Rendiconti
del
Seminario Matematico
della
Università di Padova

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*Rendiconti del Seminario Matematico della Università di Padova*, tome 100 (1998), p. 1-11
<http://www.numdam.org/item?id=RSMUP_1998__100__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 100 (1998)

# A Note on Groups with Hamiltonian Quotients.

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ABSTRACT - The *norm* $\kappa(G)$ of a group $G$ is the subgroup of those elements which normalise every subgroup of $G$. An ascending series of subgroups $\kappa_i(G)$ may be defined in a familiar way by iteration, even transfinitely. It is known that, in a 2-group $G$, every section $\kappa_{i+1}(G)/\kappa_i(G)$ is abelian or Hamiltonian, and only the top-most section can ever be Hamiltonian. We produce what may be the first example of a non-Hamiltonian 2-group in which this top section is Hamiltonian.

## 1. – Introduction.

The *norm* $\kappa(G)$ of a group $G$ is the subgroup of elements normalising every subgroup of $G$. An ascending series of subgroups may be defined by iteration: write $\kappa_0(G) = 1$ and, for $i \geq 1$, $\kappa_i(G)/\kappa_{i-1}(G) := \kappa(G/\kappa_{i-1}(G))$. This definition can be extended transfinitely in the usual way. Since $\kappa(G)$ is Dedekindian (it normalises all of its own subgroups) it is, by a result of Dedekind [4], either abelian or Hamiltonian, and that is a direct product of a quaternion group of order eight and a periodic abelian group without elements of order 4. Schenkman [9] shows that $\kappa(G) \subseteq \zeta_2(G)$. A result of Baer [1] says that $G = \kappa(G)$ if $G$ is a 2-group and $\kappa(G)$ is Hamiltonian. It follows that, for a 2-group $G$, if ever a factor $\kappa_{i+1}(G)/\kappa_i(G)$ is Hamiltonian, then $\kappa_{i+1}(G) = G$. However, it seems to be difficult to find non-Hamiltonian 2-groups having a Hamiltonian factor in this series. In this article we construct one such example, proving the following theorem.

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THEOREM 1.1. *Let $G$ be a 2-group with the following properties:*

(*a*) *$G/\kappa(G)$ is Hamiltonian;*

(*b*) *no proper section of $G$ also has property* (*a*).

*Then $G$ is of order $2^7$ and, up to isomorphism, there is just one such group.*

We have as an almost immediate corollary.

COROLLARY 1.2. *In a 2-group $X$ suppose that $\kappa_2(X) = X$. Then $X/\kappa(X)$ is Hamiltonian if and only if $X$ has a section isomorphic to the group $G$.* ∎

These results make no assumptions about finiteness, though local finiteness is an immediate consequence of the fact that $\kappa_2(G) = G$. Since in a nilpotent group the norm coincides with the Wielandt subgroup, this result may be thought of as a step towards the classification of 2-groups with Wielandt length 2. For odd primes $p$ the classification of $p$-groups with Wielandt length 2 has been completed by Ormerod [8], who shows that such groups may be described in terms of groups of class 2 and a small number of exceptions. The difficulties for 2-groups are known to be considerable and, in a sense, Theorem 1 adds to them.

The restriction to groups of length 2 in Theorem 1 seems difficult to remove. What we have been able to do, however, is to reduce the general problem to one concerning finite 2-groups.

THEOREM 1.3. *Let $j$ be a limit ordinal, and $i$ a non-negative integer. There is a 2-group $G$ in which $G/\kappa_{j+i}(G)$ is Hamiltonian if and only if there is a finite 2-group $H$ for which $H/\kappa_i(H)$ is Hamiltonian.*

## 2. – Proof of Theorem 1.1.

First notice that $\kappa(G)$ is abelian or else, by the theorem of Baer [1], $\kappa(G)$ would be $G$. Since $G/\kappa(G)$ is Hamiltonian there is a pair of elements $x, y$ for which $[x, y] \notin \kappa(G)$. Hence there is an element $c$ of $G$ for which $c^{[x,y]} \notin \langle c \rangle$. From property (*b*) it follows at once that $G = \langle x, y, c \rangle$. Since $G$ is a finitely generated nilpotent 2-group it is finite. Note also that $t := [x, y, c] \neq 1$. In particular this means that $c \notin C_G(G')$, hence $c \notin \zeta_2(G)$, and therefore $c \notin \kappa(G)$ by Schenkman [9].

