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Homogenization of linear and nonlinear ordinary differential equations with time depending coefficients

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Homogenization of Linear and Nonlinear Ordinary Differential Equations with Time Depending Coefficients.

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ABSTRACT - We study the effective behaviour of solutions of a linear equation $\partial_t u^\varepsilon + a^\varepsilon(t, x)\, u^\varepsilon = 0$ and a non linear Riccati non homogeneous one $\partial_t u^\varepsilon + a^\varepsilon(t, x)(u^\varepsilon)^2 = f(t, x)$, $f > 0$, when $a^\varepsilon \rightharpoonup a$ in L^∞ weak* and data at time $t = 0$ is non oscillating. In the first case the limit is an integro-differential equation and the memory term is described by a resolvent Volterra equation. Existence and uniqueness for the solution of the effective equation are proved by passing to a kinetic formulation. On the other side the Riccati equation reveals a phenomena of instantaneous memory, described throught an asymptotic approach that consists in looking for an expansion by introducing a suitable parameter.

1. - Introduction.

This paper is concerned with the homogenization of some ordinary differential equations of linear and nonlinear type, involving highly oscillating coefficients, in which there is persistence of oscillations of the solutions and memory effects appear.

Let $T > 0$ be fixed, Ω be an open set of $\boldsymbol{R}^n$ and let $a^\varepsilon(t, x)$ be a sequence in $L^\infty((0, T) \times \Omega)$, satisfying $0 \leqslant \alpha_- \leqslant a^\varepsilon(t, x) \leqslant \alpha_+$ a.e., converging to $a(t, x)$ in $L^\infty((0, T) \times \Omega)$ weak*.

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We consider the following two equations:

$$\partial_t u^\varepsilon + a^\varepsilon(t, x) u^\varepsilon = 0 \quad \text{with } u^\varepsilon(0, x) = u_0(x), \tag{1.1}$$

$$\partial_t u^\varepsilon + a^\varepsilon(t, x)(u^\varepsilon)^2 = f(t, x) \quad \text{with } u^\varepsilon(0, x) = 0, \tag{1.2}$$

where $u_0(x) \in L^\infty$ and f is a positive bounded function.

The interest in (1.1), (1.2) is motivated by the paper of L. Tartar [17], in which, under the additional assumption of equicontinuity in t for the sequence a^ε, the effective behaviour of (1.1) is described by the use of Volterra equations; moreover for both problems an asymptotic approach is presented, that consists in looking for a power series expansion with respect to a suitable parameter.

In particular, Tartar obtained that, up to a subsequence, the weak limit u of solutions u^ε satisfies an integro-differential equation, with a memory kernel:

$$\partial_t u(t, x) + a(t, x)\, u(t, x) - \int_0^t K(t, s, x)\, u(s, x)\, ds = 0$$

where $K(t, s, x)$ is given by:

$$K(t, s, x) = \frac{\partial D(t, s, x)}{\partial s} - a(s, x)\, D(t, s, x)$$

and $D(t, s, x)$ is the solution of a Volterra equation involving the weak* limits of $a^\varepsilon u^\varepsilon$ and $a u^\varepsilon$.

When $a^\varepsilon(t, x) = a(t, x/\varepsilon)$ is periodic with respect to x, the homogenization of equation (1.1) has been carried out by M. L. Mascarenhas in [13], while for the problem (1.2), the case $f = 0$ with $u^\varepsilon(0, x) = u_0(x) > 0$ has been studied by Y. Amirat, K. Hamdache and A. Ziani in [7].

In the latter paper, the authors showed also that the Volterra kernel $K(t, s, x)$ related to (1.1) when a^ε does not depend upon t is exactly the one obtained by L. Tartar in [16] by the method of Nevanlinna-Pick (see [15]).

Following the ideas of Tartar, we furnish here a characterization of the memory terms appearing in (1.1) and (1.2), without assuming the equicontinuity of the coefficients a^ε and using, when possible, the Young measures associated to the oscillating sequences.

As for the homogenized problem related to (1.1), we obtain an equivalent formulation of the kind in [6] which allows us to prove that the solution u is unique, thus the whole sequence u^ε converges towards u.

The two problems will reveal a very different behaviour from the point of view of homogenization, the first one showing a non local memory effect, while in the latter one, the persistence of oscillations let appear a memory of instantaneous response type.

We point out that the result found for equation (1.2) generalizes the one of Y. Amirat-K. Hamdache-A. Ziani [7] concerning the case in which the source term is zero and the initial condition is $u^\varepsilon(0, x) = u_0(x) > 0$.

The method exploited for (1.1) could be applied to the case of a linear transport equation having time depending coefficients oscillating in a transverse direction (shear flow) and will be the object of a next paper.

We recall here that the homogenization of linear transport equations with oscillatory velocity field having a shear structure and time independing has been treated in the papers [1], [2], [3]-[7], [10], [11], [14], [16].

Aknowledgments. I would like to thank Kamel Hamdache for useful conversations and helpful encouragement.

2. – A linear problem.

Let $T > 0$ and let Ω be an open set of $\boldsymbol{R}^n$. We consider the linear differential equation:

$$(2.1)\qquad \begin{cases} \partial_t u^\varepsilon(t, x) + a^\varepsilon(t, x)\, u^\varepsilon(t, x) = 0 & \text{in } (0, T) \times \Omega\,, \\ u^\varepsilon(0, x) = u_0(x) & \text{in } \Omega\,, \end{cases}$$

where a^ε is a sequence of measurable functions that satisfies:

$$(2.2)\qquad \begin{aligned} &0 \leqslant \alpha_- \leqslant a^\varepsilon(t, x) \leqslant \alpha_+ \qquad \text{a.e. in } (0, T) \times \Omega\,, \\ &a^\varepsilon \rightharpoonup a \qquad \text{in } L^\infty((0, T) \times \Omega) \text{ weak}^*\,, \end{aligned}$$

and $u_0(x) \in L^\infty(\Omega)$.

In order to characterize the equation satisfied by the weak* limit of the solutions u^ε, we study the equation with a parameter γ:

$$(2.3)\qquad \begin{cases} \partial_t u^\varepsilon(t, x; \gamma) + \gamma a^\varepsilon(t, x)\, u^\varepsilon(t, x; \gamma) = 0 & \text{in } (0, T) \times \Omega\,, \\ u^\varepsilon(0, x) = u_0(x) & \text{in } \Omega\,, \end{cases}$$

whose explicit solution is

$$u^{\varepsilon}(t, x; \gamma) = u_0(x)\, e^{-\gamma \int_0^t a^{\varepsilon}(\sigma, x)\, d\sigma}.$$

We will do an analysis based on analyticity properties in the parameter γ, remarking that the previous problem corresponds to the case $\gamma = 1$.

Let us define the function:

$$h^{\varepsilon}(t, s, x; \gamma) = e^{-\gamma \int_s^t a^{\varepsilon}(\sigma, x)\, d\sigma}. \tag{2.4}$$

The bounds (2.2) imply that the sequence $(h^{\varepsilon}(t, s, x; \gamma))_{\varepsilon} \subset \subset W^{1,\infty}((0, T) \times (0, T); L^{\infty}(\Omega))$ is uniformly bounded with bounded derivatives in t and s; moreover $\partial_t h^{\varepsilon}(t, \cdot, x; \gamma))_{\varepsilon} \subset W^{1,\infty}((0, T); L^{\infty}((0, T) \times \Omega))$ and has a sequence of derivatives in s which is bounded in $L^{\infty}((0, T) \times (0, T) \times \Omega)$.

Thus we have

$$h^{\varepsilon}(t, s, x; \gamma) \rightharpoonup h(t, s, x; \gamma) = \langle \pi_{t, s, x}(d\lambda), e^{-\gamma\lambda} \rangle,$$

in $L^{\infty}((0, T) \times (0, T) \times \Omega)$ weak*, where $\pi_{t, s, x}(d\lambda)$ is the family of Young measures associated to the sequence $\left(\int_s^t a^{\varepsilon}(\sigma, x)\, d\sigma \right)$, which has support in $\Lambda_T = [0, T\alpha_+]$.

