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Asymptotics of Solutions to Stokes and Navier-Stokes Equations in Domains with Paraboloidal Outlets to Infinity.

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ABSTRACT - The stationary Stokes and Navier-Stokes equations of a viscous incompressible fluid with additional flux conditions are considered in domains Ω with $m \geqslant 1$ outlets to infinity, which have in some coordinate systems the following form

$$\Omega_i = \{x: \ |x'| < g_0 x_n^{1-\gamma}, \ x_n > 0, \ \gamma \in (0, 1)\}.$$

The complete asymptotic decomposition is constructed for the solution of the Stokes problem in the case when the right-hand side has either a compact support or a special series representation. For the solution of the nonlinear Navier-Stokes problem the asymptotic decomposition is constructed in the case of zero right-hand side. The obtained asymptotic decompositions are justified in weighted Hölder spaces.

1. - Introduction.

The solvability of the boundary and initial-boundary value problems for Stokes and Navier-Stokes equations have been studied in many papers and monographs (e.g. [8], [31], [3]). The existence theory which is developed there concerns mainly the domains with compact boundaries (bounded or exterior). However, many physically important problems are related to domains with noncompact boundaries (for example, the fluid flow in channels and pipes). Therefore, it is not suprising that during the last 17 years the special attention was given

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to problems in such domains (e.g. [1], [2], [4], [9]-[11], [5], [22], [26], [27]-[30], etc.).

On the other hand, during the last three decades there was developed the theory of linear elliptic boundary value problems in domains having singular points on the boundary (e.g. [7], [12], [13], [21] and the references cited there). The asymptotics of the solutions to elliptic problems is most well studied in domains with conical points or, equivalently, in domains with cylindrical and conical outlets to infinity ([21], [7], [13]). In this paper we study the asymptotics of the solutions to the steady Stokes and Navier-Stokes problems in domains with paraboloidal outlets to infinity, having in some system of coordinates the form

$$\Omega_i = \left\{x \in \mathbb{R}^n:\ |x'| < g_0 x_n^{1-\gamma},\ x_n > 1\right\}, \qquad 0 < \gamma < 1 . \tag{1.1}$$

Notice that papaboloidal outlets to infinity are an intermediate case between cylindrical ($\gamma = 1$) and conical ($\gamma = 0$) outlets (e.g. Remark 2.4).

General elliptic boundary value problems in domains with singularity points were investigated in [12], [14], where the coercive estimates and asymptotics of solutions (in the case of exponentially vanishing right-hand sides) were obtained. In [15] the results from [12], [14] were applied to the Dirichlet problem for a scalar elliptic operator of the second order near the peak type point of the boundary. The abstract form of the asymptotic formulas in [14] looks very consistent. However, their realization for the Stokes system led to cumbersome calculations which the authors did not succeed to overcome. That is why we have chosen a different approach related to the asymptotic analysis of elliptic problems in slender domains.

Here we consider the Stokes

$$\begin{cases} -\nu\Delta \vec{u} + \nabla p = \vec{f} & \text{in } \Omega, \\ \operatorname{div} \vec{u} = 0 & \text{in } \Omega, \\ \vec{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

and Navier-Stokes

$$\begin{cases} -\nu\Delta \vec{u} + (\vec{u}\cdot\nabla)\,\vec{u} + \nabla p = \vec{f} & \text{in } \Omega, \\ \operatorname{div} \vec{u} = 0 & \text{in } \Omega, \\ \vec{u} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

problems in the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with $m \geqslant 1$ outlets to infinity Ω_i which have in certain coordinate systems the form (1.1) and we look for the solutions satisfying the additional flux conditions

$$\int\limits_{\sigma_i(t)} \vec{u} \cdot \vec{n}\, ds = F_i\,, \qquad i = 1, \ldots, m\,, \qquad \sum_{i=1}^{m} F_i = 0\,, \tag{1.4}$$

where

$$\sigma_i(t) = \{x \in \Omega\colon\ x_n = t = \text{const}\}\,.$$

In order to obtain the solvability of the Stokes problem (1.2), (1.4) and the coercitive estimates for the solutions with zero fluxes ($F_i = 0$) in weighted Sobolev and Hölder spaces, we first derive the estimates of the Dirichlet integral over the subdomains of Ω, by using the differential inequalities techniques (so called «techniques of the Saint-Venant's principle») developed by O. A. Ladyzhenskaya and V. A. Solonnikov [11], [29] and then we improve the «weighted reguliarity» of the solutions, applying the method proposed by V. G. Maz'ya, B. A. Plamenevskii [12]. To find the solutions with nonzero fluxes F_i, we look for the velocity field $\vec{u}$ in the form

$$\vec{u} = \vec{A} + \vec{v}\,, \tag{1.5}$$

where $\vec{A}$ is a solenoidal vector function, satisfying the flux conditions (1.4) and the estimates

$$|D_x^\alpha \vec{A}(x)| \leqslant C(|\vec{F}|)\, g_i(x_n)^{-n+1-|\alpha|}\,, \tag{1.6}$$

where $|\vec{F}|^2 = \sum_{i=1}^{m} F_i^2$. Then for $(\vec{v}, p)$ we get the Stokes problem (1.2), (1.4) with zero fluxes ($F_i = 0$, $i = 1, \ldots, m$) and the new right-hand side $\vec{f} + \nu \Delta \vec{A}$. The mentioned results are obtained in [23], [24] for two and three-dimensional domains Ω, having the outlets to infinity of the form

$$\Omega_i = \{x \in \mathbb{R}^n\colon\ |x'| < g_i(x_n),\ x_n > 1\}\,, \tag{1.7}$$

where $|x'| \equiv |x_1|$ if $n = 2$, $|x'| \equiv \sqrt{x_1^2 + x_2^2}$ if $n = 3$, and $g_i(t)$ are functions satisfying the conditions

$$|g_i(t) - g_i(t')| \leqslant M_i\,|t - t'|\,, \qquad \forall t, t' > 0; \qquad g_i(t) \geqslant g_0 > 0\,, \tag{1.8}$$

$$\lim_{t \to \infty} g_i'(t) = 0\,, \qquad |g_i'(t)| \leqslant M_i\,, \qquad i = 1, \ldots, m\,. \tag{1.9}$$

The nonlinear Navier-Stokes problem (1.3), (1.4) in domains having outlets to infinity of the form (1.7) was studied in [25]. For three-dimensional domains Ω it is proved in [25] under the additional assumptions on g_i:

$$\int_0^\infty g_i(t)^{-4/3}\,dt = \infty\,, \qquad i = 1, \ldots, m\,, \tag{1.10}$$

$$|g_i'(t)\, g_i(t)^{1/3}| \leqslant \delta \ll 1 \qquad \text{for } t > k_0\,, \quad i = 1, \ldots, m\,, \tag{1.11}$$

that the weak solution of (1.3), (1.4) with the unbounded Dirichlet integral is regular and has the same decay properties as the solution of the linear Stokes problem (1.2), (1.4). This result is proved for arbitrary large data and is based on estimates of the Saint-Venant's type obtained for the weak solution of (1.3), (1.4) by O. A. Ladyzhenskaya, V. A. Solonnikov [11] and on bootstrap arguments, which use the results for the linear Stokes problem (1.2), (1.4). If the conditions (1.10), (1.11) are violated and in the two-dimensional case, the analogous results were proved in [25] for sufficiently small data by means of the Banach contraction principle. Notice that the decay estimates obtained in [23], [24], [25] have the same character as that for the divergence free vector field $\vec{A}$ (see (1.6)). This is related to the decomposition of the velocity $\vec{u}$ in the form (1.5). Since $\vec{A}$ is arbitrary, the right-hand side $\vec{f} + \nu \Delta \vec{A}$ is decaying at infinity not sufficiently fast, even if $\vec{f}$ has a compact support. Therefore, one can not expect the improved decay rate for the perturbation $\vec{v}$.

In this paper for the domains Ω, having the outlets to infinity of the form (1.1), i.e. $g_i(t) = g_0 t^{1-\gamma}$, we construct the formal asymptotics of the solutions and we prove the better decay estimates for the remainder. For example, in the three-dimensional case the obtained asymptotical solution for the Stokes problem with zero right-hand side has the form

$$\begin{cases} P^{[N]}(x) = x_3^{\lambda_0} \sum\limits_{k=0}^{N} x_3^{-2k\gamma}\big(q_k^{(0)} + x_3^{-2\gamma} Q_k(x_3^{\gamma-1}x')\big)\,, \\ U_3^{[N]}(x) = x_3^{\lambda_0-\gamma+1} \sum\limits_{k=0}^{N} x_3^{-2(k+1)\gamma}\, U_{3,k}(x_3^{\gamma-1}x')\,, \\ U_j^{[N]}(x) = x_3^{\lambda_0-3\gamma+1} \sum\limits_{k=0}^{N} x_3^{-(2k+1)\gamma}\, U_{j,k}(x_3^{\gamma-1}x')\,, \qquad j = 1, 2\,, \end{cases} \tag{1.12}$$

where $q_k^{(0)}$ are constants and $\lambda_0 = 4\gamma - 3$, $\gamma \neq 3/4$. If $\gamma = 3/4$, the representation for the pressure $P^{[N]}$ contains the logarithmic term. Estimates for the discrepancies which are left by this approximate solution $(\vec{U}^{[N]}, P^{[N]})$ in the Stokes equations improve when we increase N and, therefore, we get the «good» decay estimates for the remainder $(\vec{v} = \vec{u} - \vec{U}^{[N]}, q = p - P^{[N]})$.

