MARCO LUIGI BERNARDI

FABIO LUTEROTTI

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On Some Schroedinger-Type Variational Inequalities.

MARCO LUIGI BERNARDI - FABIO LUTEROTTI (*)

1. - Introduction.

Many results are well known for various classes of evolution variational inequalities (see e.g., in particular, Lions [16], Brezis [6], Barbu [2]). However, by our knowledge and before our recent note [4], we found no references for evolution variational inequalities concerning Schroedinger-type operators. On the other hand, it must be noted that Schroedinger-type evolution equations were widely studied by several authors, both in the linear case, and in various different nonlinear cases. For the linear case, we can refer e.g. to Lions [15], Lions and Magenes [17], Carroll [7], Pozzi [20], [21], Kato [13]. Some nonlinear cases were firstly studied by Pozzi [21], Bardos and Brèzis [3], Lions [16]. Moreover, from the mid-seventies, several authors investigated different problems concerning various Schroedinger-type nonlinear partial differential equations. For the sake of brevity, we mention here only a few papers (with their bibliographies, where other important references can be found): Cazenave [8], Cazenave and Lions [9], Ginibre and Velo [10], [11], Kato [14], Merle [19].

This paper deals with a class of abstract Schroedinger-type evolution variational inequalities, where the unilateral constraints concern

(*) Indirizzo degli AA.: Dipartimento di Elettronica per l'Automazione (Sezione Matematica), Facoltà di Ingegneria, Università degli Studi di Brescia, Via Branze 38, I-25123 Brescia, Italy.

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the time derivative of the unknown function. We investigate, precisely, the following Cauchy problem:

(1.1) $u'(t) \in \mathcal{K}$, for a.e. $t \in]0, T[$;

(1.2) for a.e. $t \in]0, T[$: $\operatorname{Im}(u'(t) + iA(t)u(t) - f(t), v - u'(t)) \geqslant 0$,

$$\forall v \in \mathcal{K};$$

(1.3) $u(0) = u_0$,

where $0 < T < +\infty$; $V \subseteq H \equiv H^* \subseteq V^*$ is the standard *complex* Hilbert triplet; $(\cdot, \cdot)$ denotes the *antiduality* pairing between V^* and V; $\mathcal{K}$ is a closed non-empty convex subset of V; «$t \to A(t)$» is a suitably smooth operator function such that, for every $t \in [0, T]$, $A(t) \in \mathcal{L}(V, V^*)$, and $A(t)$ is an hermitian and strictly V-coercive operator; $u_0 \in V$ is given; $f(t)$ is some given V^*-valued function on $[0, T]$. We deal with strong solutions $u(t)$ of (1.1)-(1.2)-(1.3); i.e., we ask for some V-valued $u(t)$ on $[0, T]$, such that $u'(t)$ is also a V-valued function. For the sake of simplicity, we do not consider here more general formulations of the problem (1.1)-(1.2)-(1.3) (e.g., the ones involving multivalued operators).

The problem (1.1)-(1.2)-(1.3) concerns a variational inequality for the linear abstract Schroedinger-type operator $S(u(t)) \equiv u'(t) + iA(t)u(t)$. We could also consider other types of variational inequalities related to the operator $S(\cdot)$. It must be noted, for example, that the problems with unilateral constraints on $u(t)$ seem to be more complex, as in the case of hyperbolic variational inequalities.

We remark that, in our recent note [4], we announced a first existence and uniqueness result for the problem (1.1)-(1.2)-(1.3), under some stronger hypotheses, in particular on the assumption that $A(t) \equiv \equiv A$ does *not* depend on t. An interesting matter is that such result (with a suitable specialization of V, H, $\mathcal{K}$, A) implies, as a corollary, a «standard» result for a wide class of hyperbolic variational inequalities (see Remark 3 in [4]; see, in particular, Luterotti [18] for the details, and for other remarks and results in this direction). This fact also justifies our interest in the problem (1.1)-(1.2)-(1.3).

We will prove, in sections 2 and 3, an existence and uniqueness result for the problem (1.1)-(1.2)-(1.3), under some general hypotheses. A suitable procedure of penalization will be used as a main tool, for the existence proof. It must be noted that some serious technical difficulties arise from the fact that $A(t)$ *actually* depends on t. To overcome such difficulties (*without* additional assumptions on $A(t)$), a main tool will be to rewrite the inequality (1.2) in a suitable integral form (see section 3, step a) (in particular (3.1)), and step e)). Such device (which was formerly employed by Lions [15], chap. 8, in the context of various classes of linear abstract differential equations) is also very useful in the case of certain hyperbolic variational inequalities with time-dependent operator coefficients (see [5]).

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2. – Notation. Statement of the main result. Proof of the uniqueness result.

Let (as e.g. in Lions and Magenes [17])

$$V \subseteq H \equiv H^* \subseteq V^*, \qquad \text{with } V \text{ separable}, \tag{2.1}$$

be the standard *complex* Hilbert triplet. $(\cdot, \cdot)$ denotes both the scalar product in H and the *antiduality* pairing between V^* and V. $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$ denote respectively the norms in V, H, and V^*.

Let also

$$\mathcal{K} \text{ be a closed non-empty convex subset of } V. \tag{2.2}$$

Throughout the paper, T *denotes a real number, such that* $0 < T < +\infty$. Let now «$t \to A(t)$» be an operator function satisfying

$$\begin{aligned} &A(t) \in W^{2,1}(0, T; \mathcal{L}(V, V^*)) \text{ ((and hence } \exists M_1 > 0, \exists M_2 > 0, \\ &\exists N(t) \in L^1(0, T), \text{ with } N(t) \geq 0 \text{ for a.e. } t \in]0, T[, \\ &\text{such that: } \|A(t)v\|_* \leq M_1\|v\|, \text{ and } \|A'(t)v\|_* \leq M_2\|v\|, \\ &\forall v \in V, \ \forall t \in [0, T]; \ \|A''(t)v\|_* \leq N(t)\|v\|, \\ &\forall v \in V, \text{ for a.e. } t \in]0, T[). \end{aligned} \tag{2.3}$$

We also assume that

$$(A(t)u, v) = \overline{(A(t)v, u)}, \qquad \forall u, v \in V, \ \forall t \in [0, T]; \tag{2.4}$$

$$\exists c > 0 \text{ such that } (A(t)v, v) \geq c\|v\|^2, \qquad \forall v \in V, \ \forall t \in [0, T]. \tag{2.5}$$

We state now the main result of this paper, i.e. the following existence and uniqueness theorem for the Cauchy problem (1.1)-(1.2)-(1.3).

