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A Property Equivalent to Permutability for Groups.

A. MOHAMMADI HASSANABADI (*)

ABSTRACT - In this note we prove the following: Let m and n be positive integers and G a group such that $X_1 X_2 \dots X_n \cap \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} X_{\sigma(2)} \dots X_{\sigma(n)} \neq \emptyset$ for all subsets X_i of G where $|X_i| = m$ for all $i = 1, 2, \dots, n$; then G is finite-by-abelian-by-finite.

1. - Introduction.

Permutable groups have been studied by various people–see [1], [2], [3], [4] and [5]. Recall that a group G is called n-permutable if given any sequence $x_1, x_2, \dots, x_n$ of elements of G, then $x_1 x_2 \dots x_n = x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)}$ for some permutation $\sigma \neq 1$ of the set $\{1, 2, \dots, n\}$. Also a group is said to be permutable if it is n – permutable for some $n > 1$. The main result for groups in this class was obtained by Curzio, Longobardi, Maj and Robinson in [3] where it was shown that such groups are finite-by-abelian-by-finite.

Let m, n be positive integers. A natural extension of permutable groups, namely (m, n)-permutable groups–groups in which

$$X_1 X_2 \dots X_n \subseteq \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} X_{\sigma(2)} \dots X_{\sigma(n)}$$

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for all subsets X_i of G where $|X_i| = m$ for all $i = 1, 2, \dots, n$–was introduced in [6]. It was proved there that such a group is either n-permutable or it is finite of order bounded by a function of m and n. Here we deal with another extension of (m, n)-permutable groups.

For positive integers m and n call a group G restricted (m, n)-permutable if

$$X_1 X_2 \dots X_n \cap \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} X_{\sigma(2)} \dots X_{\sigma(n)} \neq \emptyset$$

for all subsets X_i of G where $|X_i| = m$ for all $i = 1, 2, \dots, n$. Thus restricted $(1, n)$-permutable groups are just $(1, n)$-permutable groups which are n-permutable groups.

The main result of this note is the following

THEOREM. *Suppose that m and n are positive integers, and G a restricted (m, n)-permutable group. Then G is finite-by-abelian-by-finite.*

2. – Proofs.

Throughout we assume G to be a group and denote its centre by $Z(G)$. We first prove that the centre of a restricted (m, n)-permutable-group which is not n-permutable has finite order bounded by a function of m and n.

LEMMA 1. *Suppose that m and n are positive integers and G a restricted (m, n)-permutable group which is not n-permutable. Then* $\exp(Z(G)) \leq ((mn)^{n!+1})!$

PROOF. Let $z \in Z(G)$. Let $x_1, x_2, \dots, x_n$ be elements in G such that the product $x_1 x_2 \dots x_n$ cannot be rewritten and consider the sets

$$X_i = x_i \{z, z^2, \dots, z^m\}, \qquad i = 1, 2, \dots, n .$$

Then there exists a non-trivial permutation σ such that $x_1 z^{i_1} x_2 z^{i_2} \dots \dots x_n z^{i_n} = x_{\sigma(1)} z^{j_1} x_{\sigma(2)} z^{j_2} \dots x_{\sigma(n)} z^{j_n}$ where $i_s, j_s \in \{1, 2, \dots, m\}$ for $s = 1, 2, \dots, n$; so that

$$x_1 x_2 \dots x_n = x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)} z^{\alpha_1},$$

where $-n(m-1) \leq \alpha_1 \leq n(m-1)$; $\alpha_1 \neq 0$.

Now replace z by $z^{(mn)^k}$ and let k run through $0, 1, 2, \dots, n!$. Then, there exist t and t' in $\{0, 1, \dots, n!\}$, such that $t \neq t'$, say $t > t'$ and $z^{(mn)^t \alpha} = z^{(mn)^{t'} \beta}$. So $z^{(mn)^t \alpha - (mn)^{t'} \beta} = 1$ where $-n(m-1) \leq \alpha, \beta \leq n(m-1)$; $\alpha, \beta \neq 0$ and thus $(mn)^t \alpha - (mn)^{t'} \beta \neq 0$, since otherwise $(mn)^{t-t'} = \alpha/\beta$ which is not possible. Thus $o(z) \mid (mn)^t \alpha - (mn)^{t'} \beta = \gamma$ and $|\gamma| \leq |(mn)^t \alpha| \leq (mn)^{t+1} \leq (mn)^{n!+1}$. Therefore $\exp(Z(G)) \leq ((mn)^{n!+1})!$

LEMMA 2. *Let m and n be positive integers. If G is a restricted (m, n)-permutable group, then either G is n-permutable or $|Z(G)|$ is finite bounded by* $((mn)^{n!+1})![mn(n!+1)]$.

