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# On deficient products in infinite groups

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# On Deficient Products in Infinite Groups (*).

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## 1. Introduction.

In [2] groups with the deficient squares property were completely characterized. It was shown in [3] that a group $G$ has the deficient squares property if and only if it does not contain an infinite fully-independent subset.

In this paper we investigate infinite groups with deficient products properties. To make this more precise, let $G$ be an infinite group and let $n, k \in \mathbb{N}$, $k \geq 2$. A subset of $G$ with $n$ elements will be called an *n-set* of $G$. We say that $G \in DP(n, k)$ if all $k$-tuples $X_1, X_2, \ldots, X_k$ of $n$-sets in $G$ satisfy

$$UP(X_1, \ldots, X_k) =_{\text{def}} |\cup \{X_i X_j \mid 1 \leq i, j \leq k, i \neq j\}| < (k^2 - k) n^2 .$$

In particular, $G \in DP(n)$ stands for $G \in DP(n, 2)$. Finally, we say that $G \in DP$ if $G \in DP(n, k)$ for some positive integers $n, k \in \mathbb{N}$, $k \geq 2$. Our main results are expressed in the following theorems.

THEOREM 1. *Let $G$ be an infinite group and let $n \in \mathbb{N}$. Then $G \in DP(n)$ if and only if $G$ is abelian.*

This theorem follows immediately from

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THEOREM 2. *Let $G$ be an infinite non-abelian group. Then $G$ contains two infinite product-independent subsets.*

Two subsets $A$ and $B$ of $G$ are *product-independent* if whenever $a, a' \in A$ and $b, b' \in B$, then $ab \neq b'a'$ and $ab = a'b'$ or $ba = b'a'$ only if $a = a'$ and $b = b'$.

We say that a group $G$ satisfies $G \in FIZ$ if its center is of finite index. With respect to the $DP$ property we prove

THEOREM 3. *Let $G$ be an infinite group. Then $G \in DP$ if and only if $G \in FIZ$.*

Theorem 3 follows easily from

THEOREM 4. *Let $G$ be an infinite group. Then $G$ contains $\aleph_0$ mutually product-independent infinite subsets if and only if $G \notin FIZ$.*

The proofs of Theorems 1, 2, 3 and 4 will be presented in Section 2. It is easy to see that similar proofs yield the following generalizations of Theorems 1 and 3. In order to state these generalizations, we need some additional notation. For each $k \in \mathbb{N}$, $k \geqslant 2$ we define $((n), k) = (n_1, n_2, \ldots, n_k)$, where $n_1, n_2, \ldots, n_k \in \mathbb{N}$. We say that $G \in DP((n), k)$ if all $k$-tuples $X_1, X_2, \ldots, X_k$ of subsets of $G$ with $(|X_1|, |X_2|, \ldots, |X_k|) = (n_1, n_2, \ldots, n_k)$ satisfy

$$UP(X_1, X_2, \ldots, X_k) < \sum\{n_i n_j \mid 1 \leqslant i, j \leqslant k, i \neq j\}.$$

Finally, we say that $G \in DP^*$ if $G \in DP((n), k)$ for some $((n), k)$, $k \geqslant 2$. The generalized theorems are:

THEOREM 1$'$. *Let $G$ be an infinite group. Then $G \in DP((n), 2)$ if and only if $G$ is abelian.*

THEOREM 3$'$. *Let $G$ be an infinite group. Then $G \in DP^*$ if and only if $G \in FIZ$.*

Section 3 deals with a related, but different, topic. Following [4] we say that $G \in P_n$ if $XY = YX$ for all $n$-sets $X, Y$ in $G$ and $G \in P_n^*$ if every infinite set of distinct $n$-sets of $G$ contains a pair $X, Y$ of distinct members such that $XY = YX$. In [4] it was shown that $G \in P_n$ if and only if $G$ is abelian and in [5] the second author showed that $G \in P_n^*$ for $n > 1$ if and only if $G$ is abelian. Theorems 1, 3, 1$'$ and 3$'$ are generalizations of the first mentioned result and we extend the second result by considering groups $G \in P^*$, which satisfy the property that every infinite set of distinct finite subsets of $G$ of specified sizes not less than two contains a pair of distinct members $X$, $Y$ such that $XY = YX$. We prove

THEOREM 5. *Let $G$ be an infinite group. Then $G \in P^*$ if and only if $G$ is abelian.*

The following notation and definitions will be used in this paper. The letter $G$ denotes an infinite group and $\mathbb{N}$ denotes the set of positive integers. The letters $i, j, l, m, n, \lambda, \sigma, \rho, \tau$ will denote positive integers. A subset $S$ of $G$ will be called a *Sidon set* if whenever $x, y, z, w \in S$ and $|\{x, y, z, w\}| \geqslant 3$, then $xy \neq zw$.

## 2. Groups with deficient products.

In this section we shall prove Theorems 1, 2, 3 and 4. Our proofs rely on the following important results of B. H. Neumann and of Babai and Sós.

