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Commutative Domains Large in their $\mathfrak{M}$-Adic Completions.

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Introduction.

The topic of the present paper was inspired by a question proposed by A. Orsatti. Let R be a Dedekind domain, $\mathfrak{M}$ a maximal ideal of R; let us denote by $\widehat{R}_{\mathfrak{M}}$ the completion of R in the $\mathfrak{M}$-adic topology, and by $\widehat{K}_{\mathfrak{M}}$, K, the fields of fractions of $\widehat{R}_{\mathfrak{M}}$, R, respectively. Orsatti's question was the following: if R is a Dedekind domain containing infinitely many prime ideals, is it true that the transcendence degree of $\widehat{K}_{\mathfrak{P}}$ over K is infinite for (almost) all $\mathfrak{P} \in \operatorname{Spec}(R)$?

Subsequently, Orsatti himself found that a negative answer is given by the ring $\boldsymbol{P}$ constructed by Corner in his celebrated paper [4]. Recall that $\boldsymbol{P}$ is a domain contained in $\widehat{\mathbb{Z}} = \prod_p \widehat{\mathbb{Z}}_p$, such that $|\boldsymbol{P}| = 2^{\aleph_0}$ and every ideal I of $\boldsymbol{P}$ is principal, generated by an integer n; through an examination of Corner's construction, it is easy to check (see § 1) that, for all prime numbers p, the p-adic completion of $\boldsymbol{P}$ is isomorphic to $\widehat{\mathbb{Z}}_p$, and, moreover, $\widehat{\mathbb{Q}}_p$ *is always an algebraic extension of the field of fractions of* $\boldsymbol{P}$.

In view of this property, $\boldsymbol{P}$ is said to be large in its p-adic completion, for all p; more precisely, given a commutative domain R and a

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maximal ideal $\mathfrak{M}$ of R, with $\bigcap_n \mathfrak{M}^n = \{0\}$, we shall say that R is *large* in its $\mathfrak{M}$-adic completion $\widehat{R}_{\mathfrak{M}}$ if every element of $\widehat{R}_{\mathfrak{M}}$ is algebraic over R; here we note that $\widehat{R}_{\mathfrak{M}}$ is not necessarily a domain, hence we cannot speak of $\widehat{K}_{\mathfrak{M}}$, in general.

These «large» domains are related, in some sense, with the problem of realizing torsion-free R-algebras as endomorphism algebras of R-modules. Actually, we remark that the method of realization due to Corner [4], or the localized version due to Orsatti [8], both use the key Lemma 2.1 of [4], which needs the existence of $2^{\aleph_0}$ elements of $\widehat{K}_{\mathfrak{M}}$ algebraically independent over K. Hence we conclude that these methods of realization cannot work in the case of «large» domains.

Let us also recall that G. Piva in [9] called a Dedekind domain R *admissible* if the transcendence degree of $\widehat{K}_{\mathfrak{P}}$ over K is uncountable for every prime ideal $\mathfrak{P}$ of R; he was able to extend the methods of realization of Corner and Orsatti to a class of algebras over admissible Dedekind domains ([9], Theorem C). The negative answer to Orsatti's question shows that not all the Dedekind domains are admissible, in the sense of Piva.

In the local case, valuation domains R which are large in their $\mathfrak{M}$-adic completions were investigated by Ribenboim [10]; we note that if $\mathfrak{M}$ is the maximal ideal of a valuation domain R and $\bigcap_n \mathfrak{M}^n = \{0\}$, then R is automatically a discrete valuation ring of rank one (DVR). In his recent paper [7], Okoh found other results on large DVRs; in particular, his Proposition 1.1(a) is extended by Corollary 3 of the present paper. Nagata [6] was the first one to exhibit DVRs R *non-complete* and such that $[\widehat{K}_{\mathfrak{M}} : K]$ is finite. Zanardo [11] and Arnold and Dugas [1] investigated torsion-free modules of finite rank over these kinds of rings, called Nagata valuation domains in [11], showing several peculiar results about direct decompositions and indecomposable modules.