The procedure which we use to construct the formal asymptotics is a variant of well known algorithm of constructing the asymptotics for solutions to elliptic equations in slender domains (e.g. S. A. Nazarov [17], S. N. Leora, S. A. Nazarov, A. V. Proskura [18], V. G. Maz'ja, S. A. Nazarov, B. A. Plamenevskii (Ch. 15-16) [16] for arbitrary elliptic problems and S. A. Nazarov [19], S. A. Nazarov, K. Pileckas [20] for the Stokes and Navier-Stokes equations). In order to explain the anology between the paraboloids and the slender domains, let us consider the intersection of Ω_i with the sphere S_R^2 of radius R. After the change of variables $x \to R^{-1}x = \xi$ the sphere S_R^2 goes over to the unit sphere S_1^2 and the intersection $\Omega_i \cap S_R^2$ turns out to become a domain with small, of order $O(R^{-\gamma})$, diameter. This property turns us to introduce the «transversal stretched coordinates»

$$\eta_j = x_3^{\gamma-1} x_j\,, \quad j = 1, 2\,, \qquad \eta_3 = x_3$$

while the image of the domain $\Omega_i \cap S_R^2$ is independent of R. After this, applying formally the methods from the theory of elliptic equations in slender domains, we derive for the pressure p the one dimensional Reynolds equation (see S. A. Nazarov, K. Pileckas [20]), which follows as a compatibility condition for the solvability of the two-dimensional Stokes problem (in the domain $\omega = \{\eta' \in \mathbb{R}^2 \colon \ |\eta'| < g_0\}$) for the velocity field $\vec{u}$.

The paper is organized as follows. In Section 2 we present the formal procedure of constructing the main terms of asymptotics for the solution of the Stokes problem with zero right-hand side f. In Section 3 we construct the higher terms of asymptotics. In order to construct them, we need to compensate discrepancies appearing in the equations. To this end, we consider the Stokes equations with the right-hand sides having the special form. The section is divided in eight subsections related to different cases of discrepancies, which can appear in the right-hand side. In Section 4 the obtained results are applied to construct the complete asymptotics in concrete situations. Namely, for the Stokes problem with the right-hand side, having compact support (Subsection 4.1), for the Stokes problem with the right-hand side having the special series representation (Subsection 4.2) and for the Navier-Stokes problem, having zero right-hand side (Subsection 4.3). Finally, in Section 5 we justify the obtained asymptotic decompositions for the Stokes (Subsection 5.2) and Navier-Stokes (Subsection 5.3) problems, i.e. we prove the appropriate estimates for the remainder

$$\vec{v} = \vec{u} - \vec{U}^{[N]}, \qquad q = p - P^{[N]}$$

in weighted Hölder spaces. To do this we apply to

$$\vec{v} = \vec{u} - \vec{U}^{[N]}, \qquad q = p - P^{[N]},$$

the results obtained in [23], [24], [25]. For the reader convenience we formulate these results in Subsection 5.1.

Notice, that the results obtained for the Navier-Stokes problem with zero rihgt-hand side can be generalized, with evident changes, to the case when the right-hand side has series representation. Moreover, just in the same way one can construct the asymptotics of the solutions to the Stokes and Navier-Stokes problems near the singular point of the boundary of the peak type (1) in the case when the right-hand side has series representation. Finally, we mention that all results remain valid also in the case of non-circular cross-sections of the outlets to infinity Ω_i, i.e. when $\Omega_i = \{x \in \mathbb{R}^n : x_n^{\gamma-1} x' \in S, x_n > 0\}$, where S is an arbitrary bounded domain in $\mathbb{R}^{n-1}$. Moreover, all the formal calculations needed for the above mentioned generalizations were presented in [20].

2. – The main asymptotic term; formal considerations

2.1. *Special coordinates.* In this section we construct an «approximate solution at infinity» to problem (1.2), (1.4). Let us consider the homogeneous problem (1.2), (1.4) (i.e. $\vec{f} \equiv 0$) in the outlet to infinity

$$\Omega_i = \left\{x \in \mathbb{R}^n : |x'| < g_0 x_n^{1-\gamma}, x_n > 1\right\}, \qquad 0 < \gamma < 1. \tag{2.1}$$

We pass in (1.2) to new coordinates

$$\eta_j = x_n^{\gamma-1} x_j, \qquad j = 1, \ldots, n-1, \qquad \eta_n = x_n \tag{2.2}$$

(1) I.e. if $0 \in \partial\Omega$ and in the neighbourhood of 0 the domain Ω can be represented in the form $\{x: |x'| < g(x_n), x_n \in (0, \delta)\}$ with $\lim_{x_n \to 0} g(x_n) = 0$ and $\lim_{x_n \to 0} g'(x_n) = 0$.

and, by using the evident relations

$$(2.3)\quad \begin{cases} \dfrac{\partial}{\partial x_j} = \eta_n^{\gamma-1}\dfrac{\partial}{\partial \eta_j}, \qquad \dfrac{\partial^2}{\partial x_j^2} = \eta_n^{2\gamma-2}\dfrac{\partial^2}{\partial \eta_j^2}, \qquad j=1, \ldots, n-1, \\ \dfrac{\partial}{\partial x_n} = \dfrac{\partial}{\partial \eta_n} - \displaystyle\sum_{j=1}^{n-1}(1-\gamma)\eta_j\eta_n^{-1}\dfrac{\partial}{\partial \eta_j}, \\ \dfrac{\partial^2}{\partial x_n^2} = \dfrac{\partial^2}{\partial \eta_n^2} + 2(\gamma-1)\eta_n^{-1}\displaystyle\sum_{j=1}^{n-1}\eta_j\dfrac{\partial^2}{\partial \eta_n\partial \eta_j} + (\gamma-1)(\gamma-2)\times \\ \qquad \times\, \eta_n^{-2}\displaystyle\sum_{j=1}^{n-1}\eta_j\dfrac{\partial}{\partial \eta_j} + \displaystyle\sum_{j,l=1}^{n-1}(\gamma-1)^2\eta_n^{-2}\eta_l\eta_j\dfrac{\partial^2}{\partial \eta_l\partial \eta_j}, \end{cases}$$

rewrite (1.2) in the following form:

$$(2.4)_1 \qquad -\nu(\eta_n^{2\gamma-2}\Delta' + \mathcal{D}^2)\vec{u}' + \eta_n^{\gamma-1}\nabla' p = 0 \text{ in } \Pi_+,$$

$$(2.4)_2 \qquad -\nu(\eta_n^{2\gamma-2}\Delta' + \mathcal{D}^2)u_n + \mathcal{D}p = 0 \text{ in } \Pi_+,$$

$$(2.4)_3 \qquad \eta_n^{\gamma-1}\operatorname{div}'\vec{u}' + \mathcal{D}u_n = 0 \text{ in } \Pi_+,$$

$$(2.4)_4 \qquad \vec{u} = 0 \text{ on } S_+.$$

In (2.4) we have used the notations

$$\mathcal{D} = \partial_n + (\gamma-1)\eta_n^{-1}\eta'\cdot\nabla',$$

$$\Pi_+ = \{\eta \in \mathbb{R}^n:\ |\eta'| < g_0,\ \eta_n > 1\},$$

$$S_+ = \{\eta \in \mathbb{R}^n:\ |\eta'| < g_0,\ \eta_n > 1\},$$

$$\vec{u}' = (u_1, \ldots, u_{n-1}), \qquad \partial_k = \partial/\partial\eta_k, \quad k=1, \ldots, n,$$

$$\nabla' = (\partial_1, \ldots, \partial_{n-1}), \qquad \operatorname{div}'\vec{u}' = \nabla'\cdot\vec{u}', \qquad \Delta' = \nabla'\cdot\nabla'.$$

2.2. *Structure of the asymptotics.* We look for the solution $(\vec{U}_0, P_0)$ of (2.4) in the form

$$(2.5)\quad \begin{cases} P_0(\eta', \eta_n) = q_0(\eta_n) + Q_0(\eta', \eta_n), \\ \vec{U}_0(\eta', \eta_n) = (\vec{U}_0'(\eta', \eta_n), U_{n,0}(\eta', \eta_n)), \end{cases}$$

with

$$(2.6)\qquad U_{n,0}(\eta', \eta_n) = \eta_n^{2(1-\gamma)}\partial_n q_0(\eta_n)\Phi(\eta').$$

