uralegaddi [2].

Let σ_A be the subclass of Σ which consisting of functions of the form

$$\text{(1.7)} \qquad \begin{cases} f(z) = \dfrac{1}{z} + a_1 z - a_2 z^2 + a_3 z^3 \ldots, & a_k \geq 0, \\ f(z) = \dfrac{1}{z} + \displaystyle\sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k, & a_k \geq 0, \end{cases}$$

and let $\sigma_A^*(\alpha, n) = M_n(\alpha) \cap \sigma_A$.

In this paper coefficient inequalities, distortion theorems for the class $\sigma_A^*(\alpha, n)$ are determined. Techniques used are similar to these of Silverman [3] and Uralegaddi and Ganigi [4]. Finally, the class preserving integral operators of the form

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)\, dt \qquad (c > 0) \tag{1.8}$$

is considered.

2. Coefficient inequalities.

THEOREM 1. Let $f(z) = 1/z + \sum_{k=1}^{\infty} a_k z^k$. If

$$\sum_{k=1}^{\infty} [(n+1)\delta(n+1, k+1) - (n+2-\alpha)\delta(n, k+1)]|a_k| \leqslant \leqslant (1-\alpha), \tag{2.1}$$

then $f(z) \in M_n(\alpha)$.

PROOF. Suppose (2.1) holds for all admissible values of α and n. It suffices to show that

$$\left| \frac{(n+1)\left(\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right)}{(n+1)\left(\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right) - 2(1-\alpha)} \right| < 1 \qquad \text{for } |z| < 1.$$

We have

$$\left| \frac{(n+1)\left(\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right)}{(n+1)\left(\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right) - 2(1-\alpha)} \right| =$$

$$= \left| \frac{\sum_{k=1}^{\infty} (n+1)[\delta(n+1, k+1) - \delta(n, k+1)]\, a_k z^{k+1}}{2(1-\alpha) - \sum_{k=1}^{\infty} [(n+1)\delta(n+1, k+1) - (n+3-2\alpha)\delta(n, k+1)]\, a_k z^{k+1}} \right| \leqslant$$

$$\leqslant \frac{\sum_{k=1}^{\infty}(n+1)[\delta(n+1,k+1)-\delta(n,k+1)]\,|a_k|}{2(1-\alpha)-\sum_{k=1}^{\infty}[(n+1)\delta(n+1,k+1)-(n+3-2\alpha)\delta(n,k+1)]\,|a_k|}\,.$$

The last expression is bounded above by 1, provided

$$\sum_{k=1}^{\infty}(n+1)[\delta(n+1,k+1)-\delta(n,k+1)]\,|a_k| \leqslant$$

$$\leqslant 2(1-\alpha)-\sum_{k=1}^{\infty}[(n+1)\delta(n+1,k+1)-(n+3-2\alpha)\delta(n,k+1)]\,|a_k|$$

which is equivalent to

$$(2.2)\qquad \sum_{k=1}^{\infty}[(n+1)\delta(n+1,k+1)-(n+2-\alpha)\delta(n,k+1)]|a_k| \leqslant (1-\alpha)$$

which is true by hypothesis.

For functions in $\sigma_A^*(\alpha, n)$ the converse of the above theorem is also true.

THEOREM 2. A function $f(z)$ in σ_A is in $\sigma_A^*(\alpha, n)$ if and only if

$$\sum_{k=1}^{\infty}[(n+1)\delta(n+1,k+1)-(n+2-\alpha)\delta(n,k+1)]a_k \leqslant (1-\alpha).$$