In the present paper we investigate «large» commutative domains (not necessarily Dedekind) in the non-local case. Roughly speaking, we describe the two opposite situations.

If R is noetherian, non-local, and $\mathfrak{M}$ is a maximal ideal of R, then $\widehat{R}_{\mathfrak{M}}$ can be algebraic over R (e.g. when $R = \boldsymbol{P}$ as above), but in any case $\widehat{R}_{\mathfrak{M}}$ must contain elements algebraic over R of arbitrarily large degree (Theorem 2). In particular, when $\widehat{R}_{\mathfrak{M}}$ is a domain, $[\widehat{K}_{\mathfrak{M}} : K]$ cannot be finite, as can happen in the local case.

On the other hand, without the hypothesis of noetherianity, we can have a non-complete domain R which is as large as possible in its completion, in the sense that its $\mathfrak{M}$-adic completion coincides with the localization $R_{\mathfrak{M}}$ of R at $\mathfrak{M}$; therefore $\widehat{K}_{\mathfrak{M}} = K$, in this case. Actually, we can say much more (Theorem 7): given any domain T which is complete with respect to the $\mathfrak{M}$-adic topology induced by a maximal ideal $\mathfrak{M}$ of T, there exist non-complete subrings R of T, such that $T = R_{\mathfrak{N}}$, where $\mathfrak{N} = \mathfrak{M} \cap R$, and T is the $\mathfrak{N}$-adic completion of R.

Thus Theorem 7 shows that there are plenty of domains large in their completions, if we do not ask noetherianity.

We are grateful to A. Orsatti for helpful discussions.

1. – In the sequel the symbols R, $\widehat{R}_{\mathfrak{M}}$, K, $\widehat{K}_{\mathfrak{M}}$ etc. will have the same meaning as in the introduction; of course, the symbol $\widehat{K}_{\mathfrak{M}}$ will be used only when $\widehat{R}_{\mathfrak{M}}$ is a domain. General references about $\mathfrak{M}$-adic completions may be found in [2], [6] and [3], Ch. 3. When we speak of $\mathfrak{M}$-adic topology on R we shall always mean that $\mathfrak{M}$ is a *maximal ideal* of R; we recall that, if $\mathfrak{M}$ is a maximal ideal of R and R is *complete* in the $\mathfrak{M}$-adic topology, then R is automatically a *local ring* (see e.g. [3]). As usual, if R is a ring and $\mathfrak{P}$ is a prime ideal of R, we denote by $R_{\mathfrak{P}}$ the localization of R at $\mathfrak{P}$; thus we agree with the use of the symbol $\widehat{R}_{\mathfrak{M}}$, since complete implies local, when $\mathfrak{M}$ is maximal.

We start by showing the existence of a principal ideal domain R, with infinite spectrum, such that $\widehat{K}_{(p)}$ is an algebraic extension of K for all prime elements p of R. We remark again that the idea that the following example due to Corner enjoyes this property is due to Orsatti.