Let us assume that the infinitesimals $\eta_n^k \partial_n^{k+1} q_0(\eta_n)$, $k = 1, 2, \ldots$, are equivalent (as $\eta_n \to \infty$) to $\partial_n q_0(\eta_n)$ (this assumption will be justified below). Substituting (U_0, P_0) into equations (2.4) and selecting the leading at infinity terms, we derive

$$\begin{cases} -\nu \partial_n q_0(\eta_n) \Delta' \Phi(\eta') + \partial_n q_0 = 0 & \text{in } \omega, \\ \Phi(\eta') = 0 & \text{on } \partial\omega \end{cases}$$

and

$$\begin{cases} -\nu \eta_n^{2\gamma - 2} \Delta' \vec{U}_0'(\eta') + \eta_n^{\gamma - 1} \nabla' Q_0(\eta') = 0 & \text{in } \omega, \\ \eta_n^{\gamma - 1} \operatorname{div}' \vec{U}_0'(\eta') = -\mathcal{D} U_{n,0}(\eta', \eta_n) & \text{in } \omega, \\ \vec{U}_0'(\eta') = 0 & \text{on } \partial\omega \end{cases}$$

or, what is the same,

$$\begin{cases} \nu \Delta' \Phi(\eta') = 1 & \text{in } \omega, \\ \Phi(\eta') = 0 & \text{on } \partial\omega, \end{cases} \tag{2.7}$$

$$\begin{cases} -\nu \Delta' \vec{U}_0'(\eta') + \nabla' (\eta_n^{1-\gamma} Q_0(\eta')) = 0 & \text{in } \omega, \\ \operatorname{div}' \vec{U}_0'(\eta') = G_0(\eta', \eta_n) & \text{in } \omega, \\ \vec{U}_0'(\eta') = 0 & \text{on } \partial\omega, \end{cases} \tag{2.8}$$

where $\omega = \{\eta' \in \mathbb{R}^{n-1}: \quad |\eta'| < g_0\}$,

$$G_0(\eta', \eta_n) = -\eta_n^{1-\gamma} \mathcal{D}\left(\eta_n^{2(1-\gamma)} \partial_n q_0(\eta_n) \Phi(\eta')\right). \tag{2.9}$$

Multiplying (2.7) by $\Phi(\eta')$ and integrating by parts one gets

$$\int_\omega \Phi(\eta')\, d\eta' = -\nu \int_\omega |\nabla' \Phi|^2\, d\eta' \equiv \kappa_0 < 0 .$$

The solution $\Phi(\eta')$ to (2.7) has the form

$$\Phi(\eta') = \frac{1}{2\nu(n-1)} (|\eta'|^2 - g_0^2)$$

and it is easy to compute

$$\kappa_0 = -\frac{1}{8\nu} g_0^4 \quad \text{for } n = 3 \quad \text{and} \quad \kappa_0 = -\frac{1}{3\nu} g_0^3 \quad \text{for } n = 2 . \tag{2.10}$$

The problem (2.8) has a solution $(\vec{U}_0', \eta_n^{1-\gamma} Q_0)$ if and only if the right-hand side G_0 satisfies the compatibility condition

$$\int_\omega G_0 \, d\eta' = 0 . \tag{2.11}$$

From (2.11), taking into account (2.9), (2.10), we get

$$-\eta_n^{1-\gamma} \partial_n (\eta_n^{2(1-\gamma)} \partial_n q_0(\eta_n)) \int_\omega \Phi(\eta')\, d\eta' -$$

$$-(\gamma - 1)\eta_n^{3(1-\gamma)-1} \partial_n q_0(\eta_n) \int_\omega \eta' \cdot \nabla' \Phi \, d\eta' = 0 .$$

Since

$$\int_\omega \eta' \cdot \nabla' \Phi(\eta')\, d\eta' = -(n-1) \int_\omega \Phi(\eta')\, d\eta' = -(n-1)\kappa_0 ,$$

the last relation yields

$$\begin{aligned} &- \eta_n^{1-\gamma} \partial_n (\eta_n^{2(1-\gamma)} \partial_n q_0(\eta_n)) + \\ &\qquad + (n-1)(\gamma-1)\eta_n^{3(1-\gamma)-1} \partial_n q_0(\eta_n) = 0 . \end{aligned} \tag{2.12}$$

Thus, the function $q_0(\eta_n)$ is not arbitrary; it satisfies the second order ordinary differential equation (2.12). Multiplying (2.12) by $\eta_n^{(n-2)(1-\gamma)}$, we rewrite it in the form

$$-\partial_n (\eta_n^{(n+1)(1-\gamma)} \partial_n q_0(\eta_n)) = 0 . \tag{2.13}$$

Solving (2.13), we find

$$q_0(\eta_n) = \begin{cases} \mu_1 \eta_n^{-(n+1)(1-\gamma)+1} + \mu_2 , & \gamma \neq n(n+1)^{-1}, \\ \mu_1 \ln \eta_n + \mu_2 , & \gamma = n(n+1)^{-1}. \end{cases} \tag{2.14}$$

Now, because of (2.14), (2.6)

$$\begin{aligned} &U_{n,0}(\eta', \eta_n) = \\ &= \eta_n^{-(n-1)(1-\gamma)} \Phi(\eta') \begin{cases} \mu_1 (n+1)(\gamma-1) + 1 , & \gamma \neq n(n+1)^{-1}, \\ \mu_1 , & \gamma = n(n+1)^{-1}, \end{cases} \end{aligned} \tag{2.15}$$

and the function G_0 takes the form

$$G_0(\eta', \eta_n) = -\mu_1 \eta_n^{-(n-2)(1-\gamma)-1}\, \Upsilon(\eta', \nabla')\Phi(\eta') \times \tag{2.16}$$
$$\times \begin{cases} (n+1)(\gamma-1)+1, & \gamma \neq n(n+1)^{-1}, \\ 1, & \gamma = n(n+1)^{-1}, \end{cases}$$

where the operator $\Upsilon(\eta', \nabla')$ is given by

$$\Upsilon(\eta', \nabla') = (n-1)(\gamma-1) + (\gamma-1)\eta' \cdot \nabla' . \tag{2.17}$$

Comparing the power exponents of η_n in (2.16), (2.8), we conclude that the functions $\vec{U}_0'(\eta', \eta_n)$ and $Q_0(\eta', \eta_n)$ can be taken in the form

$$\begin{cases} \vec{U}_0'(\eta', \eta_n) = \eta_n^{-(n-2)(1-\gamma)-1}\, \vec{U}_0'(\eta'), \\ Q_0(\eta', \eta_n) = \eta_n^{-(n-1)(1-\gamma)-1}\, Q_0(\eta') \end{cases} \tag{2.18}$$

and we rewrite (2.8) as follows

$$\begin{cases} -\nu\Delta' \vec{U}_0' + \nabla' Q_0 = 0 & \text{in } \omega, \\ \operatorname{div}' \vec{U}_0' = G_0 & \text{in } \omega, \\ \vec{U}_0' = 0 & \text{on } \partial\omega, \end{cases} \tag{2.19}$$

where

$$G_0(\eta') = \tag{2.20}$$
$$= -\mu_1 \Upsilon(\eta', \nabla')\, \Phi(\eta') \begin{cases} (n+1)(\gamma-1)+1, & \gamma \neq n(n+1)^{-1}, \\ 1, & \gamma = n(n+1)^{-1}. \end{cases}$$

It is a well-known fact that in a bounded domain ω with the smooth boundary $\partial\omega$ solutions of the Poisson equation (2.7) and of the Stokes system (2.19) are infinitely differentiable up to the boundary and obey the estimates

$$|\partial_j^k \Phi(\eta')| \leq C_k, \qquad j = 1, \dots, n-1, \quad k = 0, 1, \dots, \tag{2.21}$$

$$|\partial_j^k \vec{U}_0'(\eta')| + |\partial_j^k Q_0(\eta')| \leq C_k\, |\mu_1|, \tag{2.22}$$
$$j = 1, \dots n-1, \quad k = 0, 1, \dots .$$

Therefore, we obtain (see (2.15), (2.18))

$$(2.23)\quad \begin{cases} |\partial_j^k \partial_n^l U_{n,0}(\eta', \eta_n)| \leq c_{k,l} |\mu_1| \eta_n^{-(n-1)(1-\gamma)-l}, \\ \qquad\qquad k, l = 0, 1, \dots, \\ |\partial_j^k \partial_n^l \vec{U}_0'(\eta', \eta_n)| \leq c_{k,l} |\mu_1| \eta_n^{-(n-2)(1-\gamma)-1-l}, \\ \qquad\qquad k, l = 0, 1, \dots, \\ |\partial_j^k \partial_n^l Q_0(\eta', \eta_n)| \leq c_{k,l} |\mu_1| \eta_n^{-(n+1)(1-\gamma)+1-l}, \\ \qquad\qquad k = 0, \dots, l = 1, \dots . \end{cases}$$