By denoting $\Lambda = [\alpha_-, \alpha_+]$, we get:

LEMMA 2.0.1. *Let $\omega_{t, s, x}(d\lambda_1, d\lambda_2, d\lambda)$ the family of Young measures, parametrized in t, s, x, with support in $\Lambda \times \Lambda \times \Lambda_T$, associated to the vector-valued sequence:*

$$\left(a^{\varepsilon}(t, x), a^{\varepsilon}(s, x), \int_s^t a^{\varepsilon}(\sigma, x)\, d\sigma \right).$$

Then, after the extraction of a subsequence, we obtain that:

$$h(t, s, x; \gamma) = \langle \omega_{t, s, x}(d\lambda_1, d\lambda_2, d\lambda), 1_{\lambda_1} \otimes 1_{\lambda_2} \otimes e^{-\gamma\lambda} \rangle$$

and

$$\partial_t h^{\varepsilon} \rightharpoonup \partial_t h = -\gamma \langle \omega_{t, s, x}(d\lambda_1, d\lambda_2, d\lambda), \mu_1 \otimes 1_{\lambda_2} \otimes e^{-\gamma\lambda} \rangle,$$

$$\partial^2_{ts} h^{\varepsilon} \rightharpoonup \partial^2_{ts} h = -\gamma^2 \langle \omega_{t, s, x}(d\lambda_1, d\lambda_2, d\lambda), \mu_1 \mu_2 e^{-\gamma\lambda} \rangle,$$

in $L^{\infty}((0, T) \times (0, T) \times \Omega)$ weak.*

Moreover, if $\nu_{t,x}(d\lambda_1)$ *is the family of Young measures associated to the sequence* $a^\varepsilon(t, x)$, *we have that:*

$$\langle \nu_{t,x}(d\lambda_1), \cdot \rangle = \omega_{t,s,x}(d\lambda_1, d\lambda_2, d\lambda), \cdot \otimes 1_{\lambda_2} \otimes 1_\lambda \rangle. \tag{2.5}$$

PROOF. Let $\omega_{t,s,x}(d\lambda_1, d\lambda_2, d\lambda)$ be the Young measure associated to the sequence $\left(a^\varepsilon(t, x), a^\varepsilon(s, x), \int_s^t a^\varepsilon(\sigma, x)\, d\sigma\right)$ (see Tartar [15]). Then clearly the weak* limits of h^ε, $\partial_t h^\varepsilon$ and $\partial^2_{ts} h^\varepsilon$ are given as before. Moreover (2.5) follows easily.

If we denote $\pi^1_{t,s,x}(d\lambda_1, d\lambda)$, $\pi^2_{t,s,x}(d\lambda_2, d\lambda)$, $\pi^{1,2}_{t,s,x}(d\lambda_1, d\lambda_2)$ the Young measures associated to the sequences $\left(a^\varepsilon(t, x), \int_s^t a^\varepsilon(\sigma, x)\, d\sigma\right)$, $\left(a^\varepsilon(s, x), \int_s^t a^\varepsilon(\sigma, x)\, d\sigma\right)$, $(a^\varepsilon(t, x), a^\varepsilon(s, x))$, respectively, as for (2.5) we have for exemple:

$$\pi_{t,s,x}(d\lambda) = proj\,|_{\lambda_1 \in \Lambda, \lambda_2 \in \Lambda}\, \omega_{t,s,x}(d\lambda_1, d\lambda_2, d\lambda),$$

$$\pi^1_{t,s,x}(d\lambda_1, d\lambda) = proj\,|_{\lambda_2 \in \Lambda}\, \omega_{t,s,x}(d\lambda_1, d\lambda_2, d\lambda). \quad \blacksquare$$

Let us define the function

$$C(t, s, x; \gamma) = \langle \pi^1_{t,s,x}(d\lambda_1, d\lambda), (\lambda_1 - a(t, x))\, e^{-\gamma\lambda} \rangle \tag{2.6}$$

which is bounded with bounded derivative in s, analytic in the parameter γ and admits an expansion:

$$C(t, s, x; \gamma) = \sum_{k=0}^{\infty} \gamma^k C_k(t, s, x),$$

where

$$C_k(t, s, x) = \frac{(-1)^k}{k!} \langle \pi^1_{t,s,x}(d\lambda_1, d\lambda), (\lambda_1 - a(t, x))\, \lambda^k \rangle. \tag{2.7}$$

In particular, by (2.5), $C_0(t, s, x) = 0$.

For a subsequence labelled u^ε, we see that u^ε converges weakly to $u(t, x; \gamma) = u_0(x)\, h(t, 0, x; \gamma)$, and u satisfies the equation:

$$\partial_t u(t, x; \gamma) + \gamma a(t, x)\, u(t, x; \gamma) + \gamma C(t, 0, x; \gamma)\, u_0(x) = 0 \tag{2.8}$$

in $(0, T) \times \Omega$, with $C(t, s, x; \gamma)$ given by (2.6).

We can restate the result of Theorem 2.9 in Tartar [17] under the only assumption (2.2) on the sequence a^ε:

THEOREM 2.1 (Tartar [17]). *Under hypothesis* (2.2) *and after extraction of a subsequence of* $\left(a^\varepsilon(t,x),\, a^\varepsilon(s,x),\, \int_s^t a^\varepsilon(\sigma,x)\,d\sigma\right)$ *and* u^ε, *there is a kernel K defined on* $(0,T)\times(0,T)\times\Omega$ *such that, for any* $\gamma>0$, *the subsequence* $u^\varepsilon(t,x;\gamma)$ *of solutions of* (2.3) *converges weakly* to the solution* $u(t,x;\gamma)$ *of*:

$$\partial_t u(t,x;\gamma)+\gamma a(t,x)\,u(t,x;\gamma)-\gamma^2\int_0^t K(t,s,x;\gamma)\,u(s,x;\gamma)\,ds=0 \tag{2.9}$$

in $(0,T)\times\Omega$ *with* $u(0,x;\gamma)=u_0(x)$ *in* Ω.

The kernel $K(t,s,x;\gamma)$ *is measurable in t, x and analytic in* γ *and is given by*:

$$K(t,s,x;\gamma)=\frac{1}{\gamma^2}\left(\frac{\partial}{\partial s}D(t,s,x;\gamma)-\gamma a(s,x)\,D(t,s,x;\gamma)\right) \tag{2.10}$$

where, for any $\gamma>0$, $D(t,s,x;\gamma)$ *is the solution of the Volterra equation*:

$$C(t,s,x;\gamma)=\frac{1}{\gamma}\,D(t,s,x;\gamma)-\int_s^t D(t,\tau,x;\gamma)\,C(\tau,s,x;\gamma)\,d\tau \tag{2.11}$$

in $(0,T)\times(0,T)\times\Omega$, *with C defined by* (2.6).

PROOF. In order to show that the equation (2.8) can be written as in (2.9) with a kernel K, we notice that it has the form

$$-\gamma C(t,0,x;\gamma)\,u_0(x)=g(t,x;\gamma) \tag{2.12}$$

with the function g defined by

$$g(t,x;\gamma)=\partial_t u(t,x;\gamma)+\gamma a(t,x)\,u(t,x;\gamma) \qquad \text{in } (0,T)\times\Omega\,.$$

By considering (2.12), we have that $g(t,x;\gamma)$ solves the integral equation:

$$g(t,x;\gamma)+\int_0^t D(t,s,x;\gamma)\,g(s,x;\gamma)\,ds=-D(t,0,x;\gamma)\,u_0(x) \tag{2.13}$$

in which the kernel D is the solution of the resolvent equation (2.11).

From (2.11) it follows that D is bounded and has bounded derivative in s.

Moreover, noticing that the equality

$$\int_s^t D(t, \tau, x; \gamma)\, C(\tau, s, x; \gamma)\, d\tau = \int_s^t C(t, \tau, x; \gamma)\, D(\tau, s, x; \gamma)\, d\tau$$

holds in view of the unicity of the solution of an homogeneous Volterra equation, we have:

$$(2.14)\quad \left(\frac{\partial}{\partial s} - \gamma a(s, x)\right) C(t, s, x; \gamma) = \frac{1}{\gamma}\left(\frac{\partial}{\partial s} - \gamma a(s, x)\right) D(t, s, x; \gamma) - \\ - \int_s^t C(t, \tau, x; \gamma)\left(\frac{\partial}{\partial s} - \gamma a(s, x)\right) D(\tau, s, x; \gamma)\, d\tau$$

in $(0, T) \times (0, T) \times \Omega$.

Thus, after integration by parts in (2.13) and noticing that $D(t, t, x; \gamma) = 0$, we obtain:

$$\partial_t u(t, x; \gamma) + \gamma a(t, x)\, u(t, x; \gamma) - D(t, 0, x; \gamma)\, u_0(x) - \\ - \int_0^t (\partial_s D(t, s, x; \gamma) - \gamma a(s, x)\, D(t, s, x; \gamma))\, u(s, x; \gamma)\, ds = \\ = -D(t, 0, x; \gamma)\, u_0(x)$$

in $(0, T) \times \Omega$, for any $\gamma > 0$, from which we get the formula (2.9) for the kernel K.