A minimal normal subgroup $Z$ of $G$ is central and of order 2, say $Z = \langle z \rangle$. Now $Z \subseteq \kappa(G)$ and, by property $(b)$, all commutators in the group $G/Z$ are in its norm. Hence $t = c^r z$ for some integer $r$ satisfying $0 \leq r < o(c)$. Note that $z$ cannot be a power of $c$, or else we contradict the choice of $c$. Hence

(1) $$\langle c \rangle \cap \zeta_1(G) = 1 .$$

Moreover if $\langle z' \rangle$ is another minimal normal subgroup of $G$ then, for some $s$ satisfying $0 \leq s < o(c)$, we have $t = c^s z'$. Since $c^{r-s} = z^{-1} z'$, by (1) we must have $r = s$, and then $z = z'$. In other words

(2) $G$ has a unique minimal normal subgroup.

By the theorem of Schenkman [9] $\kappa(G) \subseteq \zeta_2(G)$ and therefore $G' \subseteq C_G(\kappa(G))$. Also $G' \kappa(G) = \langle [x, y] \rangle \kappa(G)$, so

(3) $G' \kappa(G)$ is abelian and $G/G' \kappa(G)$ is of exponent 2.

We are also able to prove that

(4) $G$ has class exactly 3, and $\gamma_3(G)$ has exponent 2.

Note that $G$ is generated, modulo $\kappa(G)$, by elements $u$ whose square is not in $\kappa(G)$. This is because a Hamiltonian 2-group is generated by elements of order 4. Now for every such element $u$, and some element $v$ of $\kappa(G)$, $[x, y] = u^2 v$. Hence $[x, y, u] = [v, u] \in \zeta_1(G)$. It follows that all commutators of weight 4 in $G$ are trivial, and so $G$ is of class at most 3. It is of class exactly 3 since, by the choice of $c$, $[x, y, c] \neq 1$. That $\gamma_3(G)$ is of exponent 2 now follows from (3): commutators and squares commute.

Next we prove that

(5) $G'$ has exponent dividing 4.

To see this we argue as follows. For all elements $u$, $v$ in $G$ we have from (3) that

$$1 = [u^2, v^2] = [u^2, v]^2 [u^2, v, v] = [u, v]^4 .$$

A corollary of this is that fourth powers in $G$ are central:

(6) $G/\zeta_1(G)$ is of exponent 4.

For,

$$[u^4, v] = [u^2, v]^2[u^2, v, u^2] = [u, v]^4[u, v, u]^2[u, v, u]^4 = 1$$

for all elements $u, v \in G$, by (4), (5).

Next observe that $[c, \kappa(G)]$ and $\langle c^4 \rangle$ are central, and subgroups of $\langle c \rangle$, and therefore, by (1),

(7) $$[c, \kappa(G)] = 1 \text{ and } c^4 = 1 .$$

We now prove that

(8) $u^4 = t$ whenever $u \in G$, but $u^2 \notin \kappa(G)$; and $[x, y]^2 = t$.

First of all we show that if the order of such an element $u$ is $2^a$ then $a \geqslant 3$ and $u^{2^{a-1}} = t$. For some $k \in \kappa(G)$,

$$[u^2, c] = [[x, y]k, c] = [x, y, c] \neq 1$$

since, by (7), $c$ commutes with every element of $\kappa(G)$. This shows, in particular, that $c^2 \in \kappa(G)$. It then follows that $[G, c] \subseteq \kappa(G)$ since, in the Hamiltonian group $G/\kappa(G)$, all involutions are central. Hence

$$t = [x, y, c] = [u^2, c] = [u, c]^2[u, c, u] = [u, c^2][u, c, u],$$

and the right side is a power of $u$, which must be at least $u^4$. If $u$ has order $2^a$, then $t = u^{2^{a-1}}$. Since $u^2 \notin \kappa(G)$, $a \geqslant 3$.