THEOREM 2.1. *Let* (2.1)-(2.5) *hold. Let* $u_0 \in V$ *and* $f(t) \in$

$\in W^{2,1}(0, T; V^*)$ *be given, with*

$$f(0) - iA(0)u_0 \in \mathcal{K}. \tag{2.6}$$

Then, there exists a unique $u(t) \in W^{1,\infty}(0, T; V)$, *satisfying* (1.1)-(1.2)-(1.3).

Before giving the proof of Theorem 2.1, we make some comments on such result, by presenting some simple examples in the following Remarks 2.1 and 2.2.

REMARK 2.1. Fix any $T > \pi/2$, and take: $V = H = V^* = \mathbb{C}$; $\mathcal{K} = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$; $A(t) = I$ (identity operator); $f(t) \equiv 0$; $u_0 = ib$ ($b \in \mathbb{R}$). It is clear that (2.1)-(2.5) hold; moreover, (2.6) holds, if and only if $b \geq 0$. By making some calculations, and putting $u(t) \equiv u_1(t) + iu_2(t)$ (where $u_1(t)$ and $u_2(t)$ are real-valued), we see that (1.1)-(1.2)-(1.3) can be here rewritten as

$$\begin{cases} u_2(t) = u_1'(t) \geq 0, & [u_2'(t) + u_1(t)]\, u_1'(t) = 0, \\ u_2'(t) + u_1(t) \geq 0, & \text{for a.e. } t \in]0, T[; \\ u_1(0) = 0, \quad u_2(0) = b. \end{cases} \tag{2.7}$$

First, it is clear from (2.7) that, when the «compatibility condition» (2.6) does *not* hold (i.e. when $b < 0$), there exists *no* solution $u(t) \in W^{1,\infty}(0, T; \mathbb{C})$ to (1.1)-(1.2)-(1.3). On the other hand, when $b \geq 0$, it results that the unique solution $u(t) = (u_1(t) + iu_2(t)) \in W^{1,\infty}(0, T; \mathbb{C})$ to (1.1)-(1.2)-(1.3) is given by

$$\begin{cases} u_1(t) = b \sin t, \quad 0 \leq t \leq \dfrac{\pi}{2}; & u_1(t) = b, \quad \dfrac{\pi}{2} \leq t \leq T; \\[2ex] u_2(t) = b \cos t, \quad 0 \leq t \leq \dfrac{\pi}{2}; & u_2(t) = 0, \quad \dfrac{\pi}{2} \leq t \leq T. \end{cases} \tag{2.8}$$

Thus, we observe that, when $b > 0$, $u'(t) \in L^\infty(0, T)$, but $u_2'(t)$ (and hence $u'(t)$) is *not* continuous at $t = \pi/2$. This example shows that, under the assumptions in Theorem 2.1, it is *not* true, in general, that «$t \to u'(t)$» is a continuous function, even if «$t \to f(t)$» and «$t \to A(t)$» are arbitrarily smooth functions. Let us also give briefly another example in the same direction, when $A(t)$ *actually* depends on t. Fix now any $T > 1$, and take: $V = H = V^* = \mathbb{C}$, and $\mathcal{K}$ as above; $A(t) = \exp(t) \cdot I$; $f(t) = -i(2\exp(2 - 2t) + \exp(2t) + e^2)$; $u_0 = -(e^2 + 1) + i(e^2 - 1)$. Clearly, (2.1)-(2.6) hold. By making some calculations, it can be seen here that the unique solution $u(t) \in W^{1,\infty}(0, T; \mathbb{C})$ to (1.1)-(1.2)-(1.3) is such that $u'(t)$ is *not* continuous at $t = 1$.

REMARK 2.2. By considering Theorem 2.1, we observe that we are assuming (through (2.5)) that $A(t)$ is a *strictly* V-coercive operator. Of course, we can raise the question whether our result still holds true, by assuming more generally that $A(t)$ is a *weakly* V-coercive operator, i.e. by replacing (2.5) with the following condition:

$$\exists c > 0\,, \quad \exists \lambda \geqslant 0 \text{ such that } (A(t)v, v) + \lambda |v|^2 \geqslant c\|v\|^2\,, \quad \forall v \in V,\ \forall t \in [0, T]. \tag{2.9}$$

Recall that (2.9) (instead of (2.5)) suffices for various classes of hyperbolic or parabolic variational inequalities (see e.g. [2], [6], [16]), and also in the case of linear abstract differential equations of Schroedinger-type (see e.g. [15], [17], and [20]). Now, we have here a *negative* answer to such a question. To see this, we can consider the following examples.

Firstly, fix any $T > 0$, and take: $V = H = V^* = \mathbb{C}$; $\mathcal{K} = \{z \in \mathbb{C} \mid \operatorname{Re} z \geqslant 0\}$; $A(t) \equiv 0$; $u_0 \equiv 0$; $f(t) \equiv 0$. Clearly (2.1)-(2.4), (2.6) and (2.9) hold, but (2.5) does *not* hold. By making some calculations, and putting $u(t) \equiv u_1(t) + iu_2(t)$ (with real-valued $u_1(t)$ and $u_2(t)$), we see that (1.1)-(1.2)-(1.3) can be here rewritten as

$$\begin{cases} u_1'(t) = 0\,, & u_2'(t) \geqslant 0\,, \qquad \text{for a.e. } t \in]0, T[\,; \\ u_1(0) = 0\,, & u_2(0) = 0\,. \end{cases} \tag{2.10}$$

Then, it is clear from (2.10) that (1.1)-(1.2)-(1.3) has here *infinitely* many solutions $u(t) = iu_2(t) \in W^{1,\infty}(0, T; \mathbb{C})$.