The analyticity of $K(t, s, x; \gamma)$ with respect to γ comes from the equation (2.10) in view of to the following Lemma:

LEMMA 2.2. *The solution $D(t, s, x; \gamma)$ of (2.11) is an analytic function which is represented by the expansion*:

$$D(t, s, x; \gamma) = \sum_{k=0}^{\infty} \gamma^{k+1} D_k(t, s, x),$$

where

$$(2.15)\qquad D_k(t, s, x) = C_k(t, s, x), \qquad \textit{when } k = 0, 1, 2$$

and D_k is obtained by induction for $k \geqslant 3$:

$$D_k(t, s, x) = C_k(t, s, x) + \sum_{j=0}^{k-1} \int_s^t D_{k-1-j}(t, \tau, x)\, C_j(\tau, s, x)\, d\tau . \tag{2.16}$$

PROOF. We find (2.15) and (2.16) by injecting the expansion of $C(t, s, x; \gamma)$ into the Volterra equation and by observing that $(1/\gamma)\, D(t, s, x; \gamma)$ is bounded in γ.

To show that we have indeed identified the solution $D(t, s, x; \gamma)$, we verify that the formal power series obtained has an infinite radius of convergence.

Actually, from (2.7) we deduce for C_k:

$$|C_k(t, s, x)| \leqslant (\alpha_+ - \alpha_-)\, \alpha_+^k \frac{(t-s)^k}{k!} .$$

This implies for D_1 the bound:

$$|D_1(t, s, x)| \leqslant c_k(\alpha_+ - \alpha_-)\, \alpha_+^k (t-s) .$$

When $k \geqslant 2$, we look for a bound of D_k of the form

$$|D_k(t, s, x)| \leqslant c_k(\alpha_+ - \alpha_-)\, \alpha_+^k \frac{(t-s)^k}{k!} .$$

Putting this bound into equation (2.16) implies the inequality:

$$c_k \leqslant 1 + \sum_{j=1}^{k-1} c_j \quad \text{if } k > 1$$

that gives

$$c_k \leqslant 2^{k-1} \quad \text{if } k \geqslant 1,\ (c_1 = 1). \qquad \blacksquare$$

To end the proof of Theorem 2.1, we notice that (2.15) yields $D_0(t, s, x) = 0$, so the function $K(t, s, x; \gamma)$ defined by formula (2.10) is bounded in γ. $\blacksquare$

In order to characterize the kernel K and let his structure more explicit, we define on $(0, T) \times (0, T) \times \Omega$ and for any $\gamma > 0$ the functions:

$$G(t, s, x; \gamma) = \langle \pi^2_{t, s, x}(d\lambda_2, d\lambda), (\lambda_2 - a(s, x)) e^{-\gamma\lambda} \rangle, \tag{2.17}$$

$$H(t, s, x; \gamma) = \tag{2.18}$$

$$= \langle \omega_{t, s, x}(d\lambda_1, d\lambda_2, d\lambda), (\lambda_1 - a(t, x))(\lambda_2 - a(s, x)) e^{-\gamma\lambda} \rangle.$$

REMARK 2.1. *$H(t, s, x; \gamma)$ admits an expansion:*

$$H(t, s, x; \gamma) = \sum_{k=0}^{\infty} \gamma^k H_k(t, s, x)$$

where

$$H_k(t, s, x) = \frac{(-1)^k}{k!} \langle \omega_{t, s, x}(d\lambda_1, d\lambda_2, d\lambda), (\lambda_1 - a(t, x))(\lambda_2 - a(s, x)) \lambda^k \rangle.$$

By Lemma 2.0.1 *we have*

$$H_0(t, s, x) = \langle \pi^{1,2}_{t, s, x}(d\lambda_1, d\lambda_2), (\lambda_1 - a(t, x))(\lambda_2 - a(s, x)) \rangle.$$

In particular, in view of (2.5),

$$H_0(t, t, x) = \langle \nu_{t, x}(d\lambda_1), (\lambda_1 - a(t, x))^2 \rangle$$

and

$$H_0(t, t, x) \geq 0 \quad \textit{a.e. in } (0, T) \times \Omega \tag{2.19}$$

by application of the Jensen inequality.

LEMMA 2.3. *Under hypothesis of Theorem* 2.1, *the kernel $K(t, s, x; \gamma)$ solves the family of Volterra equations indexed by t, s, x:*

$$K(t, s, x; \gamma) - \gamma \int_s^t C(t, \tau, x; \gamma) K(\tau, s, x; \gamma) \, d\tau = H(t, s, x; \gamma) \tag{2.20}$$

on $(0, T) \times (0, T) \times \Omega$ and for any $\gamma > 0$, with $K(s, s, x; \gamma) = H_0(s, s, x)$. Moreover, K has the following expansion in γ:

$$K(t, s, x; \gamma) = \sum_{i=0}^{\infty} \gamma^i K_i(t, s, x) \tag{2.21}$$

where:

$$K_i(t, s, x) = H_i(t, s, x), \quad \textit{for } i = 0, 1, \tag{2.22}$$

and $K_i(t, s, x)$ *is obtained by induction for* $i \geq 2$:

$$K_i(t, s, x) = H_i(t, s, x) + \sum_{j=0}^{i-2} \int_s^t D_{i-1-j}(t, \tau, x) H_j(\tau, s, x)\, d\tau. \tag{2.23}$$

Proof. Equation (2.20) corresponds to the (2.14) in the proof of Theorem 2.1 and gives that $K(t, s, x; \gamma)$ is defined by:

$$K(t, s, x; \gamma) = H(t, s, x; \gamma) + \int_s^t D(t, \tau, x; \gamma) H(\tau, s, x; \gamma)\, d\tau \tag{2.24}$$

where D is the solution of the Volterra equation (2.11).

The expansion (2.21) is found by putting into the equation (2.24) the expansions of functions D and H.

The formal power series thus obtained has an infinite radius of convergence, indeed:

$$|K_0(t, s, x)| \leq (\alpha_+ - \alpha_-)^2$$

and

$$|K_k(t, s, x)| \leq c'_k (\alpha_+ - \alpha_-)^2 \alpha_+^k \frac{(t-s)^k}{k!}, \tag{2.25}$$

with

$$c'_k \leq 2^{k-1} \quad \text{if } k \geq 1, \ (c'_1 = 1). \qquad \blacksquare$$

It is now possible to give to the Cauchy problem (2.9) an equivalent formulation, which will allow us to prove, by using the Volterra equations theory, existence and uniqueness results for the homogenized problem and in particular we will get that (2.9) is the effective equation associated to the homogenization problem.

Let us consider the Banach spaces $X_0 = L^\infty(\Omega)$, $X = W^{1,\infty}((0, T); L^\infty(\Omega))$, $Y = L^\infty((0, T) \times \Omega)$.

We introduce the auxiliary function $z(t, x; \gamma)$ defined on $(0, T) \times \Omega$ and for any $\gamma > 0$ by

$$z(t, x; \gamma) = \gamma \int_0^t K(t, s, x; \gamma)\, u(s, x; \gamma)\, ds. \tag{2.26}$$

THEOREM 2.2. *Under hypothesis of Theorem* 2.1, *for any* $\gamma > 0$ *the Cauchy problem* (2.9) *is equivalent to the following system*:

$$(2.27)\qquad \begin{cases} \partial_t u(t, x; \gamma) + \gamma a(t, x)\, u(t, x; \gamma) - \gamma z(t, x; \gamma) = 0\,, \\ z(t, x; \gamma) - \gamma \int_0^t C(t, s, x; \gamma)\, z(s, x; \gamma)\, ds = \\ \qquad\qquad = \gamma \int_0^t H(t, s, x; \gamma)\, u(s, x; \gamma)\, ds\,, \end{cases}$$

in $(0, T) \times \Omega$, *with the initial conditions* $(u, z)(0, x; \gamma) = (u_0(x), 0)$ *in* Ω.

If we assume

$$(2.28)\qquad H_0(t, t, x) > 0 \quad a.e.\ in\ (0, T) \times \Omega$$

for any initial data in $X_0 \times X_0$, *the system* (2.27) *admits a unique solution* $(u, z) \in X \times Y$ *and in particular* (2.9) *has a unique solution in* X.