Suppose now that $u$, $v$ and $uv$ are, all three, elements of order 4 modulo $\kappa(G)$, that $u$, $v$ have orders $2^a$, $2^b$ respectively, with $b \geqslant a \geqslant 3$, and that $a$ is as small as possible. Suppose that $b > a$. Write $w = uv^{2^{b-a}}$. Recalling (3), we find

$$w^2 = u^2 v^{2^{b-a+1}}[v^{2^{b-a}}, u] = u^2 v^{2^{b-a+1}}[v, u]^{2^{b-a}} \pmod{\gamma_3(G)} .$$

Therefore, using (3), (5),

$$w^{2^{a-1}} = u^{2^{a-1}} v^{2^{b-1}} = t^2 = 1 .$$

Then $w, v, wv$ would be a valid triple with a smaller value of $a$, a contradiction.

Therefore $a = b \geqslant 3$. If $a > 3$ then we have

$$(uv)^2 = u^2 v^2 \pmod{G'}$$

and so

$$t=(uv)^{2^{a-1}}=u^{2^{a-1}}v^{2^{a-1}}=t^2=1$$

using (3), (5), another contradiction, so $a=3$. Finally, to complete the proof of the first part of (8), we note that if $g$ is an arbitrary element of $G$ whose square is not in $\kappa(G)$, then one of $ug$ or $vg$ has the same property, whence $g^4=t$. Then note that $t=(xy)^4=(x^2y^2[y,x][y,x,y])^2=$ $=[y,x]^2$. This completes the proof of (8).

(9) $$G/\kappa(G)\cong Q_8\times C_2\,.$$

Since $G/\kappa(G)$ is Hamiltonian and can be generated by the images of $x, y, c$, it suffices to show that $G/\kappa(G)$ is not generated by the images of $x, y$. In the course of the proof of (8), we showed that $c^2\in\kappa(G)$. Hence, if $G=\langle x,y\rangle\,\kappa(G)$, then $c\in\kappa(G)\langle[x,y]\rangle\subseteq\zeta_2(G)$, a contradiction.

The next step is to prove

(10) $$[u,c]^2=c^2=1\,,\quad \textit{where } u^2\notin\kappa(G)\,.$$

From (7) and (8): $t=(uc)^4=(u^2c^2[c,u])^2=u^4[c,u]^2=t[c,u]^2$ whence $[c,u]^2=1$. But then $[c^2,u]=[c,u]^2[c,u,c]=1$. Since $G$ is generated modulo $\kappa(G)$ by elements whose squares do not lie in $\kappa(G)$, we conclude that $c^2\in\zeta_1(G)$ and therefore, by (1), that $c^2=1$.

(11) $\kappa(G)$ has exponent 2.

The elements of $\kappa(G)$, all have the form

$$x^{2e}y^{2f}[x,y]^g[x,c]^h[y,c]^i[x,y,c]^j$$

where $e, f, g, h, i, j$ are each either 0 or 1, and where exactly two of $e, f, g$ are equal to 1. It now follows from (8), (10) that every element of $\kappa(G)$ is of order at most 2.

Now write $U=\kappa(G)$ and $M=U\langle c\rangle$. Both may be regarded as modules over $\boldsymbol{Z}_2$, the field of two elements, for the group $G/M$ which is, of course, isomorphic to $Q_8$. As such $M$ is monolithic with unique minimal submodule $S=\langle t\rangle$. We aim to show that

(12) $$|U:S|=4\,.$$

Firstly note that $U$ is a monolithic module for the group $G/MG'$, which is isomorphic to $C_2\times C_2$. Hence $U$ is isomorphic to a submodule, of Loewy length at most 2, since $U \subseteq \zeta_2(G)$, of the regular $Z_2(G/MG')$ module. This means that $|U:S| \leqslant 4$. It suffices therefore to find a pair of elements of $U$ which are independent modulo $S$.