On the other hand, take now: $T > 0$, $V = H = V^* = \mathbb{C}$, $\mathcal{K}$, $A(t) \equiv 0$, and $u_0 = 0$ as above; take, moreover, $f(t) = -t$. Of course, (2.1)-(2.4), (2.6), and (2.9) hold, but (2.5) does *not* hold. By putting $u(t) = u_1(t) + iu_2(t)$ as before, it results that (1.1)-(1.2)-(1.3) can be here rewritten as

$$u_1'(t) \geqslant 0\,, \quad u_1'(t) = -t\,, \quad u_1'(t)\, u_2'(t) = 0\,, \quad u_2'(t) \geqslant 0\,, \quad \text{for a.e. } t \in]0, T[\,; \quad u_1(0) = 0\,, \quad u_2(0) = 0\,. \tag{2.11}$$

Hence, it is clear, from the first and the second condition in (2.11), that, in this case, there exists *no* solution to (1.1)-(1.2)-(1.3).

We now deal with the proof of Theorem 2.1. The proof of the existence result will be carried out in the following section 3. As regards the uniqueness, we can prove, in fact, a result (Theorem 2.2 below), which is more general than the one contained in Theorem 2.1. Towards this aim, let us consider, instead of (2.3), the following weaker assumption:

(2.12) $A(t) \in W^{1,1}(0, T; \mathscr{L}(V, V^*))$ (and hence, in particular,

$\exists M_2(t) \in L^1(0, T)$, with $M_2(t) \geq 0$ for a.e. $t \in]0, T[$,

such that $\|A'(t)v\|_* \leq M_2(t)\|v\|$, $\forall v \in V$, for a.e. $t \in]0, T[$).

(Also remark that, in the next Theorem 2.2, (2.6) is not assumed; moreover, a less regular $f(t)$ is taken).

THEOREM 2.2. *Let* (2.1), (2.2), (2.12), (2.4), *and* (2.5) *hold. Take any* $f(t) \in L^1(0, T; V^*)$, *and any* $u_0 \in V$. *Let* $u(t) = u_1(t)$ *and* $u(t) = u_2(t)$ *both belong to* $W^{1,\infty}(0, T; V)$, *and satisfy* (1.1)-(1.2)-(1.3) (*with the same previous* $f(t)$ *and* u_0). *Then* $u_1(t) = u_2(t)$, $\forall t \in [0, T]$.

PROOF. Let us define $w(t) \equiv u_1(t) - u_2(t)$. Consider now (1.2) with $u(t) = u_1(t)$ (resp. $u(t) = u_2(t)$), and take $v = u_2'(t)$ (resp. $v = u_1'(t)$). (This is allowed, since $u(t) = u_1(t)$ and $u(t) = u_2(t)$ satisfy (1.1)). Then, by adding up the resulting inequalities, we obtain:

$$
\begin{aligned}
(2.13)\qquad 0 &\leq \operatorname{Im}(u_1'(t) + iA(t)u_1(t) - f(t), u_2'(t) - u_1'(t)) + \\
&+ \operatorname{Im}(u_2'(t) + iA(t)u_2(t) - f(t), u_1'(t) - u_2'(t)) = \\
&= -\operatorname{Im}(w'(t) + iA(t)w(t), w'(t)), \qquad \text{for a.e. } t \in]0, T[.
\end{aligned}
$$

Take now *any* $s \in [0, T]$, and integrate (2.13) from 0 to s. Thanks to (2.12), (2.4), (2.5), and to $w(0) = 0$, we get:

$$
\begin{aligned}
(2.14)\qquad 0 &\geq \int_0^s \operatorname{Im}(w'(t) + iA(t)w(t), w'(t))\,dt = \\
&= \int_0^s \operatorname{Re}(A(t)w(t), w'(t))\,dt = \\
&= \frac{1}{2}\int_0^s \frac{d}{dt}(A(t)w(t), w(t))\,dt - \frac{1}{2}\int_0^s (A'(t)w(t), w(t))\,dt \geq \\
&\geq \frac{c}{2}\|w(s)\|^2 - \frac{1}{2}\int_0^s (A'(t)w(t), w(t))\,dt.
\end{aligned}
$$

By also using (2.12), we can deduce that

$$\|w(s)\|^2 \leqslant c^{-1} \int_0^s M_2(t) \|w(t)\|^2 \, dt \,, \qquad \forall s \in [0, T]. \tag{2.15}$$

Hence, thanks to the generalized Gronwall lemma, we obtain that $\|w(s)\| \leqslant 0$, $\forall s \in [0, T]$, i.e. $u_1(s) = u_2(s)$, $\forall s \in [0, T]$.

3. – Proof of the existence result in Theorem 2.1.

We now prove the existence result in Theorem 2.1. Our proof is based on the method of penalization (see Lions [16], chap. 3, as a main reference; also see [5]), and consists of several steps.

a) A preliminary remark. We observe at once that, in our existence proof, (1.2) can be replaced by the following condition:

(3.1) $\exists p \in \mathbb{R}$, such that

$$\text{(I)} \quad \int_0^T \exp(pt) \, \mathrm{Im}\,(u'(t) + iA(t)\,u(t) - f(t), v(t) - u'(t)) \, dt \geqslant 0,$$

$$\forall v(t) \in L^2(0, T; V) \text{ such that } v(t) \in \mathcal{K} \text{ for a.e. } t \in]0, T[.$$

Indeed, under the assumptions in Theorem 2.1 (and also under more general hypotheses), *it is equivalent to require that* $u(t)$ $(\in W^{1,\infty}(0, T; V))$ *satisfies either* (1.1)-(1.2) *or* (1.1)-(3.1). Firstly, it is obvious that, if $u(t)$ satisfies (1.2), then (3.1) also holds ($\forall p \in \mathbb{R}$). Conversely, suppose that $u(t)$ satisfies (1.1)-(3.1). Then, we can see that (1.2) also holds, by adapting a procedure given in Remarque 7.9 of chap. 3 in Lions [16], for the case of hyperbolic variational inequalities (also see [5], section 3). Such procedure is here based on the consideration of the Lebesgue points for both the V^*-valued function «$t \to \exp(pt)[u'(t) + iA(t)\,u(t) - f(t)]$» and the scalar function «$t \to \exp(pt)\,\mathrm{Im}\,(u'(t) + iA(t)\,u(t) - f(t), u'(t))$». Without entering into details (also see [16] and [5]), we can obtain that (1.2) holds, by using some well known facts concerning Lebesgue points (also in the case of vector-valued functions; see e.g. Hille and Phillips [12]).