PROOF. Clearly if u satisfies (2.9) and (2.20) holds for K, then z defined by (2.26) is in Y and (u, z) satisfies (2.27).

Now let (u, z) be a solution of (2.27) in $W^{1,\infty}(0, T; L^\infty(\Omega)) \times \times L^\infty((0, T) \times \Omega)$ with $z(0, x; \gamma) = 0$ and let (2.20) hold.

Then the function $v(t, x; \gamma) = \gamma \int_0^t K(t, s, x; \gamma)\, u(s, x; \gamma)\, ds$, where $K(t, s, x; \gamma)$ is the solution of (2.20) belongs to Y and putting (2.20) into the second equation in (2.27) gives

$$z(t, x; \gamma) - \gamma \int_0^t C(t, s, x; \gamma)\, z(s, x; \gamma)\, ds =$$
$$= v(t, x; \gamma) - \gamma \int_0^t C(t, s, x; \gamma)\, v(s, x; \gamma)\, ds\,.$$

So $(z - v)(t, x; \gamma)$ solves the homogeneous integral equation

$$(z - v)(t, x; \gamma) = \gamma \int_0^t C(t, s, x; \gamma)(z - v)(s, x; \gamma)\, ds$$

with the initial condition $(z - v)(0, x; \gamma) = 0$, which, for any $\gamma > 0$

admits only the solution zero in $L^\infty((0,T)\times\Omega)$. Thus we get that $z(t,x;\gamma)$ is given by (2.26).

Let us now consider $U(t,x)=(u(t,x),z(t,x))\in Y\times Y$, $U_0(x)\in X_0\times\times X_0$ and $F(t,x;\gamma)\in L^1((0,T)\times\Omega)$.

By integrating the first equation with respect to time, we can write the system (2.27) under the general form:

$$U(t,x)-\gamma\int_0^t R(t,s,x;\gamma)\,U(s,x)\,ds=U_0(x)+\int_0^t F(s,x;\gamma)\,ds \tag{2.29}$$

where $R(t,s,x;\gamma)$ is the bounded operator acting in $Y\times Y$ represented by

$$R(t,s,x;\gamma)=\begin{pmatrix}-a(s,x) & 1\\ H(t,s,x;\gamma) & C(t,s,x;\gamma)\end{pmatrix}.$$

By (2.2) and the definition of C and H, we have that, for any $\gamma>0$,

$$\|R\|_{\mathcal{L}(Y\times Y)}\leqslant\left\|\begin{pmatrix}0 & 1\\ (\alpha_+-\alpha_-)^2 & (\alpha_+-\alpha_-)\end{pmatrix}\right\|_{\mathcal{L}(Y\times Y)}.$$

For any given $U_0(x)\in X_0\times X_0$ and F, (2.29) is an inhomogeneous integral equation in $Y\times Y$ depending on a parameter $\gamma>0$.

Thus, by a generalization of the fixed point theorem, for any $\gamma>0$ the equation (2.29) has a unique solution, being a suitable power of the operator R a contraction in $Y\times Y$.

Moreover the solution $U(t,x;\gamma)$ is given by

$$\begin{aligned}U(t,x;\gamma)&=U_0(x)+\int_0^t F(s,x;\gamma)\,ds+\gamma\int_0^t S(t,s,x;\gamma)\,ds\,U_0(x)+\\&+\int_0^t\left(\gamma\int_s^t S(t,\sigma,x;\gamma)\,d\sigma\right)F(s,x;\gamma)\,ds\end{aligned} \tag{2.30}$$

where $S(t,s,x;\gamma)$ is the solution of the resolvent Volterra equation

$$S(t,s,x;\gamma)-\gamma\int_s^t S(t,\tau,x;\gamma)\,R(\tau,s,x;\gamma)\,d\tau=R(t,s,x;\gamma)$$

corresponding to the initial data $S(s,s,x;\gamma)=R(s,s,x;\gamma)$.

Thanks to hypothesis (2.28), for any $\gamma > 0$ the equation above has a unique solution in $L^\infty((0, T)^2 \times \Omega)^4$, explicitely given by

$$S(t, s, x; \gamma) = \begin{pmatrix} -G(t, s, x; \gamma) - a(s, x)\langle \pi_{t, s, x}(d\lambda), e^{-\gamma\lambda}\rangle & e^{-\gamma A(t, s, x)} \\ H(t, s, x; \gamma) + a(s, x)\, C(t, s, x; \gamma) & 0 \end{pmatrix}$$

and $G(t, s, x; \gamma)$, $H(t, s, x; \gamma)$ are defined in (2.17), (2.18).

By integrating we obtain

$$U(t, x; \gamma) = U_0(x) + \int_0^t F(s, x; \gamma)\, ds + Q(t, 0, x; \gamma)\, U_0(x) + \tag{2.31}$$
$$+ \int_0^t Q(t, s, x; \gamma)\, F(s, x; \gamma)\, ds$$

where

$$Q(t, s, x; \gamma) = \begin{pmatrix} \langle \pi_{t, s, x}(d\lambda), e^{-\gamma\lambda}\rangle - 1 & \gamma \int_s^t e^{-\gamma A(t, \sigma, x)}\, d\sigma \\ -C(t, s, x; \gamma) & 0 \end{pmatrix}.$$

For the equivalence previously seen, it follows that $U \in W^{1,\infty}((0, T); L^\infty(\Omega)) \times L^\infty((0, T) \times \Omega)$ is indeed a solution of (2.27). ∎

We finally write the homogenized equation related to the problem (2.1), obtained by taking $\gamma = 1$:

THEOREM 2.3. *Under hypotheses* (2.2) *and* (2.28), *there is a kernel* $K(t, s, x)$ *defined in* $(0, T) \times (0, T) \times \Omega$ *such that the whole sequence* $u^\varepsilon(t, x)$ *of solutions of* (2.1) *converges weakly to the unique solution* $u(t, x)$ *of*:

$$\partial_t u(t, x) + a(t, x)\, u(t, x) - \int_0^t K(t, s, x)\, u(s, x)\, ds = 0 \quad \text{in } (0, T) \times \Omega \tag{2.32}$$

with $u(0, x) = u_0(x)$ *in* Ω.

If $C(t, s, x) := C(t, s, x; 1)$, $H(t, s, x) := H(t, s, x; 1)$, $z(t, x) := := z(t, x; 1)$ *then* $K(t, s, x)$ *is the solution of the Volterra equation:*

$$K(t, s, x) - \int_s^t C(t, \tau, x) K(\tau, s, x)\, d\tau = H(t, s, x).$$

and the problem (2.1) *is equivalent to the system:*

$$\text{(2.33)} \quad \begin{cases} \partial_t u(t, x) + a(t, x) u(t, x) - z(t, x) = 0, \\ z(t, x) - \int_0^t C(t, s, x) z(s, x)\, ds = \int_0^t H(t, s, x) u(s, x)\, ds \end{cases}$$

in $(0, T) \times \Omega$, *with the initial conditions* $u(0, x) = u_0(x)$, $z(0, x) = 0$ *in* Ω.

PROOF. By integration in time, the system is equivalent to the integral one in Theorem 2.5, for which existence and uniqueness results hold. ■

Let us now assume that the coefficients $a^\varepsilon(t, x)$ are absolutely continuous in time and satisfy

$$\text{(2.34)} \quad \begin{aligned} &\alpha_- \leq a^\varepsilon(t, x) \leq \alpha_+, \quad \beta_- \leq \partial_t a^\varepsilon(t, x) \leq \beta_+ \quad \text{a.e. } (0, T) \times \Omega, \\ &a^\varepsilon(t, x) \rightharpoonup a(t, x), \quad \partial_t a^\varepsilon(t, x) \rightharpoonup \partial_t a(t, x) \\ &\qquad \text{in } L^\infty((0, T) \times \Omega) \text{ weak}^*. \end{aligned}$$

In this case, we can make use of the Young measure associated to the sequence $\left(\partial_t a^\varepsilon(t, x), a^\varepsilon(t, x), a^\varepsilon(s, x), \int_s^t a^\varepsilon(\sigma, x)\, d\sigma\right)$:

LEMMA 2.4. *Let* $\sigma_{t,s,x}(d\mu, d\lambda_1, d\lambda_2, d\lambda)$ *the family of Young measures indexed by t, s and x, with support in* $\Lambda \times \Lambda \times \Lambda \times \Lambda_T$, *associated to the vector-valued sequence:*

$$\left(\partial_t a^\varepsilon(t, x), a^\varepsilon(t, x), a^\varepsilon(s, x), \int_s^t a^\varepsilon(\sigma, x)\, d\sigma\right).$$