EXAMPLE 1. Let $\boldsymbol{P}$ be the subring of $\widehat{\mathbb{Z}} = \prod_p \widehat{\mathbb{Z}}_p$ constructed in Lemma 1.5 of Corner's paper [4]. We recall the properties of $\boldsymbol{P}$ which we need: it is an integral domain, and a pure subring of $\widehat{\mathbb{Z}}$; every ideal of $\boldsymbol{P}$ is principal, generated by an integer n; moreover the only integers which are invertible in $\boldsymbol{P}$ are ± 1; therefore, in particular, $\operatorname{Spec}(\boldsymbol{P})$ is infinite. For all prime numbers p, let $\pi_p \colon \widehat{\mathbb{Z}} \to \widehat{\mathbb{Z}}_p$ be the canonical projection; note that $\pi_p(\boldsymbol{P})$ is isomorphic to $\boldsymbol{P}$ for all p: in fact any nonzero element x of $\boldsymbol{P}$ is of the form $x = n\varepsilon$, with $n \in \mathbb{Z}$ and ε a unit of $\boldsymbol{P}$, so that $\pi_p(\varepsilon)$ is necessarily a unit of $\widehat{\mathbb{Z}}_p$, and therefore $\pi_p(x) = n\pi_p(\varepsilon)$ cannot be zero. Let us now show that $\widehat{\mathbb{Z}}_p$ is the p-adic completion of $\pi_p(\boldsymbol{P}) \cong \boldsymbol{P}$. From $\mathbb{Z} \subsetneq \pi_p(\boldsymbol{P})$ it follows that $\pi_p(\boldsymbol{P})$ is dense in $\widehat{\mathbb{Z}}_p$. It is then enough to show that the p-adic topology on $\pi_p(\boldsymbol{P})$ coincides with the induced topology of $\widehat{\mathbb{Z}}_p$, i.e. $\pi_p(\boldsymbol{P}) \cap p^m \widehat{\mathbb{Z}}_p = p^m \pi_p(\boldsymbol{P})$, for all $m \in \mathbb{N}$. Recall that $\boldsymbol{P}$ is pure in $\widehat{\mathbb{Z}}$, whence $\boldsymbol{P} \cap p^m \widehat{\mathbb{Z}} = p^m \boldsymbol{P}$ for all m. Let $\pi_p(x) = p^m z$, with $x \in \boldsymbol{P}$ and $z \in \widehat{\mathbb{Z}}_p$; then $x = p^m z + \eta$, with $\eta \in \prod_{q \neq p} \widehat{\mathbb{Z}}_q$; we have $\eta = p^m \delta$, since $\widehat{\mathbb{Z}}_q = p\widehat{\mathbb{Z}}_q$ for all $q \neq p$; thus $x \in p^m \widehat{\mathbb{Z}} \cap \boldsymbol{P} = p^m \boldsymbol{P}$

and so $\pi_p(x) \in p^m \pi_p(\boldsymbol{P})$. This argument shows that $\pi_p(\boldsymbol{P}) \cap p^m \widehat{\mathbb{Z}}_p = p^m \pi_p(\boldsymbol{P})$, as desired.

It remains to check that $\widehat{\mathbb{Q}}_p$ (the field of fractions of $\widehat{\mathbb{Z}}_p$) is an algebraic extension of K, the field of fractions of $\pi_p(\boldsymbol{P})$. But this follows directly from Corner's construction: $\pi_p(\boldsymbol{P})$ contains a transcendence basis of $\widehat{\mathbb{Q}}_p$ over $\mathbb{Q}$, for all p (see [4], p. 696), and therefore $\widehat{\mathbb{Q}}_p$ must be algebraic over K.

Let us remark that, *a priori*, in the above example it could be possible that $\widehat{\mathbb{Q}}_p = K$ for some p. This possibility is excluded by our next result (see also Prop. 1.1 of [7]).

Recall that a ring R is said to be a *Krull ring* if it satisfies the three following conditions (see [6], § 33, p. 115):

(i) for every minimal prime ideal $\mathfrak{P}$, $R_{\mathfrak{P}}$ is a DVR;

(ii) $R = \cap R_{\mathfrak{P}}$, the intersection being taken over all minimal prime ideals;

(iii) any nonzero element of R lies in only a finite number of minimal prime ideals.

If R is a noetherian domain, then its integral closure $\overline{R}$ (in the field of fractions K of R) is not necessarily noetherian (see [6], Example 5, p. 207), but it is in any case a Krull ring ([6], T. 33.10, p. 118). This result will be needed in the following Theorem 2.

If $R \subset T$ are rings, not necessarily domains, and $u \in T$ is algebraic over R, then the *degree* of u will be the minimal degree of a nonzero polynomial $f(X) \in R[X]$ such that $f(u) = 0$.