Moreover, the simple computations using (2.10), imply

$$(2.24)\quad \int_\omega U_{n,0}(\eta', \eta_n)\, d\eta' =$$

$$= \mu_1 \kappa_0 \begin{cases} ((n+1)(\gamma-1)+1)\, \eta_n^{-(n-1)(1-\gamma)}, & \gamma \neq n(n+1)^{-1}, \\ \eta_n^{-(n-1)(n+1)^{-1}}, & \gamma = n(n+1)^{-1}. \end{cases}$$

2.3. *Estimates of the dicrepances.* Let us define

$$(2.25)\quad \begin{cases} \vec{u}_0(x) = \vec{U}_0(x' x_n^{\gamma-1}, x_n), \\ p_0(x) = P_0(x' x_n^{\gamma-1}, x_3) = q_0(x_n) + Q_0(x' x_n^{\gamma-1}, x_n). \end{cases}$$

By the construction

$$\operatorname{div} \vec{u}_0(x) = 0 \text{ in } \Omega_i, \qquad \vec{u}_0(x) = 0 \text{ on } \partial\Omega_i \backslash \sigma_i(0)$$

and

$$\int_{\sigma_i} \vec{u}_0 \cdot \vec{n}\, dx' = \mu_1 \kappa_0 \begin{cases} (n+1)(\gamma-1)+1, & \gamma \neq n(n+1)^{-1}, \\ 1, & \gamma = n(n+1)^{-1}. \end{cases}$$

Thus, taking

$$(2.26)\quad \mu_1 = \begin{cases} F_i((n+1)(\gamma-1)+1)^{-1} \kappa_0^{-1}, & \gamma \neq n(n+1)^{-1}, \\ F_i \kappa_0^{-1}, & \gamma = n(n+1)^{-1}, \end{cases}$$

we have

$$(2.27)\quad \int_{\sigma_i} \vec{u}_0 \cdot \vec{n}\, dx' = F_i .$$

Furthermore, direct computations using (2.3), (2.4), (2.14), (2.15), (2.18), (2.23) and the condition $\gamma > 0$, show that $\vec{u}^{(0)}$, $p^{(0)}$ satisfy the Stokes system

$$\begin{cases} -\nu\Delta \vec{u}_0 + \nabla p_0 = \vec{H}_0 & \text{in } \Omega_i, \\ \operatorname{div} \vec{u}_0 = 0 & \text{in } \Omega_i, \\ \vec{u}_0 = 0 & \text{on } \partial\Omega_i \setminus \sigma_i(0), \end{cases} \tag{2.28}$$

with the right-hand side $\vec{H}_0$, subject to the estimates

$$|D_x^\alpha H_{j,0}(x)| \leqslant c|F_i|x_n^{-(n+1+|\alpha|)(1-\gamma)-(3+\alpha_n)\gamma}, \qquad j = 1, \ldots, n-1, \tag{2.29}$$

$$|D_x^\alpha H_{n,0}(x)| \leqslant c|F_i|x_n^{-(n+1+|\alpha|)(1-\gamma)-(2+\alpha_n)\gamma}, \tag{2.30}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$. The functions $\vec{u}_0$, p_0 themselves obey the inequalities

$$|D_x^\alpha \vec{u}_0'(x)| \leqslant c|F_i|x_n^{-(n-1+|\alpha|)(1-\gamma)-\gamma}, \qquad |\alpha| \geqslant 0, \tag{2.31}$$

$$|D_x^\alpha u_{n,0}(x)| \leqslant c|F_i|x_n^{-(n-1+|\alpha|)(1-\gamma)}, \qquad |\alpha| \geqslant 0, \tag{2.32}$$

$$|D^\alpha p_0(x)| \leqslant c|F_i|x_n^{-(n+|\alpha|)(1-\gamma)}, \qquad |\alpha| \geqslant 1, \tag{2.33}$$

$$|p_0(x)| \leqslant c\begin{cases} |F_i|x_n^{-(n+1)(1-\gamma)+1} + c_1, & \gamma \neq n(n+1)^{-1}, \\ |F_i|\ln x_n + c_1, & \gamma = n(n+1)^{-1}. \end{cases} \tag{2.34}$$

REMARK 2.1. *Inequality* (2.32) *coincides with* (1.6), *where we take* $g_i(x_n) = g_0 x_n^{1-\gamma}$, *while* (2.31) *states better decay at infinity for the components* $u_{j,0}(x)$, $j = 1, \ldots, n-1$, *of the velocity field* $\vec{u}$; *we have in* (2.31) *an additional vanishing factor* $x_n^{-\gamma}$. *The discrepancy* $\vec{H}_0$ *also has at infinity better decay as* $\Delta\vec{A}$. *We have in* (2.29) *an additional vanishing factor* $x_n^{-\gamma(3+\alpha_n)}$ *and in* (2.30) *we have the factor* $x_n^{-\gamma(2+\alpha_n)}$. *This is the case, since we have already compenseted the principal at infinity terms in equations* (1.2).

REMARK 2.2. *Equation* (2.13) *describing* q_0 *is similar to the Reynolds equation, which is well known in the theory of lubrication* (*see the lists of references in* [19], [20]).

REMARK 2.3. *In* [26] *it was shown that divergence free vector fields with the finite Dirichlet integral may have the nonzero fluxes over the sections σ_i of the outlet to infinity Ω_i, having the form* (1.7), *if and only if there holds the condition*

$$\int^{\infty} g_i(t)^{-(n-1)}\,dt < \infty\,.$$

In the case $g_i(t) = g_0 t^{1-\gamma}$ this yields

$$\int^{\infty} t^{-(n+1)(1-\gamma)}\,dt < \infty\,.$$

One has

$$\int^{\infty} t^{-(n+1)(1-\gamma)}\,dt = \infty\,, \qquad \text{if } \gamma < n(n+1)^{-1}$$

and

$$\int^{\infty} t^{-(n+1)(1-\gamma)}\,dt < \infty\,, \qquad \text{if } \gamma > n(n+1)^{-1}.$$

In the limit case $\gamma = n(n+1)^{-1}$ the logarithmic term appears in the asymptotic representation for the pressure function P.

REMARK 2.4. *The formulas* (2.14), (2.15), (2.18) *together with* (2.26), (2.27) *declare the continuous dependence on γ of the power exponent of $x_n = \eta_n$ in the formal asymptotic representation $(\vec{u}^F, p^F)$ of the solution $(\vec{u}, p)$ with the prescribed flux $F \neq 0$. For example, at $n = 3$ we have*

$$p^F(x) = O(x_n^{4\gamma-3})\,, \qquad |\vec{u}^F(x)| = O(x_n^{2\gamma-2}) \qquad \text{as } x_n \to \infty \tag{2.35}$$

(*if $\gamma = 3/4$, then $x_n^{4\gamma-3}$ in* (2.35) *is replaced by* $\ln x_n$). *The relations* (2.35) *remain valid also for cylindrical* ($\gamma = 1$) *outlets to infinity* (Poiseuille flow) *and for conical* ($\gamma = 0$) *ones* (2). *Notice that p^F is a bounded function only under the condition $\gamma < 3/4$, and at $\gamma = 1/2$ the decays of p^F and $|\vec{u}^F|$ are of the same order, while $p^F(x) = o(|\vec{u}^F(x)|)$ for $\gamma < 1/2$ and $|\vec{u}^F(x)| = o(p^F(x))$ for $\gamma > 1/2$ as $x_n \to \infty$.*

(2) In the case $\gamma = 0$ the relations (2.35) are proved by a different argumentation.