Then, for any $\gamma > 0$, *the functions* $C(t, s, x; \gamma)$ *and* $H(t, s, x; \gamma)$ *defined* *by* (2.6) *and* (2.18) *can be written as*:

$C(t, s, x; \gamma) =$

$$= \langle \sigma_{t,s,x}(d\mu, d\lambda_1, d\lambda_2, d\lambda), 1_\mu \otimes (\lambda_1 - a(t, x)) \otimes 1_{\lambda_2} \otimes e^{-\gamma\lambda} \rangle,$$

$H(t, s, x; \gamma) =$

$$= \langle \sigma_{t,s,x}(d\mu, d\lambda_1, d\lambda_2, d\lambda), 1_\mu \otimes (\lambda_1 - a(t, x))(\lambda_2 - a(s, x)) e^{-\gamma\lambda} \rangle.$$

Moreover, their derivative in t is bounded, and write respectively:

(2.35) $\partial_t C(t, s, x; \gamma) =$

$$= \langle \sigma_{t,s,x}(d\mu, d\lambda_1, d\lambda_2, d\lambda), (\mu - \partial_t a(t, x)) \otimes 1_{\lambda_1} \otimes 1_{\lambda_2} \otimes e^{-\gamma\lambda} \rangle -$$

$$- \gamma \langle \sigma_{t,s,x}(d\mu, d\lambda_1, d\lambda_2, d\lambda), 1_\mu \otimes \lambda_1 (\lambda_1 - a(t, x)) e^{-\gamma\lambda} \rangle,$$

(2.36) $\partial_t H(t, s, x; \gamma) =$

$$= \langle \sigma_{t,s,x}(d\mu, d\lambda_1, d\lambda_2, d\lambda), (\mu - \partial_t a(t, x)) \otimes 1_{\lambda_1} \otimes (\Lambda_2 - a(s, x)) e^{-\gamma\lambda} \rangle -$$

$$- \gamma < \sigma_{t,s,x}(d\mu, d\lambda_1, d\lambda_2, d\lambda), 1_\mu \otimes \lambda_1 (\lambda_1 - a(t, x))(\lambda_2 - a(s, x)) e^{-\gamma\lambda} \rangle.$$

$\partial_t K(t, s, x; \gamma)$ *is also bounded and is defined by the equation*:

(2.37) $\partial_t K(t, s, x; \gamma) =$

$$= \gamma \int_s^t \partial_t C(t, \tau, x; \gamma) K(\tau, s, x; \gamma) \, d\tau + \partial_t H(t, s, x; \gamma)$$

in $(0, T) \times (0, T) \times \Omega$.

Thus, we find that by using the auxiliary function $z(t, x; \gamma)$ defined in (2.26), the Cauchy problem (2.9) can be formulated as an equivalent system in which a derivative in time is allowed on z:

THEOREM 2.4. *Under hypotheses* (2.28) *and* (2.34), *for any* $\gamma > 0$ *the problem* (2.9) *is equivalent to the following system:*

$$(2.38) \quad \begin{cases} \partial_t u(t, x; \gamma) + \gamma a(t, x) u(t, x; \gamma) - \gamma z(t, x; \gamma) = 0\,, \\ \partial_t z(t, x; \gamma) - \gamma H_0(t, t, x) u(t, x; \gamma) = \\ = \gamma \displaystyle\int_0^t \partial_t C(t, s, x; \gamma) z(s, x; \gamma) \, ds + \gamma \int_0^t \partial_t H(t, s, x; \gamma) u(s, x; \gamma) \, ds \end{cases},$$

in $(0, T) \times \Omega$, *with the initial conditions* $u(0, x; \gamma) = u_0(x)$, $w(0, x; \gamma) = 0$ *in* Ω *and* $H_0(t, t, x)$ *given by* (2.19).

Under the assumption $H_0(t, t, x) > 0$ *a.e. in* $(0, T) \times \Omega$, *we have that for any initial data in* $X_0 \times X_0$, *the system* (2.38) *has a unique solution in* $W^{1,\infty}(0, T; L^\infty(\Omega)) \times W^{1,\infty}(0, T; L^\infty(\Omega))$.

PROOF. Let X_0, X, Y be the Banach spaces defined in the proof of Theorem 2.2.

Clearly if u satisfies (2.9) and (2.37) holds, then $z(t, x; \gamma) \in W^{1,\infty}(0, T; L^\infty(\Omega))$ and (u, z) satisfies (2.38).

Now let (u, z) be a solution of (2.38) in $X \times X$ with $z(0, x) = 0$ and let us take the function $v(t, x; \gamma) = \gamma \int_0^t K(t, s, x; \gamma)\, u(s, x; \gamma)\, ds$.

In view of (2.37) we have $v \in X$ and

$$\partial_t v(t, x; \gamma) = \gamma H_0(t, t, x)\, u(t, x; \gamma) + \gamma \int_0^t \partial_t K(t, s, x; \gamma)\, u(s, x; \gamma)\, ds\,.$$

Putting (2.37) into the second equation in (2.38) gives

$$\partial_t (z - v)(t, x; \gamma) - \gamma \int_0^t \partial_t C(t, s, x; \gamma)(z - v)(s, x; \gamma)\, ds = 0$$

so $(z - v)(t, x; \gamma)$ solves the homogeneous integral equation

$$(z - v)(t, x; \gamma) = \gamma \int_0^t C(t, s, x; \gamma)(z - v)(s, x; \gamma)\, ds$$

with the initial condition $(z - v)(0, x; \gamma) = 0$, which admits only the solution zero in $L^\infty((0, T) \times \Omega)$, for any $\gamma > 0$.

Thus we get that $z(t, x; \gamma)$ is given by (2.26).

Let now $U_0 \in X_0 \times X_0$, $F(t, x; \gamma) \in L^1((0, T) \times \Omega)$. By integration in time, the system is equivalent to the one in (2.29) for which existence and uniqueness results hold in $X \times Y$.

Moreover, thanks to (2.34) we have that the resolvent kernel $R(t, s, x; \gamma)$ is absolutely continuous in t, thus the general system corresponding to (2.38) writes

$$\partial_t U - \gamma R(t, t, x)\, U(t, x; \gamma) - \gamma \int_0^t \partial_t R(t, s, x; \gamma)\, U(s, x; \gamma)\, ds = F(t, x; \gamma)$$

where

$$R(t, t, x) = \begin{pmatrix} -a(t, x) & 1 \\ H_0(t, t, x) & 0 \end{pmatrix}$$

and

$$\partial_t R(t, s, x; \gamma) = \begin{pmatrix} 0 & 0 \\ \partial_t H(t, s, x; \gamma) & \partial_t C(t, s, x; \gamma) \end{pmatrix}.$$

Clearly, the solution $S(t, s, x; \gamma)$ inherits the same property of absolute continuity in t, from which it follows that U obtained in (2.31) is in $X \times X$. ■

3. - A non linear problem.

Let $T > 0$ and Ω be an open set of $\boldsymbol{R}^n$. We consider the following model equation:

$$\text{(3.1)} \qquad \begin{cases} \dfrac{\partial}{\partial t} u^\varepsilon(t, x) + a^\varepsilon(t, x)\{u^\varepsilon(t, x)\}^2 = f(t, x) & \text{in } (0, T) \times \Omega\,, \\ u^\varepsilon(0, x) = 0 & \text{in } \Omega\,. \end{cases}$$

We assume that the coefficients a^ε and the function f are measurable and satisfy:

$$\text{(3.2)} \qquad \begin{aligned} & 0 < \alpha_- \leqslant a^\varepsilon(t, x) \leqslant \alpha_+ \quad \text{for a.e. } (t, x) \in (0, T) \times \Omega\,, \\ & a^\varepsilon \rightharpoonup a \quad \text{in } L^\infty((0, T) \times \Omega) \text{ weak*}\,, \end{aligned}$$

$$\text{(3.3)} \qquad 0 < F_- \leqslant f(t, x) \leqslant F_+ \quad \text{for a.e. } (t, x) \in (0, T) \times \Omega\,,$$

for some α_-, α_+, F_-, F_+.