To this end note that, for every element $u \in G \backslash MG'$,

$$1 \neq t = [c, u^2] \in S .$$

We see also that $[c, u]$ is not in $S$. For, $[c, u] \in S$ means that $[c, u] = [c, u^2]$ whence $c^u = c^{u^2}$ and therefore $1 = [c, u] = [c, u^2]$, a contradiction. Now if $[c, x]$ and $[c, y]$ were dependent modulo $S$ we would have $[c, x][c, y] \in S$. From this we would get $c^x c^y \in S$ which would mean that $[c, yx^{-1}] \in S$, again a contradiction. This completes the proof of (12).

This also completes the proof of the first part of the theorem since now, from (9), (12), $|G| = 2^7$. For the second part we rely on the library of groups of order $2^7$ (Newman and O'Brien [7]) to prove the existence and uniqueness of a group with the properties $(a)$ and $(b)$. There are 2328 groups in the list and, rather than try to search them all, we use some of the properties that we have found to eliminate non-starters. An outline of the steps used in this search follows.

Firstly the fact that $G$ has three generators reduces the list to 833 groups. The fact that $G'$ is isomorphic to $C_4 \times C_2 \times C_2$ further reduces it to 122 groups. Since an element $g$ of $G$ has order 8 if and only if $g \notin \langle G', c \rangle$, a group of order 32, there are 96 elements of order 8 in $G$. This reduces the list to four groups. Since there is an elementary abelian subgroup of order 8 not containing $c$, there are at least eight elements of order 2. This reduces the list to two groups. Finally the centre of $G$ is of order 2, leaving a unique candidate. This group, which we call $\Gamma$, is number 801 in the library. By calculation we find, indeed, that $\Gamma/\kappa(\Gamma) \cong Q_8 \times C_2$. (Newman and O'Brien have an unpublished routine for doing this.) ■

A power-commutator presentation of $\Gamma$ is the following. The generators are $x_1, x_2, \ldots, x_7$, and the relations are

$$x_1^2 = x_4 x_5 , \quad x_2^2 = x_4 , \quad x_3^2 = 1 , \quad x_4^2 = x_7 , \quad x_5^2 = x_6^2 = x_7^2 = 1 ,$$

$$[x_2, x_1] = x_4 , \quad [x_3, x_1] = x_5 , \quad [x_3, x_2] = x_6 ,$$

$$[x_4, x_1] = [x_4, x_3] = [x_5, x_1] = [x_6, x_1] = [x_6, x_2] = x_7 ,$$

and all commutators of pairs of generators not listed are trivial, and the group made of class 3.

The proof of Theorem 1.1 gives enough properties of the group we sought, to enable it to be identified by computer. However we have not much insight into its structure. We are grateful to the referee for suggesting a method of construction which does give more insight. The construction we now give is a variation on his; we suppress most of the details.

We begin with two copies of the «nameless» group of order 16 with a cyclic subgroup of order 8 (Huppert [5] Satz I.14.9 (3)). Let $H_i = \langle n_i, s_i \colon n_i^8 = s_i^2 = 1, s_i^{-1} n_i s_i = n_i^5 \rangle$ $(i = 1, 2)$. Set $H = H_i \times H_2$ and define automorphisms $\alpha$, $\beta$ of $H$ as follows: $\alpha(n_i) = n_i s_i$, $\alpha(s_i) = s_i$ $(i = 1, 2)$, $\beta(n_1) = n_1^3$, $\beta(s_1) = s_1 n_1^4$, $\beta(n_2) = n_2$ and $\beta(s_2) = s_2$. It is easy to check that $\alpha$ and $\beta$ are commuting automorphisms of $H$ of order 2 so that $A = \langle \alpha, \beta \rangle$ is elementary abelian of order 4. We let $K$ be the semi-direct product of $H$ by $A$, and in $K$ we take $M = \langle n_1, s_1, n_2\beta, s_2, \alpha \rangle$ and $N = \langle n_1^2 n_2^2 s_2 \rangle = \langle n_1^2 (n_2\beta)^2 s_2 \rangle$. It is easy to see that $N$ is normal in $H$ and contained in $M$. Our claim is that $M/N \cong \Gamma$ and that $\kappa(M/N) = \langle s_1 N, s_2 N, n_1^6 N \rangle$. The proof of this seems non-trivial and was confirmed for us by Newman by computer.