So, in our proof (see, later on, the step e)), it will be convenient to replace (1.2) with (3.1), where some $p < -M_2 \cdot c^{-1}$ is fixed (recall the assumptions (2.3) and (2.5)).

b) Approximation by the method of penalization. Let $P_{\mathcal{K}}$ be the projection operator from V to $\mathcal{K}$; let moreover J be the canonical antiduality operator from V onto V^* (defined by $(Ju, v) = ((u, v))$, $\forall u, v \in V$, where $((\cdot, \cdot))$, denotes the scalar product in V). Then, we define

$$\beta(v) = J(v - P_{\mathcal{K}} v), \qquad \forall v \in V. \tag{3.2}$$

We can verify that β is a penalty operator connected with $\mathcal{K}$. Now, we are working, in fact, in a *complex* Hilbert spaces framework. However, by reviewing and adapting some proofs performed in the *real* case (see e.g. Lions [16], Baiocchi and Capelo [1]), we can easily obtain that β has the following properties.

(3.3) $\beta(\cdot)\colon V \to V^*$ is a strongly Lipschitz continuous

(and hence bounded and hemicontinuous) operator, with $\ker(\beta) = \mathcal{K}$;

moreover, β has the following *monotonicity* property:

$$\operatorname{Re}(\beta(u) - \beta(v), u - v) \geq 0, \qquad \forall u, v \in V. \tag{3.4}$$

We can also remark that, if e.g. $v(t) \in W^{1,\infty}(0, T; V)$, it results:

$$\operatorname{Re}\big((\beta(v(t)))', v'(t)\big) \geq 0, \qquad \text{for a.e. } t \in]0, T[. \tag{3.5}$$

Clearly, (3.5) follows from (3.3) and (3.4), by using a standard differential quotients argument.

Now, *let us take any integer* $k \geq 1$. We could approximate (1.1)-(1.2)-(1.3) (i.e. (1.1)-(3.1)-(1.3)) by means of the following «penalized» problem:

$$\begin{cases} \text{(I)}\ u_k'(t) + iA(t)u_k(t) + ik\beta(u_k'(t)) = f(t), \quad \text{on }]0, T[; \\ \text{(II)}\ u_k(0) = u_0. \end{cases} \tag{3.6}$$

However, (3.6)(I) is an implicit differential equation. On the other hand, we can deduce *formally*, from (3.6), that $u_k'(0) = f(0) - iA(0)u_0 - ik\beta(u_k'(0))$. Hence, thanks also to (2.6), we see that another «more convenient» approximation of (1.1)-(1.2)-(1.3) is given by the following problem:

$$\begin{cases} \text{(I)}\ \dfrac{1}{k} iu_k''(t) + u_k'(t) + iA(t)u_k(t) + ik\beta(u_k'(t)) = f(t), \\ \hfill \text{on }]0, T[; \\ \text{(II)}\ u_k(0) = u_0, \quad u_k'(0) = f(0) - iA(0)u_0. \end{cases} \tag{3.7}$$

Next, we deduce *formally*, from (3.7) and (2.6), that $u_k''(0) = 0$. Then, by differentiating *formally* (3.7)(I), we can also consider the following problem:

$$(3.8)\quad \begin{cases} \text{(I)}\ \dfrac{1}{k} i u_k^{(3)}(t) + u_k''(t) + iA(t)u_k'(t) + iA'(t)u_k(t) + \\ \qquad\qquad + ik(\beta(u_k'(t)))' = f'(t), \quad \text{on }]0, T[\,; \\ \text{(II)}\ u_k(0) = u_0\,, \quad u_k'(0) = f(0) - iA(0)u_0\,, \quad u_k''(0) = 0\,. \end{cases}$$

Now, we can prove that, *when we fix any integer* $k \geq 1$,

(3.9) the problem (3.7) has a unique solution

$$u_k(t) \in W^{1,\infty}(0, T; V), \text{ with } u_k''(t) \in L^\infty(0, T; H).$$

As regards the uniqueness, such result can be obtained by using a «natural» procedure. In fact, let $u_{k1}(t)$ and $u_{k2}(t)$ both satisfy (3.7), and define $w(t) = u_{k1}(t) - u_{k2}(t)$. Hence, from (3.7), it results that $w(0) = w'(0) = 0$, and that

$$(3.10)\quad \frac{1}{k} i(w''(t), w'(t)) + |w'(t)|^2 + i(A(t)w(t), w'(t)) +$$
$$+ ik(\beta(u_{k1}'(t)) - \beta(u_{k2}'(t)), u_{k1}'(t) - u_{k2}'(t)) = 0\,, \quad \text{for a.e. } t \in]0, T[\,.$$

Next, we consider the imaginary parts in (3.10), and we take into account (3.4). Hence, by integrating the resulting inequality from 0 to t ($0 < t \leq T$), and using the properties of $A(t)$ and the Gronwall lemma, we can deduce that $w(t) = 0$, $\forall t \in [0, T]$.

As regards the existence result in (3.9), we can use the Faedo-Galerkin method. So (recall (2.1)), we choose

(3.11) a countable basis $\{w_j\}(1 \leq j \leq +\infty)$ of V,

such that u_0 and $u_1 \overset{\text{def}}{=} f(0) - iA(0)u_0$ both belong to span $[w_1, w_2]$.