THEOREM A (B. H. Neumann [6]). *Let $G$ be a group. Then $G \in FIZ$ if and only if $G$ does not contain an infinite subset $U$ satisfying $uv \neq vu$ whenever $u, v \in U$ and $u \neq v$.*

THEOREM B (L. Babai and V. T. Sós [1]). *If $U$ is an infinite subset of a group $G$, then $U$ contains an infinite Sidon set.*

We proceed with our

LEMMA 1. *Let $G$ be an infinite group and let $k$ denote a positive integer or $\aleph_0$. Then there exist $k$ infinite sequences $A_i = (a_1^i, a_2^i, \ldots)$, $i = 1, 2, \ldots, k$ of elements of $G$ such that*

$$A_\lambda A_\sigma \cap A_\rho A_\tau = \emptyset \quad \text{if } |\{\lambda, \sigma, \rho, \tau\}| \geqslant 3 \tag{2.1}$$

*and*

$$a_i^\lambda a_j^\sigma = a_k^\lambda a_l^\sigma \quad \text{for } \lambda \neq \sigma \text{ if and only if } i = k \text{ and } j = l. \tag{2.2}$$

*If $G \notin FIZ$, we may choose the $A_i$ in such a way, that in addition to the above mentioned properties they will satisfy*

$$A_\lambda A_\sigma \cap A_\sigma A_\lambda = \emptyset \quad \textit{for } \lambda \neq \sigma. \tag{2.3}$$

PROOF. Suppose, first, that $G \notin FIZ$. By Theorem A there exists an infinite sequence $B = (g_1, g_2, \ldots)$ of distinct elements of $G$ such that

$g_i g_j \neq g_j g_i$ if $i \neq j$. By Theorem B there exists an infinite subsequence $U = (u_1, u_2, \ldots)$ of $B$ such that $u_i u_j = u_k u_l$ implies that either $i = k$ and $j = l$ (i.e. $u_i u_j = u_i u_j$) or $i = j$ and $k = l$ (i.e. $u_i^2 = u_k^2$). Partition the set $U$ into $k$ disjoint infinite subsequences $A_i = (a_1^i, a_2^i, \ldots)$, $i = 1, 2, \ldots, k$. Then (2.1), (2.2) and (2.3) hold. The proof in the case $G \notin FIZ$ is complete.

If $G \in FIZ$, then $Z(G)$ is of infinite order and by Theorem B $Z(G)$ contains an infinite sequence $U = (u_1, u_2, \ldots)$ of distinct elements such that $u_i u_j = u_k u_l$ implies that either $i = k$ and $j = l$ (i.e. $u_i u_j = u_i u_j$) or $i = j$ and $k = l$ (i.e. $u_i^2 = u_k^2$) or $i = l$ and $j = k$ (i.e. $u_i u_j = u_j u_i$). Construct the sequences $A_i = (a_1^i, a_2^i, \ldots)$ for $i = 1, 2, \ldots, k$ as before. Then (2.1) and (2.2) hold, as claimed. ■

We are ready now to prove Theorem 2.

Proof of Theorem 2. If $G \notin FIZ$, then Theorem 2 follows from Lemma 1, with $k = 2$. So suppose that $G \in FIZ$. Then $Z(G)$ is infinite, $G'$ is finite and $Z(G)/H$ is infinite, where $H = Z(G) \cap G'$. Let $T$ be a transversal of $H$ in $Z(G)$ and let $U$ be an infinite subset of $T$ such that $\{uH \mid u \in U\}$ is a Sidon set in $Z(G)/H$. This means that if $u, v, w, t \in U$ satisfy $uvH = wtH$ then $|\{uH, vH, wH, tH\}| < 3$, hence also $|\{u, v, w, t\}| < 3$. Now split $U$ into two infinite disjoint subsets $R$ and $S$; fix elements $x, y \in G$ with $xy \neq yx$ and finally set $A = xR$, $B = yS$. If $a = xr$, $a' = xr'$, $b = ys$ and $b' = ys'$, with $r, r' \in R$ and $s, s' \in S$, satisfy $ab = b'a'$, then $xrys = ys'xr'$, which implies $[x, y]\ rs = s'r'$ and $[x, y] \in H$. But then $rsH = s'r'H$ and it follows that $|\{r, s, r', s'\}| < 3$ and $r = r'$, $s = s'$, which implies $[x, y] = 1$, a contradiction. Thus whenever $a, a' \in A$ and $b, b' \in B$, then $ab \neq b'a'$. Suppose, now, that $ab = a'b'$ (or $ba = b'a'$). Then $xrys = xr'ys'$ (or $ysxr = ys'xr'$), which implies $rs = r's'$ (or $sr = s'r'$) and as above $r = r'$, $s = s'$, yielding $a = a'$ and $b = b'$. Hence $A$ and $B$ are infinite product-independent subsets of $G$ and the proof of Theorem 2 is complete. ■

Theorem 1 follows immediately from Theorem 2.

We proceed with a proof of Theorem 4, from which Theorem 3 follows easily.