PROOF. Suppose that G is a counterexample. Then since by Lemma 1 $\exp(Z(G)) \leq ((mn)^{n!+1})!$

$$(*) \qquad Z(G) = \langle z_1 \rangle \times \langle z_2 \rangle \times \dots$$

is a direct product of cyclic groups of size at most $\exp(Z(G))$. Let $x_1, x_2, \dots, x_n$ be elements in G such that the product $x_1 x_2 \dots x_n$ cannot be rewritten and consider

$$X_1 = x_1 \{z_1, \dots, z_m\},$$

$$X_2 = x_2 \{z_{m+1}, \dots, z_{2m}\},$$

$$\vdots$$

$$X_n = x_n \{z_{(n-1)m}, \dots, z_{nm}\}.$$

Then $x_1 x_2 \dots x_n = x_{\sigma_1(1)} x_{\sigma_1(2)} \dots x_{\sigma_1(n)} z_1'$ for some $\sigma_1 \in S_n \setminus 1$ and $z_1' \in \langle z_1 \rangle \times \dots \times \langle z_{nm} \rangle$; $z_1' \neq 1$. Now use the next nm direct factors in $(*)$ to obtain

$$x_1 x_2 \dots x_n = x_{\sigma_2(1)} x_{\sigma_2(2)} \dots x_{\sigma_2(n)} z_2'$$

with $\sigma_2 \in S_n \setminus 1$ and $z_2' \in \langle z_{nm+1} \rangle \times \dots \times \langle z_{2nm} \rangle$; $z_2' \neq 1$.

If $|Z(G)| > ((mn)^{n!+1})![mn(n!+1)]$ then the number of factors in (*) is at least $(n!+1)\, mn$ and we may continue the above process to obtain

$$x_1 x_2 \dots x_n = x_{\sigma_3(1)} x_{\sigma_3(2)} \dots x_{\sigma_3(n)} z_3'$$

for some $\sigma_3 \in S_n \backslash 1$ and $z_3' \in \langle z_{2nm+1} \rangle \times \ldots \times \langle z_{3mn} \rangle$; $z_3' \neq 1$

$$\vdots$$

$$x_1 x_2 \ldots x_n = x_{\sigma_{n!+1}(1)} x_{\sigma_{n!+1}(2)} \ldots x_{\sigma_{n!+1}(n)} z'_{n!+1}$$

for some $\sigma_{n!+1} \in S_n \backslash 1$ and $z'_{n!+1} \in \langle z_{n!(mn)+1} \rangle \times \ldots \times \langle z_{(n!+1)mn} \rangle$, $z'_{n!+1} \neq 1$. Thus there exist i and j, $1 \leqslant i, j \leqslant n! + 1$ such that $i \neq j$ and $z_i' = z_j'$ which is not possible. This completes the proof.

We next want to prove our key lemma that the FC-centre of a non-trivial restricted (m, n)-permutable group is not trivial, and we find it easier to show this first for a general version of the restricted (m, n)-permutable groups.

Let $m_1, m_2, \ldots, m_n$ be positive integers and call a group G restricted $(m_1, \ldots, m_n)$-permutable if $X_1 X_2 \ldots X_n \cap \bigcup_{\sigma \in S_n \backslash 1} X_{\sigma(1)} X_{\sigma(2)} \ldots X_{\sigma(n)} \neq \emptyset$ for all subsets X_i of size m_i in G, $i = 1, 2, \ldots, n$. Then we have

LEMMA 3. *Suppose that G is a non-trivial restricted $(m_1, \ldots, m_n)$-permutable group. Then the FC-centre of G is non-trivial.*

PROOF. We use induction on the sum $s = m_1 + \ldots + m_n$. If $s = n$ then $m_i = 1$ for all i and so G is n-permutable and the result follows from [3]. So assume that the result holds for all $s < t$ and suppose that $m_1 + m_2 + \ldots + m_n = t$ and G is restricted $(m_1, m_2, \ldots, m_n)$-permutable. Then by induction there exist subsets $Y_1, Y_2, \ldots, Y_n$ of G such that $|Y_1| + |Y_2| + \ldots + |Y_n| = t - 1$ and

$$(**) \qquad Y_1 Y_2 \ldots Y_n \cap \bigcup_{\sigma \in S_n \backslash 1} Y_{\sigma(1)} Y_{\sigma(2)} \ldots Y_{\sigma(n)} = \emptyset .$$