THEOREM 2. *Let R be a non local noetherian domain, and let $\mathfrak{M}$ be a maximal ideal of R. Then for all integers $n > 0$ there exists an element $u \in \widehat{R}_{\mathfrak{M}}$ which is algebraic over R, of degree greater than n, and such that $R[u]$ is a domain.*

PROOF. Since $\mathfrak{M}$ is a maximal ideal and R is not local, there exists a non-unit $\mu \in R$ such that $\mu \equiv 1 \pmod{\mathfrak{M}}$. Now, for all prime numbers q different from the characteristic of R, the polynomial $X^q - 1$ in $(R/\mathfrak{M})[X]$ has 1 as a simple root; therefore, by Hensel's Lemma, the polynomial $X^q - \mu \in R[X]$ has a root $\eta_q \in \widehat{R}_{\mathfrak{M}}$. Let us now fix a positive integer $n > 0$; we shall show that there exists a prime number $p > n$, different from the characteristic, such that $X^p - \mu$ is irreducible over K; then $u = \eta_p \in \widehat{R}_{\mathfrak{M}}$ will be the required element. By contradiction, let us assume that $X^q - \mu$ is reducible over K for all large enough primes q; it is then known from field theory that μ is a q-th power in K (see e.g. [5]): $\mu = \theta_q^q$, for some $\theta_q \in K$. Since $X^q - \mu \in R[X]$, we then obtain that the θ_q lie in the integral closure $\overline{R}$ of R in K. Since R is noetherian, $\overline{R}$ is a Krull ring by the above recalled result. Now, μ is not a unit of $\overline{R}$, since it is not a unit of R, and therefore μ is contained in a minimal prime ideal $\mathfrak{P}$ of $\overline{R}$, by (ii); moreover $\overline{R}_{\mathfrak{P}}$ is a DVR by (i). We conclude that μ is not a unit of $\overline{R}_{\mathfrak{P}}$, but μ is a q-th power in $\overline{R}_{\mathfrak{P}}$ for all q large enough, since the θ_q lie in $\overline{R} \subseteq \overline{R}_{\mathfrak{P}}$; this fact is clearly impossible in a DVR, and yields the required contradiction.

It remains to show that $R[u]$ is a domain. Let us consider the ideal $\mathfrak{Z}$ generated by $X^p - \mu$ in $R[X]$; since $X^p - \mu$ is monic, the division algorithm shows that

$$\mathfrak{Z}K[X] \cap R[X] = \mathfrak{Z},$$

whence $\mathfrak{Z}$ is a prime ideal, consisting of those $f(X) \in R[X]$ such that $f(u) = 0$. We conclude that $R[u] \cong R[X]/\mathfrak{Z}$ is a domain, as desired. ■

COROLLARY 3. *Let R be a non local noetherian domain, $\mathfrak{M}$ a maximal ideal of R such that $\widehat{R}_{\mathfrak{M}}$ is a domain; then $\widehat{K}_{\mathfrak{M}}$ is neither a finite nor a pure transcendental extension of K.*

It is clear that the hypothesis that R is not local is essential in the preceding theorem (otherwise R could be complete in the $\mathfrak{M}$-adic topology). However we also remark that Nagata [6] proved the existence of a *non complete* DVR R such that the degree $[\widehat{K}_{\mathfrak{M}} : K]$ is finite; moreover Ribenboim [10] showed that a DVR satisfying this property must be of prime characteristic, and $\widehat{K}_{\mathfrak{M}}$ must be a purely inseparable extension of K (these conditions are of course satisfied by Nagata's example).

This situation is very far from the one examined in Theorem 2: from its proof we actually infer that, when $\widehat{R}_{\mathfrak{M}}$ is a domain, $\widehat{K}_{\mathfrak{M}}$ is never a purely inseparable extension of K.

2. – The main purpose of this second section is to show the somewhat surprising fact of the existence of domains R non-complete in the $\mathfrak{M}$-adic topology, whose completion is $R_{\mathfrak{M}}$.