## 3. - Proof of Theorem 1.2.

Let $j$ be a limit ordinal and $P$ be a $p$-group with the following properties:

(13) (i) $\zeta_j(P) = P$; (ii) $\zeta_{k+1}(P)/\zeta_k(P)$ has finite exponent $(k < j)$.

The existence of such a group is guaranteed by an example of McLain [6] in which the upper central factors are all elementary abelian. We use this group to prove the following result.

LEMMA 3.1. *Let $K$ be a finite $p$-group, $P$ a group satisfying* (13), *and write $W := P \mathrm{wr} K$. Let $k$ be an ordinal $\leqslant j$.*

(*a*) *If $k$ is a limit ordinal then*

$$\zeta_k(W) = \zeta_k(P)^K .$$

(*b*) *In general there is a positive integer $n = n(k)$ such that*

$$\zeta_k(W) \subseteq \zeta_k(P)^K \subseteq \zeta_{k+n}(W) .$$

PROOF. We start by proving $(b)$ when $k$ is finite. The case $k=0$ is clear. When $k=1$ it is easy to see that $\zeta_1(W)\subseteq\zeta_1(P)^K$; and the second inclusion holds because $\zeta_1(P)\, wrK$ is nilpotent by a result of Baumslag [2]. Now suppose that $k\geqslant 1$ and that $(b)$ is already established for $k$; that is that $\zeta_k(W)\subseteq N:=\zeta_k(P)^K\subseteq\zeta_{k+n}(W)$ for some positive integer $n$. Then

$$\zeta_{k+1}(W)\,N/N\subseteq\zeta_1(W/N)\subseteq\zeta_1(PN/N)^K\subseteq\zeta_m(W/N)$$

for some positive integer $m$, whence

$$\zeta_{k+1}(W)\subseteq\zeta_{k+1}(P)^K\subseteq\zeta_{k+n+m}(W)$$

as required to complete the induction, proving $(b)$ when $k$ is finite.

It follows that

$$\zeta_\omega(W)=\bigcup_{i=1}^{\infty}\zeta_i(W)\subseteq\bigcup_{i=1}^{\infty}\langle\zeta_i(P)\rangle^K\subseteq\bigcup_{i=1}^{\infty}\zeta_{i+n(i)}(W)=\zeta_\omega(W)$$

and therefore $\zeta_\omega(W)=\zeta_\omega(P)^K$, proving $(a)$ when $j=\omega$.

We prove $(a)$ by induction over the limit ordinals $i<j$. Let $k_0$ be a limit ordinal satisfying $\omega<k_0\leqslant j$ and suppose that $(a)$ is true for all limit ordinals $<k_0$. Let $l$ be the union of all these limit ordinals $k_0$. If $l=k_0$ then, by induction,

$$\zeta_{k_0}(W)=\bigcup_{i<k_0}\zeta_i(W)=\langle\bigcup_{i<k_0}\zeta_i(P)\rangle^K=\zeta_{k_0}(P)^K.$$

Next suppose that $l<k_0$. Then $k_0=l+\omega$. Let $N=\zeta_l(P)^K$. Then, by induction, $\zeta_l(W)=N$. By what we proved above $\zeta_\omega(W/N)=\zeta_\omega(PN/N)^K$, and hence $\zeta_{l+\omega}(W)=\zeta_{l+\omega}(P)^K$, as required to complete the induction and the proof of $(a)$.

The proof of $(b)$ is completed similarly. ■

In a very similar way we prove the following lemma.