Let

$$S = (Y_{n-1}^{-1} \ldots Y_1^{-1}) \left(\bigcup_{\sigma \in S_n \backslash 1} Y_{\sigma(1)} \ldots Y_{\sigma(n)} \right) \bigcup_{\substack{\sigma \in S_n \backslash 1 \\ 1 \leqslant i \leqslant n}} ,$$

$$Y_{\sigma(i-1)}^{-1} \ldots Y_{\sigma(1)}^{-1} Y_1 \ldots Y_n Y_{\sigma(n)}^{-1} \ldots Y_{\sigma(i+1)}^{-1},$$

with the convention that $Y_{\sigma(i-1)}^{-1} \ldots Y_{\sigma(1)}^{-1} = 1$ if $i = 1$ and $Y_{\sigma(n)}^{-1} \ldots Y_{\sigma(i+1)}^{-1} = 1$ if $i = n$. Then S is a finite subset of G.

Now for any $a \in G \backslash S$, define $X_i = Y_i$; $i = 1, 2, \ldots, n-1$ and $X_n = Y_n \cup \{a\}$. Then $X_1 X_2 \ldots X_n \cap \bigcup_{\sigma \in S_n \backslash 1} X_{\sigma(1)} X_{\sigma(2)} \ldots X_{\sigma(n)} \neq \emptyset$. This together with $(**)$ and the choice of a imply that there exist $x_i \in X_i$; $i =$

$=1, 2, \dots, n-1$ such that $x_1 x_2 \dots x_{n-1} a = x'_{\sigma(1)} \dots x'_{\sigma(i-1)} a x'_{\sigma(i+1)} \dots x'_{\sigma(n)}$ for some $\sigma \in S_n \backslash 1$ and $x_j' \in X_j$. This gives $a g_\sigma a^{-1} = f_\sigma^{-1} c$ where $f_\sigma = x'_{\sigma(1)} \dots x'_{\sigma(i-1)}$, $g_\sigma = x'_{\sigma(i+1)} \dots x'_{\sigma(n)}$ and $c = x_1 x_2 \dots x_{n-1}$. Now there exist only finitely many choices for f_σ and g_σ, and so G is the union of finitely many cosets of centralizers of g_τ's ($\tau \in S_n \backslash \{1\}$). Therefore, by a famous theorem of B. H. Neumann [7], one of the centralizers is of finite index and the proof is complete.

As an immediate corollary to Lemma 3 we have

LEMMA 4. *Let m and n be positive integers. Then a non-trivial restricted (m, n)-permutable group has non-trivial FC-centre.*

We are now able to give the proof of the main result.

PROOF OF THE THEOREM. By Lemma 4 there exists an element $x_1 \in G \backslash \{1\}$ such that $[G: C_G(x_1)]$ is finite. If $G_1 := C_G(x_1)$ is n-permutable then G is finite-by- abelian-by-finite. Thus we may assume that G_1 is not n-permutable. So $Z(G_1)$ is finite by Lemma 2, and $\langle x_1 \rangle \leqslant Z(G_1)$ is finite. Therefore $G_1 / \langle x_1 \rangle \neq 1$ and, again by Lemma 4, there exists $x_2 \in G_1 \backslash \langle x_1 \rangle$ such that $[G_1 / \langle x_1 \rangle : C_{G_1/\langle x_1 \rangle}(x_2)]$ is finite.

Write $V/\langle x_1 \rangle := C_{G_1/\langle x_1 \rangle}(x_2)$. Then $[V: C_{G_1}(x_2)]$ is finite, since $\langle x_1 \rangle$ is finite, and $[G_1: C_{G_1}(x_2)]$ is finite. If $G_2 := C_{G_1}(x_2)$ is n-permutable, we are done. So suppose that G_2 is not n-permutable. Continuing the above process we obtain sequences $x_1, x_2, \dots$ of distinct elements of G and $G_1, G_2, \dots$ of subgroups of G such that for each $i = 1, 2, \dots$; $\langle x_1, x_2, \dots, x_i \rangle \leqslant Z(G_i)$. Now if G_j is n-permutable for some j then by [3] G_j and therefore G is finite-by-abelian-by-finite. Otherwise, by Lemma 2, $Z(G_i)$ is boundedly finite for all i and so the process must stop after a bounded number of times. This means that there exists some positive integer l such that G_l is an n-permutable group. This completes the proof.

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