Existence of solution u^ε then comes from Carathéodory theory (see [12]). Moreover, hypotheses (4.2) and (4.3) imply that

$$\text{(3.4)} \qquad 0 \leqslant \left(\frac{F_-}{\alpha_+}\right)^{1/2} \tanh\left((\alpha_+ F_-)^{1/2} t\right) \leqslant$$

$$\leqslant u^\varepsilon(t, x) \leqslant \left(\frac{F_+}{\alpha_-}\right)^{1/2} \tanh\left((\alpha_- F_+)^{1/2} t\right)$$

for all $t \geqslant 0$ and for a.e. $x \in \Omega$. Unicity of u^ε is assured by a comparison result (see [12]).

For each f one can extract a subsequence such that u^ε converges weakly to a function u in $W^{1,\infty}(0, T; L^\infty(\Omega))$.

Here we look for a characterization of the homogenized equation satisfied by u.

We will see that the homogenization process produces a phenomena with memory of instantaneous response type, where the memory term will depend on f. The same behaviour has been found by Amirat, Hamdache, Ziani in [7] in the case $f = 0$, with an initial condition $u^\varepsilon(0, x) = u_0(x) > 0$.

The problem (3.1) has been considered by Tartar in [17], where the approach of introducing a suitable parameter is presented.

Thus we consider firstly the problem:

$$\text{(3.5)} \qquad \begin{cases} \dfrac{\partial}{\partial t} u^\varepsilon(t, x; \gamma) + \gamma a^\varepsilon(t, x)\{u^\varepsilon(t, x; \gamma)\}^2 = f(t, x) & \text{in } (0, T) \times \Omega\,, \\ u^\varepsilon(0, x) = 0 & \text{in } \Omega\,, \end{cases}$$

and do an expansion in the parameter γ.

We look for an expansion:

$$u^\varepsilon(t, x; \gamma) = U_0(t, x) + \sum_{k=1}^{\infty} \gamma^k U_k^\varepsilon(t, x)$$

where the first term U_0 is independent of ε and is a solution of:

$$\text{(3.6)} \qquad \begin{cases} \dfrac{\partial}{\partial t} U_0 = f & \text{in } (0, T) \times \Omega\,, \\ U_0(0, x) = 0 & \text{in } \Omega\,. \end{cases}$$

The following terms can be computed by induction, the terms U_1^ε and U_2^ε being solutions of:

$$\text{(3.7)} \qquad \begin{cases} \dfrac{\partial}{\partial t} U_1^\varepsilon + a^\varepsilon U_0^2 = 0 & \text{in } (0, T) \times \Omega\,, \\ U_1^\varepsilon(0, x) = 0 & \text{in } \Omega\,, \end{cases}$$

$$\text{(3.8)} \qquad \begin{cases} \dfrac{\partial}{\partial t} U_2^\varepsilon + 2a^\varepsilon U_0 U_1^\varepsilon = 0 & \text{in } (0, T) \times \Omega\,, \\ U_2^\varepsilon(0, x) = 0 & \text{in } \Omega\,, \end{cases}$$

and the general term U_k^ε, for $k \geq 3$, satisfying:

$$(3.9)\qquad \begin{cases} \dfrac{\partial}{\partial t} U_k^\varepsilon + 2a^\varepsilon U_0 U_{k-1}^\varepsilon + a^\varepsilon \sum_{j=1}^{k-2} U_j^\varepsilon U_{k-1-j}^\varepsilon = 0 & \text{in } (0, T) \times \Omega\,, \\ U_k^\varepsilon(0, x) = 0 & \text{in } \Omega\,. \end{cases}$$

We point out that $0 < U_0(t, x) \leq F_+ t$ for all $t \in (0, T]$.

From (3.7) and (3.8) we deduce the explicit expression of U_1^ε, U_2^ε:

$$(3.10)\qquad U_1^\varepsilon = -\int_0^t a^\varepsilon(s, x)\, U_0^2(s, x)\, ds\,,$$

$$(3.11)\qquad U_2^\varepsilon = \int_0^t \frac{1}{U_0(s, x)}\, \partial_s (U_1(s, x)^2)\, ds =$$

$$= \frac{1}{U_0(t, x)}\, (U_1^\varepsilon(t, x))^2 + \int_0^t \frac{f(s, x)}{U_0^2(s, x)}\, (U_1(s, x))^2\, ds\,.$$

We have:

Proposition 3.1. *For $k \geq 2$, U_k^ε satisfies the equation*:

$$(3.12)\qquad \frac{\partial}{\partial t} U_k^\varepsilon = \sum_{l=1}^{k-1} (-1)^{l+1} \frac{1}{U_0^l} \times$$

$$\times \frac{\partial}{\partial t} \left(\sum_{j_1=1}^{k-l} U_{j_1}^\varepsilon \sum_{j_2=1}^{k-l-j_1+1} U_{j_2}^\varepsilon \cdots \sum_{j_l=1}^{k-j_1-\ldots-j_{l-1}-1} U_{j_l}^\varepsilon U_{k-j_1-\ldots-j_l}^\varepsilon \right).$$

Proof. For $k = 2$ we have

$$\frac{\partial}{\partial t} U_2^\varepsilon = \frac{1}{U_0} \frac{\partial}{\partial t} ((U_1^\varepsilon)^2)\,.$$

Let us assume (3.12) true for any index lower than k; then for U_k^ε we find that:

$$\frac{\partial}{\partial t} U_k^\varepsilon = -2a^\varepsilon U_0 U_{k-1}^\varepsilon - a^\varepsilon \sum_{j=1}^{k-2} U_j^\varepsilon U_{k-1-j}^\varepsilon =$$

$$= \frac{1}{U_0} \frac{\partial}{\partial t} \left(\sum_{j=1}^{k-1} U_j^\varepsilon U_{k-j}^\varepsilon \right) - \frac{2}{U_0} \sum_{j=1}^{k-2} U_j^\varepsilon \frac{\partial U_{k-j}^\varepsilon}{\partial t} + \frac{1}{U_0^2} \frac{\partial U_1^\varepsilon}{\partial t} \sum_{j=1}^{k-2} U_j^\varepsilon U_{k-1-j}^\varepsilon$$

and by replacing the expression of $\partial U_{k-j}^{\varepsilon}/\partial t$, $j=1, \ldots, k-2$ given by the induction, we get:

$$\frac{\partial}{\partial t} U_k^{\varepsilon} = \frac{1}{U_0} \frac{\partial}{\partial t}\left(\sum_{j=1}^{k-1} U_j^{\varepsilon} U_{k-j}^{\varepsilon}\right) - \frac{2}{U_0} \sum_{j=1}^{k-2} U_j^{\varepsilon} \sum_{l=1}^{k-j-1} (-1)^{l+1} \frac{1}{U_0^l} \times$$

$$\times \frac{\partial}{\partial t}\left(\sum_{j_1=1}^{k-j-l} U_{j_1}^{\varepsilon} \sum_{j_2=1}^{k-j-l-j_1+1} U_{j_2}^{\varepsilon} \cdots \sum_{j_l=1}^{k-j-l-j_1-\ldots-j_{l-1}+l-1} U_{j_l}^{\varepsilon} U_{k-j-j_1-\ldots-j_l}^{\varepsilon}\right) +$$

$$+ \frac{1}{U_0^2} \frac{\partial U_1^{\varepsilon}}{\partial t}\left(\sum_{j=1}^{k-2} U_j^{\varepsilon} U_{k-1-j}^{\varepsilon}\right).$$

The terms in $1/U_0^2$ are:

$$-\frac{2}{U_0^2} \sum_{j=1}^{k-2} U_j^{\varepsilon} \frac{\partial}{\partial t}\left(\sum_{j_1=1}^{k-j-1} U_{j_1}^{\varepsilon} U_{k-j-j_1}^{\varepsilon}\right) + \frac{1}{U_0^2} \frac{\partial U_1^{\varepsilon}}{\partial t} \sum_{j=1}^{k-2} U_j^{\varepsilon} U_{k-1-j}^{\varepsilon} =$$

$$= -\frac{1}{U_0^2} \frac{\partial}{\partial t}\left(\sum_{j=1}^{k-2} U_j^{\varepsilon} \sum_{j_1=1}^{k-j-1} U_{j_1}^{\varepsilon} U_{k-j-j_1}^{\varepsilon}\right) -$$

$$-\frac{1}{U_0^2} \sum_{j=1}^{k-2} \frac{\partial U_j^{\varepsilon}}{\partial t} \sum_{j_1=1}^{k-j-1} U_{j_1}^{\varepsilon} U_{k-j-j_1}^{\varepsilon} + \frac{1}{U_0^2} \frac{\partial U_1^{\varepsilon}}{\partial t} \sum_{j=1}^{k-2} U_j^{\varepsilon} U_{k-1-j}^{\varepsilon} =$$

$$= -\frac{1}{U_0^2} \frac{\partial}{\partial t}\left(\sum_{j=1}^{k-2} U_j^{\varepsilon} \sum_{j_1=1}^{k-j-1} U_{j_1}^{\varepsilon} U_{k-j-j_1}^{\varepsilon}\right) -$$

$$-\frac{1}{U_0^2} \sum_{j=2}^{k-2} \frac{\partial U_j^{\varepsilon}}{\partial t} \sum_{l=1}^{k-j-1} U_l^{\varepsilon} U_{k-j-l}^{\varepsilon}.$$

By still replacing the expression of $\partial U_j^{\varepsilon}/\partial t$ given by the induction, we get (3.12). ■

REMARK 3.2. *From* (3.12) *one can see that each* U_k^{ε} *can be expressed by powers of* U_1^{ε}.