Proof of Theorem 4. If $G \notin FIZ$, then $G$ contains $\aleph_0$ mutually product-independent infinite subsets by Lemma 1. On the other hand, if $G \in FIZ$, then $|G : Z(G)| = n$ for some $n \in \mathbb{N}$. Let $A_1, A_2, \ldots, A_{n+1}$ be infinite subsets of $G$. Then there exist $s \in G$ and $i, j \in \mathbb{N}$ such that $1 \leq i, j \leq n + 1, i \neq j, A_i \cap sZ(G) \neq \emptyset$ and $A_j \cap sZ(G) \neq \emptyset$. It follows that

$A_i$ and $A_j$ are not product-independent, and in particular $G$ does not contain $\aleph_0$ mutually product-independent infinite subsets. ∎

Finally we prove Theorem 3, in which the $DP$-groups are characterized.

PROOF OF THEOREM 3. Suppose, first, that $G \in FIZ$ and $|G : Z(G)| = m$. Then $G \in DP(1, m+1)$ since given $m+1$ elements $g_1, g_2, \ldots, g_{m+1}$ of $G$, at least two of them belong to the same coset of $Z(G)$ in $G$ and therefore commute with each other. This implies that

$$UP(\{g_1\}, \{g_2\}, \ldots, \{g_{m+1}\}) < (m+1)^2 - (m+1)$$

and hence $G \in DP(1, m+1)$, as claimed. But this implies that $G \in DP$, thus completing the proof in one direction.

Suppose, now, that $G \in DP$. Then $G \in DP(n, k)$ for some positive integers $n, k$ with $k \geqslant 2$. It follows immediately by Theorem 4 that $G \in FIZ$. ∎

## 3. Sequences with equal products.

In this section we shall prove Theorem 5.

PROOF OF THEOREM 5. If $G$ is abelian then clearly $G \in P^*$. So suppose that $G \in P^*$ and $G$ is non-abelian. We shall reach a contradiction from our assumptions. Let $(n_1, n_2, \ldots)$ be an infinite sequence of integers greater or equal to two. Pick $x, y \in G$ such that $xy \neq yx$.

If $G \notin FIZ$ then by Lemma 1 applied to $G$ and $k = \aleph_0$, the subsequences $U_i = (a_1^i, a_2^i, \ldots, a_{n_i}^i)$ of the infinite sequences $A_i = (a_1^i, a_2^i, \ldots)$, $i = 1, 2, \ldots$ satisfy $U_i U_j \cap U_j U_i = \emptyset$ for $i \neq j$ and hence $G \notin P^*$, a contradiction.

So suppose that $G \in FIZ$. Then $Z(G)$ is infinite and by Lemma 1 applied to $Z(G)$ and $k = 3$, there exist three disjoint infinite sequences $(a_1, a_2, \ldots)$, $(b_1, b_2, \ldots)$ and $(t_1, t_2, \ldots)$ of distinct elements of $Z(G)$ such that $a_i b_j \neq a_j b_i$ whenever $i \neq j$. Define the following infinite sequence of sequences:

$$U_1 = (t_1, t_2, \ldots, t_{n_1 - 2}, a_1 x, b_1 y),$$

$$U_2 = (t_{n_1 - 1}, t_{n_1}, \ldots, t_{n_1 + n_2 - 4}, a_2 x, b_2 y),$$

$$U_3 = (t_{n_1 + n_2 - 3}, t_{n_1 + n_2 - 2}, \ldots, t_{n_1 + n_2 + n_3 - 6}, a_3 x, b_3 y)$$

and so on. Suppose that for some $i \neq j$ we have $U_i U_j = U_j U_i$. Since $t_k, a_l, b_m \in Z(G)$ for all $k, l, m \in \mathbb{N}$, $x, y, xy \notin Z(G)$ and $a_i b_j \neq a_j b_i$, we must have $a_i x b_j y = b_j y a_i x$, which implies $xy = yx$, a contradiction. Thus $G \notin P^*$, a final contradiction. ■

## REFERENCES


[1] L. Babai - V. T. Sós, *Sidon sets in groups and induced subgraphs of Cayley graphs*, Europ. J. Combinatorics, **6** (1985), pp. 101-114.

[2] M. Herzog - P. Longobardi - M. Maj, *On a combinatorial problem in group theory*, Israel J. Math., to appear.

[3] M. Herzog - C. M. Scoppola, *On deficient squares groups and fully-independent subsets*, preprint.

[4] J. C. Lennox - A. M. Hassanabadi - J. Wiegold, *Some commutativity criteria*, Rend. Sem. Mat. Univ. Padova, 84 (1990), pp. 135-141.

[5] F. Menegazzo, *A property equivalent to commutativity for infinite groups*, Rend. Sem. Mat. Univ. Padova, 87 (1992), pp. 299-301.

[6] B. H. Neumann, *A problem of Paul Erdös on groups*, J. Austral. Math. Soc. (Series A), 21 (1976), pp. 467-472; *Selected Works of B. H. Neumann and Hanna Neumann*, vol. 5, pp. 1003-1008.


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