PROOF. In view of Theorem 1 it suffices to show the only if part. Suppose that

$$(2.3)\qquad \mathrm{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}-2\right\}=$$

$$=\mathrm{Re}\left\{\frac{-1/z+\sum_{k=1}^{\infty}(-1)^{k-1}[\delta(n+1,k+1)-2\delta(n,k+1)]a_k z^k}{1/z+\sum_{k=1}^{\infty}(-1)^{k-1}\delta(n,k+1)a_k z^k}\right\}<$$

$$<-\frac{n+\alpha}{n+1}\,.$$

Choose values of z on the real axis so that $\left(\frac{D^{n+1}f(z)}{D^n f(z)}-2\right)$ is real. Upon clearing the denominator in (2.3) and letting $z \to -1$ through real values, we obtain

$$(n+1) - \sum_{k=1}^{\infty} (n+1)[\delta(n+1, k+1) - 2\delta(n, k+1)]\, a_k \geqslant$$

$$\geqslant (n+\alpha)\left[1 + \sum_{k=1}^{\infty} \delta(n, k+1)\, a_k\right]$$

which is equivalent to

$$\sum_{k=1}^{\infty} [(n+1)\,\delta(n+1, k+1) - (n+2-\alpha)\,\delta(n, k+1)]\, a_k \leqslant (1-\alpha).$$

This completes the proof of Theorem 2.

COROLLARY 1. If $f(z) \in \sigma_A^*(\alpha, n)$, then

$$a_k \leqslant \frac{1-\alpha}{(n+1)\,\delta(n+1, k+1) - (n+2-\alpha)\,\delta(n, k+1)} \qquad (k \geqslant 1).$$

Equality holds for the functions of the form

$$f_k(z) = \frac{1}{z} +$$

$$+ (-1)^{k-1} \frac{1-\alpha}{(n+1)\,\delta(n+1, k+1) - (n+2-\alpha)\,\delta(n, k+1)}\, z^k .$$

3. Distortion theorems.

THEOREM 3. If $f(z) = 1/z + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k$ $(a_k \geqslant 0)$, is in $\sigma_A^*(\alpha, n)$, then for $0 < |z| = r < 1$,

$$\text{(3.1)} \quad \frac{1}{r} - \frac{1-\alpha}{(n+1)\,\delta(n+1, 2) - (n+2-\alpha)\,\delta(n, 2)}\, r \leqslant |f(z)| \leqslant$$

$$\leqslant \frac{1}{r} + \frac{1-\alpha}{(n+1)\,\delta(n+1, 2) - (n+2-\alpha)\,\delta(n, 2)}$$

with equality for the function

$$\text{(3.2)} \quad f(z) = \frac{1}{z} +$$

$$+ \frac{1-\alpha}{(n+1)\,\delta(n+1, 2) - (n+2-\alpha)\,\delta(n, 2)}\, z \qquad \text{at } z = r, ir.$$

PROOF. Suppose $f(z)$ is in $\sigma_A^*(\alpha, n)$. In view of Theorem 2, we have

$$[(n+1)\delta(n+1,2)-(n+2-\alpha)\delta(n,2)]\sum_{k=1}^{\infty}a_k \leqslant$$

$$\leqslant \sum_{k=1}^{\infty}[(n+1)\delta(n+1,k+1)-(n+2-\alpha)\delta(n,k+1)]a_k \leqslant (1-\alpha)$$

which evidently yields

$$\sum_{k=1}^{\infty}a_k \leqslant \frac{1-\alpha}{(n+1)\delta(n+1,2)-(n+2-\alpha)\delta(n,2)}.$$

Consequently, we obtain

$$|f(z)| \leqslant \frac{1}{r}+\sum_{k=1}^{\infty}a_k r^k \leqslant \frac{1}{r}+r\sum_{k=1}^{\infty}a_k \leqslant$$

$$\leqslant \frac{1}{r}+\frac{1-\alpha}{(n+1)\delta(n+1,2)-(n+2-\alpha)\delta(n,2)}r.$$

Also

$$|f(z)| \geqslant \frac{1}{r}-\sum_{k=1}^{\infty}a_k r^k \geqslant \frac{1}{r}-r\sum_{k=1}^{\infty}a_k \geqslant$$

$$\geqslant \frac{1}{r}-\frac{1-\alpha}{(n+1)\delta(n+1,2)-(n+2-\alpha)\delta(n,2)}r.$$

Hence the results of (3.1) follow.