3. - The higher order terms; formal procedure

3.1. *Structure of general discrepancy term.* In order to construct the complete asymptotics series, we need to learn how to compensate each of the discrepancy terms and to show that a new discrepancy, appearing after compensation, has the similar form with a smaller exponent of η_n. To this end, we consider the equations (2.4) with the right-hand sides, having the special form

$$(3.1)_1 \qquad -\nu(\eta_n^{2\gamma-2}\Delta' + \mathbb{D}^2)\vec{U}' + \eta_n^{\gamma-1}\nabla' P = \eta_n^{\Lambda-1-\gamma}\vec{\mathcal{F}}'(\eta') \quad \text{in } \Pi_+ ,$$

$$(3.1)_2 \qquad -\nu(\eta_n^{2\gamma-2}\Delta' + \mathbb{D}^2) U_n + \mathbb{D}P = \eta_n^{\Lambda-1}\mathcal{F}_n(\eta') \quad \text{in } \Pi_+ ,$$

$$(3.1)_3 \qquad \eta_n^{\gamma-1} div' \vec{U}' + \mathbb{D}U_n = 0 \quad \text{in } \Pi_+ ,$$

$$(3.1)_4 \qquad \vec{U} = 0 \quad \text{on } S_+ ,$$

where $\vec{\mathcal{F}}'$, $\mathcal{F}_n$ are arbitrary functions. We will satisfy the equations (3.1) in main, if we put

$$(3.2) \qquad \begin{cases} P(\eta', \eta_n) = q_* \eta_n^{\Lambda} + \eta_n^{\Lambda-2\gamma} Q(\eta'), \qquad q_* = \text{const}, \\ \vec{U}'(\eta', \eta_n) = \eta_n^{\Lambda+1-3\gamma}\vec{U}'(\eta'), \\ U_n(\eta', \eta_n) = \eta_n^{\Lambda+1-2\gamma} U_n(\eta') \end{cases}$$

where $U_n(\eta')$ is represented as a sum

$$(3.3) \qquad U_n(\eta') = q_* \Lambda \Phi(\eta') + U_n^{(0)}(\eta'),$$

$\Phi(\eta')$ is the solution of the problem (2.7) and $U_n^{(0)}$ satisfies the equations

$$(3.4) \qquad \begin{cases} -\nu\Delta' U_n^{(0)} = \mathcal{F}_n & \text{in } \omega , \\ U_n^{(0)} = 0 & \text{on } \partial\omega \end{cases}$$

and $(\vec{U}', Q)$ is the solution to

$$(3.5) \qquad \begin{cases} -\nu\Delta'\vec{U}' + \nabla' Q = \vec{\mathcal{F}}' & \text{in } \omega , \\ \text{div}' \vec{U}' = -\mathbb{D}(\Lambda) U_n & \text{in } \omega , \\ \vec{U}' = 0 & \text{on } \partial\omega , \end{cases}$$

where

$$\mathbb{D}(\Lambda) = \Lambda + 1 - 2\gamma + (\gamma - 1)\eta' \cdot \nabla' .$$

The solvability condition for the problem (3.4)

$$\int_\omega \mathbb{D}(\Lambda) U_n(\eta')\, d\eta' = 0$$

gives us the constant q_*:

$$\Lambda \kappa_0 [(\Lambda + 1 - 2\gamma) + (1 - \gamma)(n - 1)] q_* = - \int_\omega \mathbb{D}(\Lambda) U_n^{(0)}(\eta')\, d\eta' . \tag{3.6}$$

Thus, if

$$\Lambda \neq 0 , \qquad \text{and } \Lambda \neq (\gamma - 1)(\mathrm{n} - 1) - 1 + 2\gamma , \tag{3.7}$$

the constant q_* is uniquely determined from (3.6).

The discrepancies $\vec{H}'(\eta', \eta_n)$, $H_n(\eta', \eta_n)$ left by functions (3.2) in the equations $(3.\,1)_1$ and $(3.\,1)_2$ can be written in the form

$$\left\{\begin{aligned} \vec{H}'(\eta', \eta_n) &= \nu \mathscr{D}^2 (\eta_n^{\Lambda + 1 - 3\gamma} \vec{U}'(\eta')) = \\ &= \eta_n^{(\Lambda - 2\gamma) - 1 - \gamma} \nu \mathbb{D}(\Lambda - \gamma - 1) \mathbb{D}(\Lambda - \gamma) \vec{U}'(\eta') = \\ &= \eta_n^{(\Lambda - 2\gamma) - 1 - \gamma} \vec{\mathscr{F}}'(\eta'), \\ H_n(\eta', \eta_n) &= \nu \mathscr{D}^2 (\eta_n^{\Lambda + 1 - 2\gamma} U_n(\eta')) - \mathscr{D}(\eta_n^{\Lambda - 2\gamma} Q(\eta')) = \\ &= \eta_n^{(\Lambda - 2\gamma) - 1} (\nu \mathbb{D}(\Lambda - 1) \mathbb{D}(\Lambda) U_n(\eta') - \mathbb{D}(\Lambda - 1) Q(\eta')) = \\ &= \eta_n^{(\Lambda - 2\gamma) - 1} \mathscr{F}_n(\eta') . \end{aligned}\right. \tag{3.8}$$

It is easy to see that the new right-hand sides have the same form as it was in (3.1) with the only difference that the decay exponent Λ is changed to $\hat{\Lambda} = \Lambda - 2\gamma$. Therefore, this process can be extended and we can look for the approximate solution $(\vec{U}, P)$ to problem (3.1) in the form of series in powers of η_n:

$$\left\{\begin{aligned} P(\eta', \eta_n) &\sim \sum_{\lambda \in \mathfrak{M}} \eta_n^{\lambda} (q_\lambda + \eta_n^{-2\gamma} Q_\lambda(\eta')) , \qquad q_\lambda = \text{const.} , \\ \vec{U}'(\eta', \eta_n) &\sim \sum_{\lambda \in \mathfrak{M}} \eta_n^{\lambda + 1 - 3\gamma} \vec{U}'_\lambda(\eta') , \\ U_n(\eta', \eta_n) &\sim \sum_{\lambda \in \mathfrak{M}} \eta_n^{\lambda + 1 - 2\gamma} U_{n,\lambda}(\eta') , \end{aligned}\right. \tag{3.9}$$

where $\mathfrak{M}$ is certain set of indeces. From the above considerations it follows that

$$\lambda \in \mathfrak{M} \Rightarrow \lambda - 2\gamma \in \mathfrak{M}. \tag{3.10}$$

3.2. *The first exceptional case.* Let us suppose that

$$\Lambda = 0, \qquad \Lambda \neq (\gamma - 1)(n-1) - 1 + 2\gamma. \tag{3.11}$$

Then we look for the solution $(\vec{U}, P)$ in the form

$$\begin{cases} P(\eta', \eta_n) = q_* \ln \eta_n + \eta_n^{-2\gamma} Q(\eta'), \\ \vec{U}'(\eta', \eta_n) = \eta_n^{1-3\gamma} \vec{U}'(\eta'), \qquad U_n(\eta', \eta_n) = \eta_n^{1-2\gamma} U_n(\eta'). \end{cases} \tag{3.12}$$

For $U_n(\eta') = q_* \Phi(\eta') + U_n^{(0)}(\eta')$ and $(\vec{U}'(\eta'), Q(\eta'))$ we get the same equations (3.4), (3.5); the relation (3.6) for q_* is changed into

$$\kappa_0 [(1 - 2\gamma) + (1 - \gamma)(n-1)] q_* = - \int_\omega \mathrm{D}(0)\, U_n^{(0)}(\eta')\, d\eta'$$

and for the discrepancies $\vec{H}'(\eta', \eta_n)$, $H_n(\eta', \eta_n)$ we have the formulas (3.8) at $\Lambda = 0$.

3.3. *The second exceptional case.* Let

$$\Lambda = (\gamma - 1)(n-1) - 1 + 2\gamma, \qquad \Lambda \neq 0. \tag{3.13}$$

We take

$$\begin{cases} P(\eta', \eta_n) = q_* \eta_n^{\Lambda} \ln \eta_n + \eta_n^{\Lambda - 2\gamma} Q(\eta', \ln \eta_n), \\ \vec{U}'(\eta', \eta_n) = \eta_n^{\Lambda + 1 - 3\gamma} \vec{U}'(\eta', \ln \eta_n), \\ U_n(\eta', \eta_n) = \eta_n^{\Lambda + 1 - 2\gamma} U_n(\eta', \ln \eta_n), \end{cases} \tag{3.14}$$

where

$$\begin{cases} Q(\eta', \ln \eta_n) = Q^{(1)}(\eta') \ln \eta_n + Q^{(0)}(\eta'), \\ \vec{U}'(\eta', \ln \eta_n) = \vec{U}'^{(1)}(\eta') \ln \eta_n + \vec{U}'^{(0)}(\eta'), \\ U_n(\eta', \ln \eta_n) = q_* (1 + \Lambda \ln \eta_n) \Phi(\eta') + U_n^{(0)}(\eta'). \end{cases} \tag{3.15}$$

Substituting the function (3.15) into equations (3.1) and collecting the coefficients of the same powers on η_n^μ and $\eta_n^\mu \ln \eta_n$, we find that $U_n^{(0)}(\eta')$ is subject to the equation (3.4); $(U'^{(0)}(\eta'), Q^{(0)}(\eta'))$ and $(U'^{(1)}(\eta'), Q^{(1)}(\eta'))$ are solutions to

$$(3.16)_0 \quad \begin{cases} -\nu\Delta' \vec{U}'^{(0)} + \nabla' Q^{(0)} = \vec{\mathscr{F}}' & \text{in } \omega, \\ \operatorname{div}' \vec{U}'^{(0)} = -q_* \mathbb{D}(\Lambda)\Phi - q_* \Lambda \mathbb{D}(2\gamma)\Phi - \mathbb{D}(\Lambda) U_n^{(0)} & \text{in } \omega, \\ \vec{U}'^{(0)} = 0 & \text{on } \partial\omega, \end{cases}$$

and

$$(3.16)_1 \quad \begin{cases} -\nu\Delta' \vec{U}'^{(1)} + \nabla' Q^{(1)} = 0 & \text{in } \omega, \\ \operatorname{div}' \vec{U}'^{(1)} = -q_* \Lambda \mathbb{D}(\Lambda)\Phi & \text{in } \omega, \\ \vec{U}'^{(1)} = 0 & \text{on } \partial\omega. \end{cases}$$