Then, for every integer $m \geq 2$, we look for $u_{km}(t) \in \text{span}[w_1, \ldots, w_m]$, solution of the following Cauchy problem:

$$(3.12)\quad \begin{cases} \text{(I)}\ \dfrac{1}{k} i(u_{km}''(t), w_j) + (u_{km}'(t), w_j) + i(A(t)u_{km}(t), w_j) + \\ \qquad\qquad + ik(\beta(u_{km}'(t)), w_j) = (f(t), w_j)\,, \quad 1 \leq j \leq m\,; \\ \text{(II)}\ u_{km}(0) = u_0\,, \quad u_{km}'(0) = f(0) - iA(0)u_0 \equiv u_1\,. \end{cases}$$

Now, thanks to (3.11) and to the properties of $A(t)$ and $\beta(\cdot)$, we can deduce, from the theory of ordinary differential systems, that

(3.13) there exists a unique $u_{km}(t) = \sum_{j=1}^{m} g_{kmj}(t)\, w_j$

$(0 \leqslant t \leqslant T)$, which is a classical solution of (3.12).

We also observe that, from (3.12) and (2.6), it results that $u''_{km}(0) = 0$. Moreover, we can differentiate (3.12)(I) (and this is correct, since, in particular, (3.3) holds). Hence, $u_{km}(t)$ also satisfies:

$$(3.14) \quad \begin{cases} \text{(I)} \ \frac{1}{k} i(u^{(3)}_{km}(t), w_j) + (u''_{km}(t), w_j) + i(A(t)\, u'_{km}(t), w_j) + \\ \qquad + i(A'(t)\, u_{km}(t), w_j) + ik\big((\beta(u'_{km}(t)))', w_j\big) = (f'(t), w_j), \\ \qquad\qquad 1 \leqslant j \leqslant m, \quad \text{for a.e. } t \in]0, T[\,; \\ \text{(II)} \ u_{km}(0) = u_0\,, \quad u'_{km}(0) = f(0) - iA(0)\, u_0 \equiv u_1\,, \quad u''_{km}(0) = 0\,. \end{cases}$$

Now, we could get some estimates for the sequence $\{u_{km}(t)\}$. The starting point for the first estimate would be to multiply both sides of (3.12)(I) by $g'_{kmj}(t)$, and to sum over j (from 1 to m). Then, we could obtain that

(3.15) $\{u_{km}(t) \,|\, k \geqslant 1,\ m \geqslant 2\}$ is bounded

(independently of k and of m) in $L^{\infty}(0, T; V)$;

$\{k^{-1/2} u'_{km}(t) \,|\, k \geqslant 1,\ m \geqslant 2\}$ is bounded

(independently of k and of m) in $L^{\infty}(0, T; H)$.

The starting point for the second estimate would be to multiply both sides of (3.14)(I) by $g''_{kmj}(t)$, and to sum over j (from 1 to m). So, we could get that

(3.16) $\{u'_{km}(t) \,|\, k \geqslant 1,\ m \geqslant 2\}$ is bounded

(independently of k and of m) in $L^{\infty}(0, T; V)$;

$\{k^{-1/2} u''_{km}(t) \,|\, k \geqslant 1,\ m \geqslant 2\}$ is bounded

(independently of k and of m) in $L^{\infty}(0, T; H)$.

Then, we could obtain that, *when we fix any integer* $k \geqslant 1$, we can extract from $\{u_{km}(t) \,|\, m \geqslant 2\}$ a subsequence $\{u_{kr}(t) \,|\ r \geqslant 1\}$, which converges (as $r \to +\infty$, in a suitable topology) to a function $u_k(t) \in$

$\in W^{1,\infty}(0, T; V)$ (with $u_k''(t) \in L^\infty(0, T; H)$), solution of the problem (3.7). So, *the result* (3.9) *would be proved.*

Moreover, thanks to (3.15)-(3.16), we could also get that we can extract from $\{u_k(t) | k \geq 1\}$ a subsequence, still denoted by $\{u_k(t) | k \geq 1\}$, which converges (as $k \to +\infty$, in a suitable topology) to a function $u(t) \in$ $\in W^{1,\infty}(0, T; V)$, solution of (1.1)-(1.2)-(1.3).

Now, *for the sake of brevity*, we carry out the detailed proof, by starting from the assumption that (3.9) holds. Then, we prove directly the two main estimates for $\{u_k(t)\}$ (as (3.15)-(3.16) for $\{u_{km}(t)\}$): the calculations are essentially the same one has to make, in order to obtain (3.15)-(3.16).

c) A priori estimates (I). We start from the result (3.9) for the problem (3.7). We fix any $z \in \mathcal{K}$, and we «multiply» (in the antiduality pairing between V^* and V) both sides of (3.7)(I) by $u_k'(t) - z$. So (being $\beta(z) = 0$), we get

$$\text{(3.17)} \quad \frac{1}{k} i(u_k''(t), u_k'(t) - z) + (u_k'(t), u_k'(t) - z) + \\ + i(A(t) u_k(t), u_k'(t) - z) + ik(\beta(u_k'(t)) - \beta(z), u_k'(t) - z) = \\ = (f(t), u_k'(t) - z), \qquad \text{a.e. on }]0, T[.$$

Next, we take the imaginary parts in (3.17). We obtain (thanks also to (2.3), (2.4), (3.4))

$$\text{(3.18)} \quad \frac{1}{k} \frac{d}{dt} |u_k'(t)|^2 - \frac{2}{k} \mathrm{Re}(u_k''(t), z) - 2\,\mathrm{Im}(u_k'(t), z) + \\ + \frac{d}{dt}(A(t) u_k(t), u_k(t)) - (A'(t) u_k(t), u_k(t)) - \\ - 2\,\mathrm{Re}(A(t) u_k(t), z) \leq 2\,\mathrm{Im}\left[\frac{d}{dt}(f(t), u_k(t))\right] - \\ - 2\,\mathrm{Im}(f'(t), u_k(t)) - 2\,\mathrm{Im}(f(t), z), \qquad \text{a.e. on }]0, T[.$$