We shall denote by χA the characteristic of a ring A; given a domain T and a maximal ideal $\mathfrak{M}$, we denote by $\pi_{\mathfrak{M}}$ the canonical projection of T onto the residue field $T/\mathfrak{M}$.

LEMMA 4. *Let T be a domain and $\mathfrak{M}$ a maximal ideal of T; let R be a subring of T and let $\mathfrak{N} = \mathfrak{M} \cap R$. Then $\mathfrak{N}$ is a maximal ideal of R if $\pi_{\mathfrak{M}}(R) = T/\mathfrak{M}$.*

PROOF. Let $a \in R\setminus\mathfrak{N}$; since $\mathfrak{M}$ is maximal in T, there exists $\beta \in T\setminus\mathfrak{M}$ such that $a\beta \equiv 1 \pmod{\mathfrak{M}}$. Since $\pi_{\mathfrak{M}}(R) = T/\mathfrak{M}$, we have $\beta = b + m$, with $b \in R$ and $m \in \mathfrak{M}$; this yields $ab - 1 \in \mathfrak{M} \cap R = \mathfrak{N}$. Since the choice of a was arbitrary, we get the desired conclusion. ∎

LEMMA 5. *Let T be a local domain with maximal ideal $\mathfrak{M}$. Let R be a subring of T such that $\mathfrak{N} = \mathfrak{M} \cap R$ is maximal in R and $T = R_{\mathfrak{N}}$. Then $\mathfrak{N}^n = R \cap (\mathfrak{M}^n)$ for all $n \in \mathbb{N}$, i.e. the $\mathfrak{N}$-adic topology of R coincides with the topology induced on R by the $\mathfrak{M}$-adic topology of T.*

PROOF. It is enough to prove that $\mathfrak{N}^n \supseteq R \cap (\mathfrak{M}^n)$ for all n. Since $T = R_{\mathfrak{N}}$, then $\mathfrak{M} = \mathfrak{N}R_{\mathfrak{N}}$, hence the above inclusion holds if we show that $r = m/s$, with $r \in R$, $s \in R\setminus\mathfrak{N}$, $m \in \mathfrak{N}^n$, implies $r \in \mathfrak{N}^n$. Equivalently, $rs \in \mathfrak{N}^n$ and $s \in R\setminus\mathfrak{N}$ yields $r \in \mathfrak{N}^n$. By induction on n, we can assume that $r \in \mathfrak{N}^{n-1}$; moreover, from $\mathfrak{N}$ maximal in R and $s \notin \mathfrak{N}$, it follows that $st = 1 + \xi$, for suitable $t \in R$ and $\xi \in \mathfrak{N}$. Therefore $r + r\xi = rst \in \mathfrak{N}^n$; since $r \in \mathfrak{N}^{n-1}$, then $r\xi \in \mathfrak{N}^n$, whence $r \in \mathfrak{N}^n$, too, as desired. ∎

LEMMA 6. *Let R be a domain, $\mathfrak{N}$ a maximal ideal of R, and let us consider the localization $R_{\mathfrak{N}}$ endowed with the $\mathfrak{M}$-adic topology, where $\mathfrak{M} = \mathfrak{N}R_{\mathfrak{N}}$. Then R is dense in $R_{\mathfrak{N}}$.*

PROOF. We must show that

$$(t + \mathfrak{M}^n) \cap R \neq \emptyset \qquad \text{for all } t \in R_{\mathfrak{N}},\ n \in \mathbb{N}.$$

The element t is of the form $t = r/s$, with $r \in R$ and $s \in R\setminus\mathfrak{N}$. Multiplying both r and s by an inverse of s mod $\mathfrak{N}$ we may assume that $s = 1 - v$, where $v \in \mathfrak{N}$. Then

$$t = r/(1 - v) \equiv r(1 + v + \ldots + v^{n-1}) \quad \mod \mathfrak{M}^n$$

where $r(1 + v + \ldots + v^{n-1}) \in R$, as desired. ∎

We are now in the position to prove the main result of this section; it shows a general property enjoyed by domains; however, we are mainly interested in the case when T is complete in its $\mathfrak{M}$-adic topology.