The solvability condition for the problem $(3.16)_0$ gives

$$(3.17) \quad \kappa_0[(\Lambda + 1 - 2\gamma) + (1-\gamma)(n-1)]q_* + $$
$$+ \Lambda\kappa_0[1 + (1-\gamma)(n-1)]q_* = -\int_\omega \mathbb{D}(\Lambda) U_n^{(0)}(\eta')\,d\eta'.$$

Because of (3.13) the first term on the left of (3.17) vanishes. Moreover, from (3.13) it follows that $\gamma \neq n(n+1)^{-1}$. Therefore, $1 + (\gamma - 1) \times$ $\times (n-1) \neq 0$ and q_* can be determined from the equation

$$(3.18) \quad \Lambda\kappa_0[1 + (1-\gamma)(n-1)]q_* = -\int_\omega \mathbb{D}(\Lambda) U_n^{(0)}(\eta')\,d\eta'.$$

The solvability condition for problem $(3.16)_1$ has the form

$$\Lambda\kappa_0[(\Lambda + 1 - 2\gamma) + (1-\gamma)(n-1)]q_* = 0$$

and it is valid automatically because of (3.13).

The discrepancies $\vec{H}'(\eta', \eta_n)$, $H_n(\eta', \eta_n)$ left by functions (3.14) in the equations $(3.1)_1$ and $(3.1)_2$ can be written in the form

$$(3.19)\qquad \begin{cases} \vec{H}'(\eta', \eta_n) = \eta_n^{(\Lambda - 2\gamma) - 1 - \gamma}(\vec{\mathcal{F}}'^{(0)}(\eta') + \vec{\mathcal{F}}'^{(1)}(\eta')\ln\eta_n), \\ H_n(\eta', \eta_n) = \eta_n^{(\Lambda - 2\gamma) - 1}(\widetilde{\mathcal{F}}_n^{(0)}(\eta') + \widetilde{\mathcal{F}}_n^{(1)}(\eta')\ln\eta_n). \end{cases}$$

3.3. *The third exceptional case.* It can happen that Λ meet both conditions (3.11), (3.13):

$$(3.20)\qquad \Lambda = (\gamma - 1)(n - 1) - 1 + 2\gamma = 0, \quad \text{i.e.}\quad \gamma = n(n+1)^{-1}.$$

In this case we take

$$(3.21)\qquad \begin{cases} P(\eta', \eta_n) = \\ \qquad = q_*(\ln\eta_n + (\ln\eta_n)^2) + \eta_n^{-2\gamma}(Q^{(1)}(\eta')\ln\eta_n + Q^{(0)}(\eta')), \\ \vec{U}'(\eta', \eta_n) = \eta_n^{1-3\gamma}(\vec{U}'^{(1)}(\eta')\ln\eta_n + \vec{U}'^{(0)}(\eta')), \\ U_n(\eta', \eta_n) = q_*\eta_n^{-1-2\gamma}(1 + 2\ln\eta_n)\Phi(\eta') + \eta_n^{1-2\gamma}U_n^{(0)}(\eta'). \end{cases}$$

Repeating the above considerations, one can find the boundary value problems of type (3.16) to determine the coefficiens $U_n^{(0)}$, $(\vec{U}'^{(0)}, Q^{(0)}(\eta'))$ and $(\vec{U}'^{(1)}, Q^{(1)}(\eta'))$. The solvability condition for the problem corresponding to $(\vec{U}'^{(0)}, Q^{(0)}(\eta'))$ will give the constant q_* and the solvability condition for $(\vec{U}'^{(1)}, Q^{(1)}(\eta'))$ is valid automatically. The discrepances $\vec{H}'(\eta', \eta_n)$, $H_n(\eta', \eta_n)$ have the same form (3.19).

3.5. *The right-hand sides, containing the logarithmic terms; case* (3.7). From (3.19) we can see that the new right-hand sides $\vec{H}'(\eta', \eta_n)$, $H_n(\eta', \eta_n)$ may contain the logarithmic terms. If we repeat the iterative procedure, the logarithmic terms will be iterated, i.e. there will appear the powers of $\ln\eta_n$. Therefore, it is necessary to consider the right-hand sides which have the form

$$(3.22)\qquad \begin{cases} \vec{\mathcal{F}}'(\eta', \eta_n) = \eta_n^{\Lambda - 1 - \gamma}\sum_{j=0}^{k}\vec{\mathcal{F}}'^{(j)}(\eta')(\ln\eta_n)^j, \\ \mathcal{F}_n(\eta', \eta_n) = \eta_n^{\Lambda - 1}\sum_{j=0}^{k}\widetilde{\mathcal{F}}_n^{(j)}(\eta')(\ln\eta_n)^j. \end{cases}$$

If Λ is subject to (3.7), the solution can be found as

$$(3.23)\quad \begin{cases} P(\eta', \eta_n) = \eta_n^{\Lambda} \sum_{j=0}^{k} q_*^{(j)} (\ln \eta_n)^j + \eta_n^{\Lambda - 2\gamma} \sum_{j=0}^{k} Q^{(j)}(\eta')(\ln \eta_n)^j, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q_*^{(j)} = \text{const}, \\ \vec{U}'(\eta', \eta_n) = \eta_n^{\Lambda + 1 - 3\gamma} \sum_{j=0}^{k} \vec{U}'^{(j)}(\eta')(\ln \eta_n)^j, \\ U_n(\eta', \eta_n) = \eta_n^{\Lambda + 1 - 2\gamma} \Phi(\eta') \sum_{j=0}^{k} a_j (\ln \eta_n)^j + \\ \qquad\qquad + \eta_n^{\Lambda + 1 - 2\gamma} \sum_{j=0}^{k} U_n^{(j)}(\eta')(\ln \eta_n)^j. \end{cases}$$

Collecting the coefficiens at the same powers of $\eta_n^{\mu}(\ln \eta_n)^j$, we derive

$$(3.24)\quad a_j = q_*^{(j)} \Lambda + (j+1) q_*^{(j+1)}, \quad j = 0, \dots, k-1, \qquad a_k = q_*^{(k)}\, \Lambda,$$

$$(3.25)\quad \begin{cases} -\nu \Delta' U_n^{(j)} = \mathcal{F}_n^{(j)} & \text{in } \omega, \\ U_n^{(j)} = 0 & \text{on } \partial\omega, \end{cases}$$

$j = 0, \dots, k$,

$$(3.26)_j\quad \begin{cases} -\nu \Delta' \vec{U}'^{(j)} + \nabla' Q^{(j)} = \vec{\mathcal{F}}'^{(j)} & \text{in } \omega, \\ \operatorname{div}' \vec{U}'^{(j)} = -a_j \mathrm{D}(\Lambda)\Phi - a_{j+1}\mathrm{D}(j+2\gamma)\Phi - \\ \qquad - \mathrm{D}(\Lambda) U_n^{(j)} - \mathrm{D}(j+2\gamma) U_n^{(j+1)} & \text{in } \omega, \\ \vec{U}'^{(j)} = 0 & \text{on } \partial\omega, \end{cases}$$

$j = 0, \dots, k-1$,

$$(3.26)_k\quad \begin{cases} -\nu \Delta' \vec{U}'^{(k)} + \nabla' Q^{(k)} = \vec{\mathcal{F}}'^{(k)} & \text{in } \omega, \\ \operatorname{div}' \vec{U}'^{(k)} = -a_k \mathrm{D}(\Lambda)\Phi - \mathrm{D}(\Lambda) U_n^{(k)} & \text{in } \omega, \\ \vec{U}'^{(k)} = 0 & \text{on } \partial\omega. \end{cases}$$

From the solvability conditions for problems $(3.26)_j$, $j = 0, \dots k$, we find the constants $q_*^{(j)}$. Together with (3.24) this gives the linear system of algebraic equations

$$
(3.27)\qquad \begin{cases}
\Lambda[(\Lambda+1-2\gamma)+(1-\gamma)(n-1)]\,q_*^{(j)}+ \\
\qquad +\big([(\Lambda+1-2\gamma)+(1-\gamma)(n-1)](j+1)+ \\
\qquad +(j+1+(\gamma-1)(n-1))\big)\,q_*^{(j+1)}+ \\
\qquad +(j+2)\Lambda(j+1+(1-\gamma)(n-1))\,q_*^{(j+2)}= \\
\qquad =-\kappa_0^{-1}\int\limits_\omega (\mathbb{D}(\Lambda)\,U_n^{(j)}+\mathbb{D}(j+2\gamma)\,U_n^{(j+1)})\,d\eta', \\
\qquad\qquad j=0,\,\ldots,\,k-1, \\
\Lambda[(\Lambda+1-2\gamma)+(\gamma-1)(n-1)]\,q_*^{(k)}=-\kappa_0^{-1}\int\limits_\omega \mathbb{D}(\Lambda)\,U_n^{(k)}.
\end{cases}
$$