LEMMA 3.2. *Suppose that $W$ is the group defined in Lemma 3.1 and let $k$ be an ordinal $\leqslant j$.*

$(a)$ *If $k$ is finite then*

$$\zeta_k(W)\subseteq\kappa_k(W)\subseteq\zeta_{2k}(W).$$

$(b)$ *If $k$ is a limit ordinal then*

$$\kappa_k(W)=\zeta_k(W).$$

PROOF. Part $(a)$ follows from Schenkman's result [9]. From $(a)$ we deduce that

$$\zeta_\omega(W) = \bigcup_{i=1}^{\infty} \zeta_i(W) \subseteq \bigcup_{i=1}^{\infty} \kappa_i(W) \subseteq \bigcup_{i=1}^{\infty} \zeta_{2i}(W) = \zeta_\omega(W)$$

and therefore $\zeta_\omega(G) = \kappa_\omega(G)$. This proves $(b)$ in the case $k = \omega$.

Again we write $l$ for the union of all the limit ordinals less than a limit ordinal $k_0$. If $k_0 = l$ then

$$\zeta_{k_0}(W) = \bigcup_{i<k_0} \zeta_i(W) = \bigcup_{i<k_0} \kappa_i(W) = \kappa_{k_0}(W).$$

On the other hand if $l < k_0$ then $k_0 = l + \omega$. By induction $\zeta_l(W) = \kappa_l(W)$ so

$$\zeta_{k_0}(W)/\zeta_l(W) = \zeta_\omega(W/\zeta_l(W)) = \kappa_\omega(W/\kappa_l(W)) = \kappa_{k_0}(W)/\zeta_l(W)$$

and so $\zeta_{k_0}(W) = \kappa_{k_0}(W)$. This completes the induction and thereby the proof of Lemma 3. ■

We are now in a position to prove Theorem 1.3. Firstly if $G$ is a 2-group in which $G/\kappa_{j+i}(G)$ is Hamiltonian, then the group $H := G/\kappa_j(G)$ satisfies $H/\kappa_i(H) \cong G/\kappa_{j+i}(G)$ and hence is Hamiltonian. We show that $H$ has a finite subgroup $L$ with $L/\kappa_i(L)$ Hamiltonian. Since $H$ is nilpotent by Schenkman [9] it is enough to find a finitely generated subgroup $L$ of $H$ for which $L/\kappa_i(L)$ is Hamiltonian.

To do this we prove the following lemma.

LEMMA 3.3. *Let $H$ be a locally finite group of finite norm length $i+1$, and suppose that $C_0$ is a finite subset of $H$ for which $C_0 \cap \kappa_i(H) = \emptyset$. There is a finite subgroup $L$ of $H$ for which $C_0 \subseteq L \backslash \kappa_i(L)$.*

PROOF. We define finite subsets $C_t$ $(0 \leqslant t \leqslant i)$ of $H$ inductively as follows. Suppose that $C_t$ has been defined for some $t$ in the range $[0, i-1]$ and that it has the property

$$C_t \cap \kappa_{i-t}(H) = \emptyset. \tag{14}$$

Note that $C_0$ satisfies this. For each $c \in C_t$ there is an element $d = d(c)$ of $H$ with the property that, for no $r \geqslant 0$, is the element $d^{-r}d^c$ in $\kappa_{i-t-1}(H)$. We define $C_{t+1} := \{d(c)^{-r}d(c)^c \colon c \in C_t, r \geqslant 0\}$. By definition $C_{t+1}$ satisfies (14). $C_{t+1}$ is finite because $H$ is locally finite.

Now let $L := \langle c, d(c) \colon c \in C_t, 0 \leqslant t \leqslant i\rangle$. We show that $L$ has the property sought. To do this we prove, by induction on $i - t$, that

$$C_t \cap \kappa_{i-t}(L) = \emptyset, \qquad 0 \leq t \leq i. \tag{15}$$

We have from (14) that $C_i \cap \kappa_0(H) = \emptyset$. This just means that $1 \notin C_i$, and hence (15) is true for $t = i$. Suppose that $0 < t \leq i$, and that (15) has been proved for it. Since none of the elements $d(c)^{-r} d(c)^c (c \in C_{t-1}, r \geq 0)$ is in $\kappa_{i-t}(L)$, we deduce that no $c \in C_{t-1}$ is in $\kappa_{i-t+1}(L)$. That is (15) is verified for $t - 1$. This completes the induction.