THEOREM 7. *Let T be a local domain, not a field, with maximal ideal $\mathfrak{M}$. Then there exists a subring R of T satisfying the following: $\mathfrak{N} = \mathfrak{M} \cap R$ is a maximal ideal of R, R is not local and $T = R_{\mathfrak{N}}$; R is not complete in the $\mathfrak{N}$-adic topology. If T is complete in the $\mathfrak{M}$-adic topology, then it is the completion of R in its $\mathfrak{N}$-adic topology.*

PROOF. We start choosing a suitable $x \in T \backslash \mathfrak{M}$. We must distinguish the cases of equal and unequal characteristics. If $\chi(T) = 0$ and $\chi(T/\mathfrak{M}) = p > 0$, we set x to be a prime number distinct from p; of course, $x \notin \mathfrak{M}$, since $p \in \mathfrak{M}$. If $\chi(T) = \chi(T/\mathfrak{M})$, then T contains a field L which is either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$. Now, if z is any nonzero element of $\mathfrak{M}$, then z is transcendental over L: in fact, if z is algebraic over L; then z is a unit in $L[z] \subseteq T$, impossible. Note that $\mathfrak{M} \neq 0$, since T is not a field, by hypothesis. Let us set $x = 1 + z$; then x is transcendental over L and $x \notin \mathfrak{M}$.

Let us now consider the family $\mathcal{F}$ of the subrings B of T satisfying the following conditions:

(i) $x \in B$;

(ii) $1/x \notin B$.

The family $\mathcal{F}$ is nonempty; if $\chi(T) \neq \chi(T/\mathfrak{M})$, then $\mathbb{Z} \in \mathcal{F}$; if $\chi(T) = \chi(T/\mathfrak{M})$, then $L[x] \in \mathcal{F}$ (recall that x is transcendental over L). Moreover $\mathcal{F}$ is clearly inductive, with respect to the inclusion order, and so $\mathcal{F}$ contains a maximal element R. Our purpose is to prove that R satisfies the requirements of our statement, from which we shall obtain the desired conclusion. Since $x \in R$ is not a unit of R, let us fix a maximal ideal $\mathfrak{P}$ of R which contains x; the localization $R_{\mathfrak{P}}$ is not necessarily a subring of T, but it is contained in the field of fractions of T. It is useful to note that $\mathfrak{P}$ does not contain the ideal $\mathfrak{N} = \mathfrak{M} \cap R$: in the eterocharacteristic case $p \in \mathfrak{N} \backslash \mathfrak{P}$, since $ax + bp = 1$, for suitable $a, b \in \mathbb{Z} \subset R$; in the equicharacteristic case we have, by construction, $x - 1 \in \mathfrak{M} \cap R \backslash \mathfrak{P}$.

Let us show various properties of R.

A) R is integrally closed in T.

By contradiction, let $u \in T \backslash R$ be integral over R. Then, in view of the maximality of R in $\mathcal{F}$, $1/x \in R[u]$. This implies that also $1/x$ is integral over R; but

$$a_0 + a_1(1/x) + \ldots + (1/x)^k = 0 \qquad \text{with } a_i \in R,$$

yields $1/x \in R$, impossible.

B) If $z \in T \backslash \mathfrak{M}$, then either $z \in R$ or $1/z \in R$.

Suppose that $z \notin R$; then $1/x \in R[z]$, by the maximality of R. From

$$(1) \qquad 1/x = b_0 + b_1 z + \ldots + b_h z^h, \qquad b_i \in R$$

we get

$$(2) \qquad (xb_0 - 1)(1/z)^h + xb_1(1/z)^{h-1} + \ldots + xb_h = 0 .$$

Since $xb_0 - 1$ is a unit of $R_{\mathfrak{P}}$, (2) implies that $1/z$ is integral over $R_{\mathfrak{P}}$; let us also recall that $1/z \in T$, since T is local. If now $1/z \notin R$, we must have $1/x \in R[1/z]$, and so $1/x$ is integral over $R_{\mathfrak{P}}$, impossible, since $x \in \mathfrak{P}R_{\mathfrak{P}}$. Thus $1/z \in R$, as desired.