THEOREM 4. If $f(z)=1/z+\sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k$ $(a_k \geqslant 0)$, is in $\sigma_A^*(\alpha, n)$, then for $0<|z|=r<1$,

$$\frac{1}{r^2}-\frac{1-\alpha}{(n+1)\delta(n+1,2)-(n+2-\alpha)\delta(n,2)} \leqslant |f'(z)| \leqslant \tag{3.3}$$

$$\leqslant \frac{1}{r^2}+\frac{1-\alpha}{(n+1)\delta(n+1,2)-(n+2-\alpha)\delta(n,2)}.$$

The result is sharp, the extremal function being of the form (3.2).

PROOF. From Theorem 2, we have

$$[(n+1)\delta(n+1,2)-(n+2-\alpha)\delta(n,2)]\sum_{k=1}^{\infty} ka_k \leqslant$$

$$\leqslant \sum_{k=1}^{\infty}[(n+1)\delta(n+1,k+1)-(n+2-\alpha)\delta(n,k+1)]\,a_k \leqslant (1-\alpha)$$

which evidently yields

$$\sum_{k=1}^{\infty} ka_k \leqslant \frac{1-\alpha}{(n+1)\delta(n+1,2)-(n+2-\alpha)\delta(n,2)}\,.$$

Consequently, we obtain

$$|f'(z)| \leqslant \frac{1}{r^2}+\sum_{k=1}^{\infty} ka_k r^{k-1} \leqslant \frac{1}{r^2}+\sum_{k=1}^{\infty} ka_k \leqslant$$

$$\leqslant \frac{1}{r^2}+\frac{1-\alpha}{(n+1)\delta(n+1,2)-(n+2-\alpha)\delta(n,2)}\,.$$

Also

$$|f'(z)| \geqslant \frac{1}{r^2}-\sum_{k=1}^{\infty} ka_k r^{k-1} \geqslant \frac{1}{r^2}-\sum_{k=1}^{\infty} ka_k \geqslant$$

$$\geqslant \frac{1}{r^2}-\frac{1-\alpha}{(n+1)\delta(n+1,2)-(n+2-\alpha)\delta(n,2)}\,.$$

This completes the proof of Theorem 4.

Putting $n=0$ in Theorem 4, we get.

COROLLARY 2. If $f(z)=1/z+\sum_{k=1}^{\infty}(-1)^{k-1}a_k z^k$, $a_k \geqslant 0$, is in $\sigma_A^*(\alpha,0)=\sigma_A^*(\alpha)$, then for $0<|z|=r<1$,

$$\frac{1}{r^2}-\frac{1-\alpha}{1+\alpha} \leqslant |f'(z)| \leqslant \frac{1}{r^2}+\frac{1-\alpha}{1+\alpha}\,.$$

The result is sharp.

We observe that our result in Corollary 2 improves the result of Uralegaddi and Ganigi [4, Theorem 3 (Equation 4)].

4. Class preserving integral operators.

In this section we consider the class preserving integral operators of the form (1.6).

THEOREM 5. If $f(z) = 1/z + \sum_{k=1}^{\infty} (-1)^{k-1} a_k z^k$, $a_k \geq 0$, is in $\sigma_A^*(\alpha, n)$ then

$$F(z) = cz^{-c-1} \int_0^z t^c f(t)\, dt = \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{c}{c+k+1} a_k z^k, \qquad c > 0$$

belongs to $\sigma_A^*(\beta(\alpha, n, c), n)$, where

$$\beta(\alpha, n, c) = \frac{(n+1)(2+\alpha c)[\delta(n+1, 2) - \delta(n, 2)] - 2(1-\alpha)\delta(n, 2)}{(n+1)(2+c)[\delta(n+1, 2) - \delta(n, 2)] - 2(1-\alpha)\delta(n, 2)}.$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{(n+1)\delta(n+1, 2) - (n+2-\alpha)\delta(n, 2)} z.$$