Now, we integrate (3.18) from 0 to t ($0 \leq t \leq T$). By using (3.7)(II), and defining $u_1 \equiv (f(0) - iA(0) u_0) \in \mathcal{K}$ (as we already did), we get

$$\text{(3.19)} \quad \frac{1}{k} |u_k'(t)|^2 - \frac{1}{k} |u_1|^2 - \frac{2}{k} \mathrm{Re}(u_k'(t), z) + \frac{2}{k} \mathrm{Re}(u_1, z) - \\ - 2\,\mathrm{Im}(u_k(t), z) + 2\,\mathrm{Im}(u_0, z) + (A(t) u_k(t), u_k(t)) -$$

$$-(A(0)u_0, u_0) - \int_0^t (A'(s)u_k(s), u_k(s))\,ds -$$

$$-2\,\mathrm{Re}\int_0^t (A(s)u_k(s), z)\,ds \leqslant 2\,\mathrm{Im}\,(f(t), u_k(t)) -$$

$$-2\,\mathrm{Im}\,(f(0), u_0) - 2\,\mathrm{Im}\int_0^t (f'(s), u_k(s))\,ds - 2\,\mathrm{Im}\int_0^t (f(s), z)\,ds\,,$$

$$\forall t \in [0, T].$$

Next, starting from (3.19), we can carry out some calculations, by using in particular (2.3) and (2.5), and assuming e.g. (without loss of generality; see (2.1)) that $|v| \leqslant \|v\|$, $\forall v \in V$. Then, by taking any η such that $0 < \eta < \min(1; c/2)$ (and considering that $k \geqslant 1$), we obtain

$$(3.20)\qquad (1-\eta)\frac{1}{k}\,|u_k'(t)|^2 + (c - 2\eta)\|u_k(t)\|^2 \leqslant 2\,|u_1|^2 +$$

$$+(2+M_1)\|u_0\|^2 + (2 + \eta^{-1}(3+T^2))\|z\|^2 +$$

$$+2\eta^{-1}\sup_{0\leqslant t\leqslant T}\|f(t)\|_*^2 + \left(\int_0^T \|f(s)\|_*\,ds\right)^2 +$$

$$+\eta^{-1}\left(\int_0^T \|f'(s)\|_*\,ds\right)^2 + \eta(1+M_1^2)\sup_{0\leqslant s\leqslant t}\|u_k(s)\|^2 +$$

$$+\int_0^t M_2\|u_k(s)\|^2\,ds\,,\qquad \forall t \in [0, T],\ \text{and for every integer } k \geqslant 1.$$

Now, by taking η sufficiently small (in particular, by *also* requiring that $\eta < c\cdot(3+M_1^2)^{-1}$), we can get from (3.20)

$$(3.21)\qquad \sup_{0\leqslant s\leqslant t}\|u_k(s)\|^2 \leqslant c_1 + c_2\int_0^t\left[\sup_{0\leqslant\tau\leqslant s}\|u_k(\tau)\|^2\right]ds\,,$$

$$\forall t \in [0, T],\ \text{and for every integer } k \geqslant 1,$$

where the positive quantities c_1 and c_2 are independent of k and of t. (In fact (see (3.20)), c_2 depends only on c, η, M_1, M_2, while c_1 depends only on c, η, M_1, T, u_1, u_0, z, f). Hence, thanks to the Gronwall lemma, we obtain from (3.21) that

(3.22) $\{u_k(t) \mid k \geq 1\}$ is bounded (independently of k) in $L^\infty(0, T; V)$.

Moreover, from (3.20) and (3.22), we also get that

(3.23) $\{k^{-1/2} u_k'(t) \mid k \geq 1\}$ is bounded

(independently of k) in $L^\infty(0, T; H)$.

d) A priori estimates (II). The starting point is here to «multiply» both sides of (3.8) (I) by $u_k''(t)$. Clearly, by considering (3.7) and (3.9), this is a *formal* procedure (which we adopt *only for the sake of brevity*). In fact (as we already observed in subsection *b*) above), we should have to start from (3.14), in order to obtain (3.16), and then we should pass to the limit with respect to m (as $m \to +\infty$). Anyway, by «multiplying» both sides of (3.8) (I) by $u_k''(t)$, and taking the imaginary parts, we obtain

$$
\begin{aligned}
(3.24)\qquad & \frac{1}{k}\frac{d}{dt}|u_k''(t)|^2 + 2\,\mathrm{Re}\,(A(t)u_k'(t), u_k''(t)) + \\
& +2\,\mathrm{Re}\,(A'(t)u_k(t), u_k''(t)) + 2k\,\mathrm{Re}\,\big((\beta(u_k'(t)))', u_k''(t)\big) = \\
& = 2\,\mathrm{Im}\left[\frac{d}{dt}(f'(t), u_k'(t)) - (f''(t), u_k'(t))\right], \qquad \text{on }]0, T[.
\end{aligned}
$$

Now, we integrate (3.24) from 0 to t ($0 \leq t \leq T$). By using (2.3), (2.4) (also for $A'(t)$), (3.5), and (3.8) (II) (always with $u_1 \equiv (f(0) - iA(0)u_0)$), we can get

$$
\begin{aligned}
(3.25)\qquad & \frac{1}{k}|u_k''(t)|^2 + (A(t)u_k'(t), u_k'(t)) - \\
& -(A(0)u_1, u_1) + 2\,\mathrm{Re}\,(A'(t)u_k(t), u_k'(t)) - \\
& -2\,\mathrm{Re}\,(A'(0)u_0, u_1) - 3\int_0^t (A'(s)u_k'(s), u_k'(s))\,ds - \\
& -2\,\mathrm{Re}\int_0^t (A''(s)u_k(s), u_k'(s))\,ds \leq 2\,\mathrm{Im}\,(f'(t), u_k'(t)) - \\
& -2\,\mathrm{Im}\,(f'(0), u_1) - 2\,\mathrm{Im}\int_0^t (f''(s), u_k'(s))\,ds, \qquad \text{for a.e. } t \in]0, T[.
\end{aligned}
$$