C) If $r \in R$ and $r \notin \mathfrak{P} \cup \mathfrak{M}$, then $1/r \in R$.

By contradiction, suppose that $1/r \notin R$. We have $R[1/r] \subsetneq T$, since $r \notin \mathfrak{M}$ implies $1/r \in T$, and therefore R maximal implies

$$(3) \qquad 1/x = c_0 + c_1(1/r) + \ldots + c_n(1/r)^n , \qquad c_i \in R ;$$

from (3) we readily get $r^n \in xR \subsetneq \mathfrak{P}$, whence $r \in \mathfrak{P}$, against the hypothesis. (Note that C) implies that $\mathfrak{P}$ and $\mathfrak{N}$ are the unique maximal ideals of R).

D) $\pi_{\mathfrak{M}}(R) = T/\mathfrak{M}$, whence $\mathfrak{N} = \mathfrak{M} \cap R$ is a maximal ideal of R, in view of Lemma 4.

Let us choose an arbitrary nonzero $\eta \in T/\mathfrak{M}$, and verify that $\eta \in \pi_{\mathfrak{M}}(R)$. Let $y \in T \backslash \mathfrak{M}$ be such that $\pi_{\mathfrak{M}}(y) = \eta$. If $y \in R$ we are done. Otherwise, $y \notin R$ implies $1/y \in R$, in view of property C). From property C) we derive that $1/y \in \mathfrak{P}$, since $1/y \notin \mathfrak{M}$ and $y = (1/y)^{-1} \notin R$. Choose now $m \in \mathfrak{N} \backslash \mathfrak{P}$; such m exists, as observed above. Then $1/y + m \in R$ and $1/y + m \notin \mathfrak{M} \cup \mathfrak{P}$, and therefore C) implies $(1/y + m)^{-1} = y/(1 + my) \in R$, whence

$$\pi_{\mathfrak{M}}(y/1 + my) = \pi_{\mathfrak{M}}(y)\pi_{\mathfrak{M}}(1 + my)^{-1} = \pi_{\mathfrak{M}}(y) = \eta \in \pi_{\mathfrak{M}}(R),$$

as desired.

E) $T = R_{\mathfrak{N}}$.

Let us observe that B) implies that $T \backslash \mathfrak{M} \subsetneq R_{\mathfrak{N}}$: in fact if $z \in T \backslash \mathfrak{M}$ and $z \notin R$, then $1/z \in R$, and $1/z \notin \mathfrak{M} \cap R = \mathfrak{N}$; therefore $(1/z)^{-1} = z \in R_{\mathfrak{N}}$. Moreover, if $z \in \mathfrak{M}$ and $z \notin R$, then $1 + z \in T \backslash \mathfrak{M} \subsetneq R_{\mathfrak{N}}$, whence $z \in R_{\mathfrak{N}}$. We conclude that $T \subsetneq R_{\mathfrak{N}}$, as we wanted.

It is now easy to reach the desired conclusions: we know that $\mathfrak{N} = \mathfrak{M} \cap R$ is a maximal ideal of R and that $T = R_{\mathfrak{N}}$; R is not complete in the $\mathfrak{N}$-adic topology, because it is not a local ring ($x \notin \mathfrak{N}$ and $1/x \notin R$); the $\mathfrak{N}$-adic topology of R coincides with the topology induced by the

$\mathfrak{M}$-adic one of T, since we are in the position to apply Lemma 5; R is a dense subset of T in the $\mathfrak{M}$-adic topology, as a consequence of Lemma 6. Therefore, if T is complete, it must be the completion of R in the $\mathfrak{N}$-adic topology. ∎

We remark that the domain R constructed in the above theorem is never noetherian, as an immediate consequence of Theorem 2.

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