The determinant of the system (3.27) is equal to

$$\Lambda^{k+1}[(\Lambda+1-2\gamma)+(1-\gamma)(n-1)]^{k+1}\neq 0$$

(see (3.7)) and $q_*^{(j)}$ are uniquely determined from (3.27). The discrepances $\vec{H}'$, H_n have the form

$$
(3.28)\qquad \begin{cases}
\vec{H}'(\eta',\eta_n)=\eta_n^{(\Lambda-2\gamma)-1-\gamma}\sum\limits_{j=0}^{k}\vec{\mathcal{F}}'^{(j)}(\eta')(\ln\eta_n)^j, \\
H_n(\eta',\eta_n)=\eta_n^{(\Lambda-2\gamma)-1}\sum\limits_{j=0}^{k}\mathcal{F}_n^{(j)}(\eta')(\ln\eta_n)^j.
\end{cases}
$$

3.6. *The right-hand sides, containing the logarithmic terms; case* (3.11). If Λ satisfies (3.11), we look for the solution $(\vec{U}, P)$ in the form

$$
(3.29)\qquad \begin{cases}
P(\eta',\eta_n)=\ln\eta_n\sum\limits_{j=0}^{k}q_*^{(j)}(\ln\eta_n)^j+\eta_n^{-2\gamma}\sum\limits_{j=0}^{k}Q^{(j)}(\eta')(\ln\eta_n)^j, \\
\qquad\qquad q_*^{(j)}=\text{const.}, \\
\vec{U}'(\eta',\eta_n)=\eta_n^{1-3\gamma}\sum\limits_{j=0}^{k}\vec{U}'^{(j)}(\eta')(\ln\eta_n)^j, \\
U_n(\eta',\eta_n)=\eta_n^{1-2\gamma}\Phi(\eta')\sum\limits_{j=0}^{k}a_j(\ln\eta_n)^j+ \\
\qquad +\eta_n^{1-2\gamma}\sum\limits_{j=0}^{k}U_n^{(j)}(\eta')(\ln\eta_n)^j.
\end{cases}
$$

The simple computations show that (3.25), (3.26) are valid at $\Lambda = 0$ and for the determination of $q_*^{(j)}$ we again obtain the system of linear algebraic equations with the determinant different from zero. The expressions for the discrepancies $\vec{H}'$, H_n are given by the same formulas (3.28).

3.7. *The right-hand sides, containing the logarithmic terms; case* (3.13). Let us consider the case (3.13). We take

$$\text{(3.30)} \quad \begin{cases} P(\eta', \eta_n) = \eta_n^{\Lambda} \ln \eta_n \sum_{j=0}^{k} q_*^{(j)} (\ln \eta_n)^j + \\ \qquad + \eta_n^{\Lambda - 2\gamma} \sum_{j=0}^{k+1} Q^{(j)}(\eta')(\ln \eta_n)^j, \qquad q_*^{(j)} = \text{const.}, \\ \vec{U}'(\eta', \eta_n) = \eta_n^{\Lambda + 1 - 3\gamma} \sum_{j=0}^{k+1} \vec{U}'^{(j)}(\eta')(\ln \eta_n)^j, \\ U_n(\eta', \eta_n) = \eta_n^{\Lambda + 1 - 2\gamma} \Phi(\eta') \sum_{j=0}^{k+1} a_j (\ln \eta_n)^j + \\ \qquad + \eta_n^{\Lambda + 1 - 2\gamma} \sum_{j=0}^{k} U_n^{(j)}(\eta')(\ln \eta_n)^j. \end{cases}$$

Then

$$\text{(3.31)} \quad a_0 = q_*^{(0)}, \qquad a_j = \Lambda q_*^{(j-1)} + (j+1) q_*^{(j)}, \qquad j = 1, \ldots, k, \qquad a_{k+1} = \Lambda q_*^{(k)};$$

the equations (3.25), (3.26) are valid for $j = 0, \ldots, k$ and for $j = k+1$ we get

$$\text{(3.32)} \quad \begin{cases} -\nu\Delta' \vec{U}'^{(k+1)} + \nabla' Q^{(k+1)} = 0 & \text{in } \omega, \\ \operatorname{div}' \vec{U}'^{(k+1)} = -a_{k+1} \mathbb{D}(\Lambda)\Phi & \text{in } \omega, \\ \vec{U}'^{(k)} = 0 & \text{on } \partial\omega. \end{cases}$$

The solvability condition for (3.26) at $j = 0, \ldots, k$ give us the the system of equations

$$
(3.33)\quad \begin{cases} \Lambda(j+1+(1-\gamma)(n-1))(q_*^{(j)}+(j+1)q_*^{(j+1)}) = \\ \qquad = -\kappa_0^{-1}\int\limits_\omega (\mathbb{D}(\Lambda)\, U_n^{(j)} + \mathbb{D}(j+2\gamma)\, U_n^{(j+1)})\, d\eta' , \\ \qquad\qquad j=0, \dots, k-1 , \\ \Lambda(j+1+(1-\gamma)(n-1))\, q_*^{(k)} = -\kappa_0^{-1}\int\limits_\omega \mathbb{D}(\Lambda)\, U_n^{(k)}\, d\eta' , \end{cases}
$$

which has the unique solution. The solvability condition for (3.32)

$$
[(\Lambda+1-2\gamma)+(1-\gamma)(n-1)]\, q_*^{(k)} = 0
$$

is valid because of (3.13). The expressions for the discrepancies have the form

$$
(3.34)\quad \begin{cases} \vec{H}'(\eta', \eta_n) = \eta_n^{(\Lambda-2\gamma)-1-\gamma} \sum\limits_{j=0}^{k+1} \vec{\mathcal{F}}'^{(j)}(\eta')(\ln \eta_n)^j , \\ H_n(\eta', \eta_n) = \eta_n^{(\Lambda-2\gamma)-1} \sum\limits_{j=0}^{k+1} \mathcal{F}_n^{(j)}(\eta')(\ln \eta_n)^j . \end{cases}
$$

3.8. *The right-hand sides, containing the logarithmic terms; case* (3.20). Finally, if we meet Λ, satisfying (3.20), we take

$$
(3.35)\quad \begin{cases} P(\eta', \eta_n) = (\ln \eta_n)^2 \sum\limits_{j=0}^{k} q_*^{(j)}(\ln \eta_n)^j + \eta_n^{-2\gamma} \sum\limits_{j=0}^{k+1} Q^{(j)}(\eta')(\ln \eta_n)^j , \\ \qquad\qquad q_*^{(k)} = \text{const.} , \\ \vec{U}'(\eta', \eta_n) = \eta_n^{1-3\gamma} \sum\limits_{j=0}^{k+1} \vec{U}'^{(j)}(\eta')(\ln \eta_n)^j , \\ U_n(\eta', \eta_n) = \\ \qquad = \eta_n^{1-2\gamma}\Phi(\eta') \sum\limits_{j=0}^{k+1} a_j(\ln \eta_n)^j + \eta_n^{1-2\gamma} \sum\limits_{j=0}^{k} U_n^{(j)}(\eta')(\ln \eta_n)^j \end{cases}
$$

and we are led to the same conclusions as in the case (3.13).

4. – Concrete problems; construction of the asymptotics.

Below we apply the described in Section 3 algorithm in order to construct the asymptotics of the solution $(\vec{u}, p)$ to the Stokes problem (1.2), (1.4) with the right–hand side $\vec{f}$, having either the compact support or admitting the special series representation. We also apply the algorithm to construct the asymptotics of the solution to the nonlinear Navier–Stokes problem (1.3), (1.4) with zero right–hand side $\vec{f}$.