Next, starting from (3.25), we can carry out some calculations, by using in particular (2.3) and (2.5). Then, by taking any η such that $0 < \eta < \min(1; c/2)$, we obtain

$$(3.26)\qquad \frac{1}{k}|u_k''(t)|^2 + (c-2\eta)\|u_k'(t)\|^2 \leqslant M_2^2\|u_0\|^2 +$$

$$+(2+M_1)\|u_1\|^2 + 2\eta^{-1}\sup_{0\leqslant t\leqslant T}\|f'(t)\|_*^2 +$$

$$+\eta^{-1}\left(\int_0^T \|f''(t)\|_*\,dt\right)^2 + \eta^{-1}M_2^2\sup_{0\leqslant t\leqslant T}\|u_k(t)\|^2 +$$

$$+\eta^{-1}\left(\int_0^T N(t)\|u_k(t)\|\,dt\right)^2 + 2\eta\sup_{0\leqslant s\leqslant t}\|u_k'(s)\|^2 +$$

$$+3M_2\int_0^t \|u_k'(s)\|^2\,ds\,,\quad \text{for a.e. } t\in[0,T], \text{ and for every integer } k\geqslant 1\,.$$

Now, by considering the right-hand side in (3.26), we take into account the previous estimate (3.22), the fact that $N(t)\in L^1(0,T)$ (see (2.3)), and the fact that $f(t)\in W^{2,1}(0,T;V^*)$ (hence $f(t)\in C^1([0,T];V^*)$ too). Then, by taking η sufficiently small (in particular, by *also* requiring that $\eta < c\cdot 4^{-1}$), we can get from (3.26)

$$(3.27)\qquad \sup_{0\leqslant s\leqslant t}\|u_k'(s)\|^2 \leqslant c_3 + c_4\int_0^t\left[\sup_{0\leqslant \tau\leqslant s}\|u_k'(\tau)\|^2\right]ds\,,$$

$$\text{for a.e. } t\in]0,T[, \text{ and for every integer } k\geqslant 1\,,$$

where the positive quantities c_3 and c_4 are independent of k and of t. Hence, thanks to the Gronwall lemma, we obtain from (3.27) that

$$(3.28)\qquad \{u_k'(t)\,|\,k\geqslant 1\} \text{ is bounded (independently of } k) \text{ in } L^\infty(0,T;V)\,.$$

Moreover, from (3.26) and (3.28), we also get that

$$(3.29)\qquad \{k^{-1/2}u_k''(t)\,|\,k\geqslant 1\} \text{ is bounded}$$

$$\text{(independently of } k) \text{ in } L^\infty(0,T;H)\,.$$

e) Passage to the limit. We consider, for every integer $k\geqslant 1$, the problem (3.7), and the corresponding result (3.9). We also consider the estimates (3.22), (3.28) and (3.29), concerning the sequence $\{u_k(t)|$

$k \geqslant 1\}$. Firstly, it is obvious that, as $k \to +\infty$,

$$k^{-1} u_k''(t) \to 0 \text{ strongly in } L^\infty(0, T; H). \tag{3.30}$$

Secondly, from (3.7)(I), (3.22), (3.28), (3.29), it is clear that, as $k \to + +\infty$,

$$\beta(u_k'(t)) \to 0 \text{ strongly in } L^\infty(0, T; V^*). \tag{3.31}$$

Moreover, thanks to (3.22), and to (3.28), we can extract from $\{u_k(t) \mid k \geqslant \geqslant 1\}$ a subsequence, still denoted by $\{u_k(t) \mid k \geqslant 1\}$, such that, as $k \to + +\infty$,

$$u_k(t) \to u(t) \text{ weakly star in } L^\infty(0, T; V), \quad \text{and also weakly in } L^2(0, T; V); \tag{3.32}$$

$$u_k'(t) \to u'(t) \text{ weakly star in } L^\infty(0, T; V), \quad \text{and also weakly in } L^2(0, T; V). \tag{3.33}$$

So, we have that $u(t) \in W^{1,\infty}(0, T; V)$. We also deduce (still denoting by $((\cdot, \cdot))$ the scalar product in V) that, as $k \to +\infty$,

$$\forall v \in V, \qquad ((u_k(t), v)) \to ((u(t), v)) \text{ uniformly on } [0, T] \quad \text{(and, in particular, } u_k(T) \to u(T) \text{ weakly in } V). \tag{3.34}$$

Now, it is clear that (1.3) holds. We have to verify that $u(t)$ satisfies (1.1), and (1.2) (or equivalently, (3.1)).

Firstly, we prove that (1.1) holds. Towards this aim, by using (3.4), we get that

$$Re \int_0^T \big(\beta(u_k'(t)) - \beta(v(t)), u_k'(t) - v(t)\big)\, dt \geqslant 0, \quad \forall v(t) \in L^2(0, T; V). \tag{3.35}$$

Next, thanks to (3.31), and to (3.33), we obtain that

$$\mathrm{Re} \int_0^T (-\beta(v(t)), u'(t) - v(t))\, dt \geqslant 0, \qquad \forall v(t) \in L^2(0, T; V). \tag{3.36}$$

We now use a standard argument. We take, in (3.36), $v(t) = u'(t) + + \lambda w(t)$, where $\lambda > 0$ and $w(t) \in L^2(0, T; V)$. Then, after division by λ, we let $\lambda \to 0^+$ in the resulting inequality; thanks to the hemicontinuity of β, we get that

$$\text{(3.37)} \qquad \operatorname{Re}\int_0^T (\beta(u'(t)), w(t))\,dt \geq 0\,, \qquad \forall w(t) \in L^2(0, T; V)\,.$$

(Clearly, (3.37) is, in fact, an equality). So, we can deduce that

$$\text{(3.38)} \qquad \beta(u'(t)) = 0 \text{ for a.e. } t \in]0, T[\,,$$

and hence (thanks to (3.3)) (1.1) holds.