The particular case $i = 0$ gives that $C_0 \cap \kappa_i(L) = \emptyset$, as required. ■

To return to the matter in hand: if $H$ is a locally finite 2-group of norm length $i$ for which $H/\kappa_i(H)$ is Hamiltonian, choose elements $c, x, y$ to satisfy $[x, y] = c \notin \kappa(H)$, and write $C_0 := \{x, y, c\}$. Then $C_0 \cap \cap \kappa_i(H) = \emptyset$. Hence, by Lemma 3.3, there is a finite subgroup $L$ of $H$ with $C_0 \cap \kappa_i(L) = \emptyset$. It follows that $L/\kappa_i(L)$ is also Hamiltonian.

This completes one direction of the proof of Theorem 1.3.

For the other direction we use the Lemmas 3.1 and 3.2. So suppose that $H$ is a finite 2-group in which $H/\kappa_i(H)$ is Hamiltonian. Let $W$ be the group constructed in Lemma 3.3. Then $B := \kappa_j(W) = \zeta_j(W)$ is the base group of $W$, and $\kappa_{j+i}(W) = B\kappa_i(H)$. Therefore $W/\kappa_{j+i}(W) \cong H/\kappa_i(H)$ is Hamiltonian. ■

COROLLARY 3.4. *Let $i$ be a non-zero ordinal and $G$ a group of norm length $i + 1$. Then $G/\kappa_i(G) \cong Q_8$ only if $i$ is a limit ordinal. Moreover if $i$ is a non-zero limit ordinal there is a group $G$ for which $G/\kappa_i(G) \cong \cong Q_8$.*

PROOF. For, if $G$ is a group of norm length $i + 1$, and $i$ is not a limit ordinal, then $X := G/\kappa_{i-1}(G)$ has norm length 2. If $X/\kappa(X) \cong G/\kappa_{i-1}(G)$ is Hamiltonian then, by Theorem 1.1, the group $\Gamma$ embeds in $X$: suppose for simplicity that $\Gamma \subseteq X$. Then $X/\kappa(X) \supseteq \Gamma\kappa(X)/\kappa(X) \cong \Gamma/(\Gamma \cap \kappa(X))$ which maps homomorphically onto $\Gamma/\kappa(\Gamma) \cong Q_8 \times C_2$. Hence $X/\kappa(X)$ is not quaternion. This takes care of the first claim. The second claim follows from Lemmas 3.1, 3.2, choosing $K = Q_8$ in Lemma 3.1. ■

## 4. – Final comments.

The motivation for investigating this question comes from the article [3]. There we defined a canonical factor group $G/\beta(G)$ of a group $G$

dual, in some loose sense, to the norm of the group. To be precise $\beta(G)$ is defined by

$$\beta(G) := \langle [S, G]\colon S \text{ not normal in } G \rangle.$$

It is a normal subgroup of $G$, modulo which all subgroups are normal. Hence $G/\beta(G)$ is Dedekindian. We proved that the Baer property [1] is dualised in that, in a 2-group, $G/\beta(G)$ is Hamiltonian only if $\beta(G) = 1$. The descending series $\beta_i(G)$ is obtained by iteration in the usual way. Then $\beta_i(G)/\beta_{i+1}(G)$ is Hamiltonian only if $\beta_{i+1}(G) = 1$. We constructed 2-groups of length $i + 1$ and with $\beta_i(G)$ Hamiltonian for arbitrary finite $i$; and proved that, in every such group, the derived length is tightly bounded in terms of $|\beta_i(G)|$.

The present article begins an attempt to prove «dual» results for the norm.

We acknowledge the computation done for us by M. F. Newman and E. A. O'Brien in searching their library of groups of order $2^7$.

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Manoscritto pervenuto in redazione l'8 maggio 1996