In order to settle the convergence of the expansion $u^{\varepsilon}(t, x; \gamma)$, we do firstly some considerations on the coefficients U_k^{ε}.

We can write, for any $k \geqslant 1$,

$$U_k^{\varepsilon}(t, x) = (-1)^k \,|U_k^{\varepsilon}| = (-1)^k U_0 V_k^{\varepsilon}(t, x)$$

where V_k^ε verifies:

(3.13) $$\begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{f}{U_0}\right) V_1^\varepsilon = a^\varepsilon U_0 & \text{in } (0,T) \times \Omega, \\ V_1^\varepsilon(0,x) = 0 & \text{in } \Omega, \end{cases}$$

(3.14) $$\begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{f}{U_0}\right) V_2^\varepsilon = 2a^\varepsilon U_0 V_1^\varepsilon & \text{in } (0, T) \times \Omega, \\ U_2^\varepsilon(0, x) = 0 & \text{in } \Omega, \end{cases}$$

(3.15) $$\begin{cases} \left(\dfrac{\partial}{\partial t} + \dfrac{f}{U_0}\right) V_k^\varepsilon = 2a^\varepsilon U_0 V_{k-1}^\varepsilon + a^\varepsilon U_0 \sum\limits_{j=1}^{k-2} V_j^\varepsilon V_{k-1-j}^\varepsilon & \\ & \text{in } (0, T) \times \Omega, \\ V_k^\varepsilon(0, x) = 0 & \text{in } \Omega. \end{cases}$$

Thus

$$u^\varepsilon(t, x; \gamma) = U_0(1 + V_\gamma^\varepsilon)$$

where

(3.16) $$V_\gamma^\varepsilon = \sum_{k \geq 1} (-1)^k \gamma^k V_k^\varepsilon .$$

LEMMA 3.0.1. *Assume that*

(3.17) $$a_+ F_+ T^2 < 1 .$$

Then:

(i) *the expansion* (3.16) *converges in* $(0, T) \times \Omega$ *for any* $\gamma \in [0, 1]$ *and the function* $U_\gamma^\varepsilon = U_0(1 + V_\gamma^\varepsilon)$ *coincides with the solution* $u^\varepsilon(t, x; \gamma)$ *of problem* (3.5);

(ii) *moreover, the function* V_γ^ε *satisfies*

$$| V_\gamma^\varepsilon(t, x) | < 1$$

uniformly in t, x *and for any* $\gamma \in [0, 1]$.

PROOF. Let

$$Z_\gamma(t) = \frac{1}{1 - \gamma a_+ F_+ (t^2/2)}$$

be the solution of the Cauchy problem:

$$(3.18)\qquad \begin{cases} \dfrac{\partial}{\partial t} Z_\gamma - \gamma a_+ F_+ t (Z_\gamma)^2 = 0 & \text{in } (0, T), \\ Z_\gamma(0) = 1 . \end{cases}$$

Under the bound (3.17), $Z_\gamma(t)$ admits a power series expansion:

$$Z_\gamma(t) = \sum_{k \geq 1} \gamma^k Z_k(t) = \sum_{k \geq 1} \gamma^k \left(\frac{a_+ F_+}{2} \right)^k t^{2k}$$

converging for any $\gamma \in [0, 1]$.

From (3.13), (3.14) and (3.15), one can deduce that for $k \geq 1$

$$V_k^\varepsilon(t, x) \leq Z_k(t)$$

for all $t \in [0, T]$, uniformly in $x \in \Omega$.

This assures the convergence of the expansion V_γ^ε.

Moreover:

$$|V_\gamma^\varepsilon(t, x)| = \sum_{k \geq 1} \gamma^k V_k^\varepsilon \leq \frac{1}{1 - \gamma a_+ F_+ (t^2/2)} - 1 < 1$$

when $\gamma a_+ F_+ t^2 < 1$. ■

As a consequence, this yields a powers expansion for the function $1/U_\gamma^\varepsilon$:

$$(3.19)\qquad \frac{1}{U_\gamma^\varepsilon} = \frac{1}{U_0} \frac{1}{1 + V_\gamma^\varepsilon} = \frac{1}{U_0} \sum_{k \geq 0} (-1)^k (V_\gamma^\varepsilon)^k$$

convergent for any $\gamma \in [0, 1]$ if (3.17) holds.

To look for the effective equation related to (3.5), we will exploit the result of Proposition 3.1.

Thanks to formula (3.12), we have:

$$\begin{aligned} \frac{\partial}{\partial t} U_\gamma^\varepsilon &= f - \gamma a^\varepsilon U_0^2 + U_0 \sum_{k \geq 2} (-1)^k \frac{1}{U_0^k} \frac{\partial}{\partial t} ((U_0 V_\gamma^\varepsilon)^k) = \\ &= f - \gamma a^\varepsilon U_0^2 + U_0 \sum_{k \geq 2} (-1)^k \left(\frac{\partial}{\partial t} + k \frac{f}{U_0} \right) (V_\gamma^\varepsilon)^k = \\ &= f - \gamma a^\varepsilon U_0^2 + U_0 \frac{\partial}{\partial t} \left(\sum_{k \geq 0} (-1)^k (V_\gamma^\varepsilon)^k \right) + f \sum_{k \geq 0} (-1)^k (k+1) (V_\gamma^\varepsilon)^k - \\ &\qquad - f \sum_{k \geq 0} (-1)^k (V_\gamma^\varepsilon)^k + \frac{\partial}{\partial t} U_\gamma^\varepsilon . \end{aligned}$$

We deduce that:

$$\gamma a^\varepsilon = \frac{1}{U_0}\frac{\partial}{\partial t}\Big(\sum_{k\geq 0}(-1)^k(V_\gamma^\varepsilon)^k\Big) +$$

$$+\frac{f}{U_0^2}\sum_{k\geq 0}(-1)^k(k+1)(V_\gamma^\varepsilon)^k - \frac{f}{U_0^2}\sum_{k\geq 0}(-1)^k(V_\gamma^\varepsilon)^k =$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{U_0}\sum_{k\geq 0}(-1)^k(V_\gamma^\varepsilon)^k\right) + \frac{f}{U_0^2}\sum_{k\geq 0}(-1)^k(k+1)(V_\gamma^\varepsilon)^k =$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{U_0}\frac{1}{1+V_\gamma^\varepsilon}\right) + \frac{f}{U_0^2}\frac{1}{(1+V_\gamma^\varepsilon)^2} = \frac{\partial}{\partial t}\left(\frac{1}{U_\gamma^\varepsilon}\right) + f\frac{1}{(U_\gamma^\varepsilon)^2}.$$

By passing to the limit in L^∞ weak* and by denoting $(V_\gamma)_k(t,x)$ the weak* limit of $(V_\gamma^\varepsilon(t,x))^k$, we get:

$$\gamma a(t,x) = \frac{\partial}{\partial t}\left(\lim_* \frac{1}{U_\gamma^\varepsilon}\right) + f\left(\lim_* \frac{1}{(U_\gamma^\varepsilon)^2}\right) = \tag{3.20}$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{U_0}\sum_{k\geq 0}(-1)^k(V_\gamma)_k\right) + \frac{f}{U_0^2}\sum_{k\geq 0}(-1)^k(k+1)(V_\gamma)_k .$$

We can now state the main result of this section

THEOREM 3.1. *Under hypotheses* (3.2), (3.3) *and* (3.17), *for any* $\gamma\in[0,1]$ *there is a subsequence, still denoted by* (u^ε), *and a function* $K_\gamma(t,x)\in L^\infty((0,T)\times\Omega)$ *such that the sequence* (u^ε) *converges in* $W^{1,\infty}(0,T;L^\infty(\Omega))$ *weak* to the unique solution* u *of the problem:*

$$\begin{cases} \dfrac{\partial}{\partial_t}u(t,x;\gamma) + (\gamma a(t,x) - K_\gamma(t,x))u(t,x;\gamma)^2 = f(t,x) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T)\times\Omega, \\ u(0,x;\gamma) = 0 \qquad\qquad\qquad\qquad\quad \text{in } \Omega. \end{cases} \tag{3.21}$$

The function K *is given by:*

$$K_\gamma(t,x) = \frac{\partial}{\partial t}\left(\lim_* \frac{1}{U_\gamma^\varepsilon} - \frac{1}{U_\gamma}\right) + f\left(\lim_* \frac{1}{(U_\gamma^\varepsilon)^2} - \frac{1}{U_\gamma^2}\right). \tag{3.22}$$

PROOF. Let $U_\gamma = U_0(1+V_\gamma) = \lim_* U_\gamma^\varepsilon$.