(We want to point out that the fact that (3.37) $\Rightarrow$ (3.38) follows from a general property of complex Hilbert spaces. Let indeed W be a complex Hilbert space, and let W^* be its antidual space. Denote by $(\cdot, \cdot)$ the antiduality pairing between W^* and W. Let $z \in W^*$ satisfy either $\operatorname{Re}(z, w) = 0, \forall w \in W$, or $\operatorname{Im}(z, w) = 0, \forall w \in W$. Then, it results that $z = 0$. This fact can be readily proved, by using e.g. the definition and the properties of the canonical antiduality operator J from W onto W^*)

We now prove that $u(t)$ also satisfies (1.2). Towards this aim, it is more convenient to show that the (equivalent) property (3.1) holds (recall the preliminary step *a*) in this section). We start by considering the functions $v(t)$ such that

$$\text{(3.39)} \qquad v(t) \in L^2(0, T; V)\,, \qquad v(t) \in \mathcal{K} \text{ for a.e. } t \in]0, T[\,.$$

We «multiply» (in the antiduality pairing between V^* and V) both sides of (3.7) (I) by $v(t) - u_k'(t)$. By taking the imaginary parts, and using (3.4), we get

$$\text{(3.40)} \qquad \operatorname{Im}(ik^{-1}u_k''(t) + u_k'(t) + iA(t)\,u_k(t) - f(t), v(t) - u_k'(t)) =$$

$$= k\operatorname{Re}(\beta(v(t)) - \beta(u_k'(t)), v(t) - u_k'(t)) \geq 0\,,$$

$\forall v(t)$ as in (3.39), and for a.e. $t \in]0, T[$.

We now fix any real number p, such that $p < -M_2 \cdot c^{-1}$ (recall the assumptions (2.3) and (2.5)). Let us define, for every $w(t) \in L^2(0, T; V)$,

$$\text{(3.41)} \qquad |||w(t)|||^2 \equiv \int_0^T \exp(pt)(-[pA(t) + A'(t)]\,w(t), w(t))\,dt\,.$$

Thanks to (2.3), (2.4), (2.5), and to the choice of p, we can readily verify that $|||\cdot|||$ is an equivalent Hilbert norm on $L^2(0, T; V)$.

Next, we multiply both sides of (3.40) by $\exp(pt)$, and we integrate the resulting inequality from 0 to T. So, we obtain

$$(3.42)\qquad \int_0^T \exp(pt)\,\mathrm{Im}\,(ik^{-1}u_k''(t) + u_k'(t) + iA(t)u_k(t) - $$

$$-f(t),\, v(t) - u_k'(t))\,dt \geqslant 0\,, \qquad \forall v(t) \text{ as in (3.39)}.$$

Now, thanks to the properties of $A(t)$ and of $u_k(t)$, (3.42) can be rewritten in the following way:

$$(3.43)\qquad \mathrm{Im}\int_0^T \exp(pt)\cdot(ik^{-1}u_k''(t) + u_k'(t) + iA(t)\,u_k(t),\, v(t))\,dt -$$

$$-\,\mathrm{Im}\int_0^T \exp(pt)\cdot(f(t),\, v(t) - u_k'(t))\,dt \geqslant$$

$$\geqslant\left[\int_0^T \exp(pt)\cdot\mathrm{Re}\,(k^{-1}u_k''(t),\, u_k'(t))\,dt + \right.$$

$$\left.+\int_0^T \exp(pt)\cdot\mathrm{Re}\,(A(t)\,u_k(t),\, u_k'(t))\,dt\right] =$$

$$= \mathrm{Re}\int_0^T \exp(pt)\cdot(k^{-1}u_k''(t),\, u_k'(t))\,dt +$$

$$+\frac{1}{2}\exp(pT)\,(A(T)\,u_k(T),\, u_k(T)) - \frac{1}{2}\,(A(0)\,u_0,\, u_0) +$$

$$+\frac{1}{2}\int_0^T \exp(pt)\cdot(-\,[pA(t) + A'(t)]\,u_k(t),\, u_k(t))\,dt\,, \quad \forall v(t) \text{ as in (3.39)}.$$

Next, we consider the above subsequence $\{u_k(t)\,|\,k \geqslant 1\}$ (satisfying (3.32), (3.33), (3.34) (and (3.30))), and we take the lim inf, as $k \to +\infty$ of both sides of the inequality (3.43). Let us observe that the lim inf of the left-hand side is, in fact, a limit. Moreover, by considering the fourth term at the right-hand side, we recall that $|||\cdot|||$ (see (3.41)) is an equivalent Hilbert norm on $L^2(0, T; V)$. So, by also using some standard arguments, we get from (3.43), as $k \to +\infty$,

$$\text{(3.44)}\quad \operatorname{Im}\int_0^T \exp(pt)\cdot(u'(t) + iA(t)u(t),\, v(t))\,dt -$$

$$-\operatorname{Im}\int_0^T \exp(pt)\cdot(f(t),\, v(t) - u'(t))\,dt \geqslant$$

$$\geqslant \frac{1}{2}\exp(pT)\cdot(A(T)u(T),\, u(T)) - \frac{1}{2}(A(0)u_0,\, u_0) +$$

$$+\frac{1}{2}\int_0^T \exp(pt)\cdot(-[pA(t) + A'(t)]u(t),\, u(t))\,dt =$$

$$= \int_0^T \exp(pt)\cdot\operatorname{Im}(u'(t) + iA(t)u(t),\, u'(t))\,dt, \qquad \forall v(t) \text{ as in (3.39)}.$$

We have thus obtained that $u(t)$ satisfies (3.1)(I) ($\forall v(t)$ as in (3.39)), for every real number p such that $p < -M_2\cdot c^{-1}$. Hence, $u(t)$ also satisfies (1.2). So, Theorem 2.1 is completely proved.

REMARK 3.1. If we assume, *in addition* to (2.3), (2.4), (2.5), that $A(t)$ *also* satisfies

$$\text{(3.45)}\qquad (A'(t)v,\, v) \leqslant 0, \qquad \forall v \in V,\ \forall t \in [0, T],$$

(in particular, if $A(t) = A$ doesn't depend on t), then the proof of Theorem 2.1 can be carried out in a much simpler way, as regards specifically the a priori estimates (I) and (II), and the passage to the limit. (In particular, in such a case, we need not use the device introduced in the previous step *a*)). However, when $A(t)$ actually depends on t, (3.45) would be a quite restrictive assumption.

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