PROOF. Suppose $f(z) \in \sigma_A^*(\alpha, n)$, then

$$\sum_{k=1}^{\infty} [(n+1)\delta(n+1, k+1) - (n+2-\alpha)\delta(n, k+1)] a_k \leq (1-\alpha).$$

In view of Theorem 2 we shall find the largest value of β for which

$$\sum_{k=1}^{\infty} \frac{[(n+1)\delta(n+1, k+1) - (n+2-\beta)\delta(n, k+1)]}{1-\beta} \cdot \frac{c}{c+k+1} a_k \leq 1.$$

It is sufficient to find the range of values of β for which

$$\frac{c[(n+1)\delta(n+1, k+1) - (n+2-\beta)\delta(n, k+1)]}{(1-\beta)(c+k+1)} \leq$$

$$\leq \frac{[(n+1)\delta(n+1, k+1) - (n+2-\alpha)\delta(n, k+1)]}{1-\alpha} \quad \text{for each } k$$

solving the above inequality for β we obtain

$$\beta \leqslant \frac{(n+1)(k+1+\alpha c)[\delta(n+1,k+1)-\delta(n,k+1)]-(k+1)(1-\alpha)\,\delta(n,k+1)}{(n+1)(k+1+c)[\delta(n+1,k+1)-\delta(n,k+1)]-(k+1)(1-\alpha)\,\delta(n,k+1)}\,.$$

For each α and c fixed let

$$F(k) = \\ = \frac{(n+1)(k+1+\alpha c)[\delta(n+1,k+1)-\delta(n,k+1)]-(k+1)(1-\alpha)\,\delta(n,k+1)}{(n+1)(k+1+c)[\delta(n+1,k+1)-\delta(n,k+1)]-(k+1)(1-\alpha)\,\delta(n,k+1)}\,.$$

Then

$$F(k+1) - F(k) = \frac{A}{B} > 0 \qquad \text{for each } k\,,$$

where

$$\begin{aligned} A = c(1-\alpha)(n+1)\{&(n+1)[\delta(n+1,k+1)-\delta(n,k+1)]\cdot \\ &\cdot[\delta(n+1,k+2)-\delta(n,k+2)]+(1-\alpha)(k+1)\cdot \\ &\cdot[\delta(n+1,k+2)-\delta(n,k+2)]\,\delta(n,k+1)-(1-\alpha)(k+2)\cdot \\ &\cdot[\delta(n+1,k+1)-\delta(n,k+1)]\,\delta(n,k+2)\} \end{aligned}$$

and

$$\begin{aligned} B = \{&(n+1)(c+k+2)[\delta(n+1,k+2)-\delta(n,k+2)]- \\ &-(k+2)(1-\alpha)\,\delta(n,k+2)\}\{(n+1)(c+k+1)\cdot \\ &\cdot[\delta(n+1,k+1)-\delta(n,k+1)]-(k+1)(1-\alpha)\,\delta(n,k+1)\}\,. \end{aligned}$$

Hence $F(k)$ is an increasing function of k. Since

$$F(1) = \frac{(n+1)(2+\alpha c)[\delta(n+1,2)-\delta(n,2)]-2(1-\alpha)\,\delta(n,2)}{(n+1)(2+c)[\delta(n+1,2)-\delta(n,2)]-2(1-\alpha)\,\delta(n,2)}$$

the result follows.

REMARK. Putting $n = 0$ in the above theorems, we have the results obtained by Uralegaddi and Ganigi [4].

Acknowledgements. The authors would like to thank the referee of the paper for his helpful suggestions.

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Manoscritto pervenuto in redazione il 14 dicembre 1992
e, in forma revisionata, il 13 luglio 1993.