By making use of the relation (3.20) we get:

$$\frac{\partial}{\partial t} U_\gamma - f + \gamma a(t, x) U_\gamma^2 = U_\gamma^2 \left\{ - \frac{\partial}{\partial t} \frac{1}{U_\gamma} - f \frac{1}{U_\gamma^2} + \gamma \right\} =$$

$$= U_\gamma^2 \left\{ \frac{\partial}{\partial t} \left(\lim{}_* \frac{1}{U_\gamma^\varepsilon} - \frac{1}{U_\gamma} \right) + f \left(\lim{}_* \frac{1}{(U_\gamma^\varepsilon)^2} - \frac{1}{U_\gamma^2} \right) \right\}. \qquad \blacksquare$$

We can display the expansion of $K_\gamma(t, x)$. For this we rewrite the formula in Proposition 3.1 in terms of V_k^ε:

$$\text{(3.23)} \qquad \left(\frac{\partial}{\partial t} + \frac{f}{U_0} \right) V_k^\varepsilon = U_0 \sum_{l=1}^{k-1} (-1)^{l+1} \left(\frac{\partial}{\partial t} + (l+1) \frac{f}{U_0} \right) \times$$

$$\times \left(\sum_{j_1=1}^{k-l} V_{j_1}^\varepsilon \sum_{j_2=1}^{k-l-j_1+1} V_{j_2}^\varepsilon \dots \sum_{j_l=1}^{k-j_1-\dots-j_{l-1}-1} V_{j_l}^\varepsilon V_{k-j_1-\dots-j_l}^\varepsilon \right).$$

Then, by writing $V_k(t, x)$ and $V_{k,l}(t, x)$ the weak* limits of functions $V_k^\varepsilon(t, x)$ and

$$V_{k,l}^\varepsilon(t, x) = \sum_{j_1=1}^{k-l} V_{j_1}^\varepsilon \sum_{j_2=1}^{k-l-j_1+1} V_{j_2}^\varepsilon \dots \sum_{j_l=1}^{k-j_1-\dots-j_{l-1}-1} V_{j_l}^\varepsilon V_{k-j_1-\dots-j_l}^\varepsilon,$$

we have:

Corollary 3.1.1. *For each $\gamma \in [0, 1]$ the function K_γ admits a convergent expansion of the form:*

$$K_\gamma(t, x) = \sum_{i=2}^{\infty} \gamma^i K_i(t, x) \qquad \text{in } (0, T) \times (0, T) \times \Omega$$

where:

$$\text{(3.24)} \qquad K_i = \frac{1}{U_0} \sum_{l=1}^{i-1} (-1)^{l+1} \left(\frac{\partial}{\partial t} + (l+1) \frac{f}{U_0} \right) \times$$

$$\times \left(V_{i,l} - \sum_{j_1=1}^{i-l} V_{j_1} \sum_{j_2=1}^{i-l-j_1+1} V_{j_2} \dots \sum_{j_l=1}^{i-j_1-\dots-j_{l-1}-1} V_{j_l} V_{i-j_1-\dots-j_l} \right).$$

PROOF. From (3.22), (3.19) and (3.23), we deduce:

$$K_\gamma(t, x) = \frac{\partial}{\partial t}\left(\frac{1}{U_0}\sum_{k\geq 0}(-1)^k((V_\gamma)_k - (V_\gamma)^k)\right) +$$

$$+\frac{f}{U_0^2}\sum_{k\geq 0}(-1)^k(k+1)((V_\gamma)_k - (V_\gamma)^k) =$$

$$= \frac{1}{U_0}\sum_{k\geq 2}(-1)^k\left(\frac{\partial}{\partial t} + k\frac{f}{U_0}\right)((V_\gamma)_k - (V_\gamma)^k)$$

where, for $k \geq 2$:

$$(V_\gamma)_k - (V_\gamma)^k =$$

$$= \sum_{i\geq k}\gamma^i\left(V_{k,i} - \sum_{j_1=1}^{k-1}V_{j_1}\sum_{j_2=1}^{k-j_1}V_{j_2}\cdots\sum_{j_{k-1}=1}^{k-j_1-\ldots-j_{k-2}-1}V_{j_{k-1}}V_{i-j_1-\ldots-j_{k-1}}\right).$$

By replacing this expression and by a change of index into the sum, we obtain (4.24). ■

REMARK 3.1.1. *We can use the parametrized measure obtained by Amirat-Hamdache-Ziani in* [3] *throught the Nevanlinna-Pick method to express the kernel K.*

To this end we consider the Young measure $\pi_{t,x}(d\lambda)$ *originated by the oscillating sequence* V_γ^ε, $\gamma = 1$:

$$V_\gamma^\varepsilon \rightharpoonup \langle \pi_{t,x}(d\lambda), \lambda\rangle = V_\gamma^\varepsilon(t, x), \qquad U_\gamma^\varepsilon \rightharpoonup U_0\langle\pi_{t,x}(d\lambda), 1+\lambda\rangle = U_0(1+V_\gamma^\varepsilon).$$

Then:

$$K(t, x) = \frac{\partial}{\partial t}\left(\left\langle\pi_{t,x}(d\lambda), \frac{1}{U_0(1+\lambda)}\right\rangle - \frac{1}{U_0\langle_{t,x}(d\lambda), 1+\lambda\rangle}\right) +$$

$$+f\left(\left\langle p_{t,x}(d\lambda), \left(\frac{1}{U_0}(1+\lambda)\right)^2\right\rangle - \left(\frac{1}{U_0\langle\pi_{t,x}(d\lambda), 1+\lambda\rangle}\right)^2\right) =$$

$$= \frac{1}{U_0}\frac{\partial}{\partial t}\left(\left\langle\pi_{t,x}(d\lambda), \frac{1}{1+\lambda}\right\rangle - \frac{1}{\langle\pi_{t,x}(d\lambda), 1+\lambda\rangle}\right) +$$

$$+\frac{f}{U_0^2}\left(\left\langle \pi_{t,x}(d\lambda), \left(\frac{1}{1+\lambda}\right)^2\right\rangle - \left(\frac{1}{\langle \pi_{t,x}(d\lambda), 1+\lambda\rangle}\right)^2 - \right.$$

$$\left. -\left\langle \pi_{t,x}(d\lambda), \frac{1}{1+\lambda}\right\rangle + \frac{1}{\langle \pi_{t,x}(d\lambda), 1+\lambda\rangle}\right).$$

We have the following relation with the measure of Nevanlinna-Pick $\omega_{t,x}(d\lambda)$ *founded in* [3]:

$$\left\langle \pi_{t,x}(d\lambda), \frac{1}{1+\lambda}\right\rangle - \frac{1}{\langle \pi_{t,x}(d\lambda), 1+\lambda\rangle} =$$

$$= \left\langle \omega_{t,x}(d\lambda), \frac{1}{1+\lambda}\right\rangle \frac{\langle \pi_{t,x}(d\lambda), 1/(1+\lambda)\rangle}{\langle \pi_{t,x}(d\lambda), 1+\lambda\rangle}$$

and

$$\left\langle \pi_{t,x}(d\lambda), \left(\frac{1}{1+\lambda}\right)^2\right\rangle - \left(\frac{1}{\langle \pi_{t,x}(d\lambda), 1+\lambda\rangle}\right)^2 \geqslant 0$$

in view of the Jensen inequality.

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