4.1. *Stokes problem with the right–hand side $\vec{f}$, having compact support.* As it is shown in Section 2, the main term of the asymptotic representation for the solution of the problem (1.2), (1.4) with $\vec{f}$, having a compact support, have the form (2.25) (see also (2.5), (2.6), (2.14), (2.15), (2.18)). It means that

$$\lambda_0 = (n+1)\gamma - n\,. \tag{4.1}$$

Hence, in virtue of (3.9) we are under the condition (3.7) and the asymptotical series for the solution $(\vec{U}, P)$ may be written in the form

$$\begin{cases} P(\eta', \eta_n) \sim \eta_n^{\lambda_0} \sum\limits_{k=0}^{\infty} \eta_n^{-2k\gamma} (q_k + \eta_n^{-2\gamma} Q_k(\eta'))\,, \\ U_n(\eta', \eta_n) \sim \eta_n^{\lambda_0} \sum\limits_{k=0}^{\infty} \eta_n^{-2(k+1)\gamma+1} U_{n,k}(\eta')\,, \\ \vec{U}'(\eta', \eta_n) \sim \eta_n^{\lambda_0} \sum\limits_{k=0}^{\infty} \eta_n^{-(2k+3)\gamma+1} \vec{U}'_k(\eta')\,, \end{cases} \tag{4.2}$$

where q_k are constants,

$$\gamma \neq n(n+1)^{-1} \tag{4.3}$$

and $U_{n,k+1}(\eta')$, $k \geqslant 1$, are represented as the sums

$$U_{n,k+1}(\eta') = q_{k+1}((\lambda_0 - 2(k+1)\gamma)\,\Phi(\eta') + U^{(0)}_{n,k+1}(\eta')\,. \tag{4.4}$$

The coefficiens $U^{(0)}_{n,k+1}(\eta')$, $k \geqslant 1$, are solutions to the problem (3.4) at $q_* = q_{k+1}$, $\Lambda = \lambda_0 - 2(k+1)\gamma$ and

$$\mathcal{F}_{n,k+1}(\eta') = \nu \mathbb{D}(\lambda_0 - 2k\gamma - 1)\,\mathbb{D}(\lambda_0 - 2k\gamma)\,U_{n,k}(\eta') - \\ - \mathbb{D}(\lambda_0 - 2k\gamma - 1)\,Q_k(\eta') \tag{4.5}$$

while $(\vec{U}'_{k+1}(\eta'), Q_{k+1}(\eta'))$, $k \geqslant 1$, are solutions to (3.5) at

$$\vec{\mathcal{F}}'_{k+1}(\eta') = \nu \mathbb{D}(\lambda_0 - (2k+1)\gamma - 1)\mathbb{D}(\lambda_0 - (2k+1)\gamma)\,\vec{U}'_k(\eta')\,. \tag{4.6}$$

The constants q_{k+1} are found in order to satisfy the solvability condition for the problem (3.5) (see (3.6), (3.7)) and are subjected to

$$(4.7)\qquad (\lambda_0 - 2(k+1)\gamma)\,\kappa_0[\lambda_0 - 2(k+2)\gamma + 1 - (\gamma - 1)(n-1)]\,q_{k+1} = \\ = -\int_\omega \mathbb{D}(\lambda_0 - 2(k+1)\gamma)\,U_n^{(0)}(\eta')\,d\eta' .$$

Notice that the functions $Q_{k+1}(\eta')$, $k > 0$, are defined from (3.5) up to an additive constants. We fix it by the normalization

$$(4.8)\qquad \int_\omega Q_{k+1}(\eta')\,d\eta' = 0 .$$

In the case $\gamma = n(n+1)^{-1}$ we put

$$(4.9)\qquad \begin{cases} P(\eta', \eta_n) \sim q_0 \ln \eta_n + \eta_n^{-2n(n+1)^{-1}} Q_0(\eta') + \\ \qquad + \sum\limits_{k=1}^{\infty} \eta_n^{-2kn(n+1)^{-1}} (q_k^{(0)} + \eta_n^{-2n(n+1)^{-1}} Q_k(\eta')) , \\ U_n(\eta', \eta_n) \sim \sum\limits_{k=0}^{\infty} \eta_n^{-2(k+1)n(n+1)^{-1}+1} U_{n,k}(\eta') , \\ \vec{U}'(\eta', \eta_n) \sim \sum\limits_{k=0}^{\infty} \eta_n^{-(2k+3)n(n+1)^{-1}+1} \vec{U}_k'(\eta') \end{cases}$$

and we are led to the same conclusions. Let

$$(4.10)\qquad \begin{cases} P^{[N]}(\eta', \eta_n) = \eta_n^{\lambda_0} \sum\limits_{k=0}^{N} \eta_n^{-2k\gamma} (q_k + \eta_n^{-2\gamma} Q_k(\eta')) , \\ U_n^{[N]}(\eta', \eta_n) = \eta_n^{\lambda_0} \sum\limits_{k=0}^{N} \eta_n^{-2(k+1)\gamma+1} U_{n,k}(\eta') , \\ \vec{U}'^{[N]}(\eta', \eta_n) = \eta_n^{\lambda_0} \sum\limits_{k=0}^{N} \eta_n^{-(2k+3)\gamma+1} \vec{U}_k'(\eta') , \qquad \gamma \neq n(n+1)^{-1}, \end{cases}$$

(or the corresponding partial sums from (4.10) if $\gamma = n(n+1)^{-1}$). We put

$$(4.11)\qquad \vec{u}^{[N]}(x) = \vec{U}^{[N]}(x' x_n^{\gamma-1}, x_n) , \qquad p^{[N]}(x) = P^{[N]}(x' x_n^{\gamma-1}, x_n) .$$

It is easy to see that $\vec{u}^{[N]}$, $p^{[N]}$ satisfy inequalities (2.31)-(2.34) and their discrepancy $\vec{H}^{[N]}$ in Stokes equations (1.2) obeys the estimates

$$(4.12)\qquad |D_x^\alpha H_j^{[N]}(x)| \leq c|F_i|\, x_n^{-(n+1+|\alpha|)(1-\gamma) - (2N+3+\alpha_n)\gamma} , \\ j = 1, \ldots, n-1 ,$$

$$(4.13)\qquad |D_x^\alpha H_n^{[N]}(x)| \leq c|F_i|\, x_n^{-(n+1+|\alpha|)(1-\gamma) - (2N+2+\alpha_n)\gamma} .$$

4.2. *The Stokes problem with the right-hand side $\vec{f}$, having the special series representation.* Let $\vec{\mathcal{F}}$ denotes the right–hand $\vec{f}$ of the Stokes system (1.2) in coordinates η. Assume that $\vec{\mathcal{F}}$ has the following form

$$
\left\{
\begin{aligned}
&\vec{\mathcal{F}}'(\eta) \sim \sum_{l=0}^{\infty} \eta_n^{\mu_* - \mu_l - 1 - \gamma} \sum_{j=0}^{\kappa_l} \vec{\mathcal{F}}_l^{\prime(j)}(\eta')(\ln \eta_n)^j, \\
&\mathcal{F}_n(\eta) \sim \sum_{l=0}^{\infty} \eta_n^{\mu_* - \mu_l - 1} \sum_{j=0}^{\kappa_l} \mathcal{F}_{n,l}^{(j)}(\eta')(\ln \eta_n)^j,
\end{aligned}
\right. \tag{4.14}
$$

where $\{\mu_l\}_{l=0}^{\infty}$ is an increasing sequence of nonnegative numbers, $\mu_0 = 0$, $\mu_l \to \infty$ as $l \to \infty$, $\vec{\mathcal{F}}_l$ are functions in $C^\infty(\omega)$.

According to (3.10) we denote by $\mathfrak{M}$ the countable set of numbers composed by the rule

$$\tau \in \{\lambda_0, \mu_* - \mu_k, l = 0, 1, \dots\} \Rightarrow \nu = \tau - 2\gamma \in \mathfrak{M}, \tag{4.15}$$

where

$$\lambda_0 = (n+1)\gamma - n. \tag{4.16}$$

We enumerate the numbers ν by decrease, i.e.

$$\nu_0 \geqslant \nu_1 \geqslant \dots \geqslant \nu_l \geqslant \dots. \tag{4.17}$$

The solution $(\vec{U}, P)$ can be found as the sums

$$
\left\{
\begin{aligned}
&P(\eta', \eta_n) \sim \sum_{k=0}^{\infty} \eta_n^{\nu_k} P_k(\ln \eta_n) + \sum_{k=0}^{\infty} \eta_n^{\nu_k - 2\gamma} Q_k(\eta', \ln \eta_n), \\
&\vec{U}'(\eta', \eta_n) \sim \sum_{k=0}^{\infty} \eta_n^{\nu_k + 1 - 3\gamma} \vec{U}_k'(\eta', \ln \eta_n), \\
&U_n(\eta', \eta_n) \sim \sum_{k=0}^{\infty} \eta_n^{\nu_k + 1 - 2\gamma} B_k(\ln \eta_n)\Phi(\eta') + \\
&\qquad\qquad + \sum_{k=0}^{\infty} \eta_n^{\nu_k + 1 - 2\gamma} U_{n,k}(\eta', \ln \eta_n).
\end{aligned}
\right. \tag{4.18}
$$

In (4.18) $P_k(\ln \eta_n)$, $Q_k(\eta', \ln \eta_n)$, $\vec{U}_k'(\eta', \ln \eta_n)$, $B_k(\ln \eta_n)$, $U_{n,k}(\eta', \ln \eta_n)$ are polynomials in $\ln \eta_n$ constructed in accordance with the scheme described in Section 3, i.e. the coefficiens of $P_k(\ln \eta_n)$, $B_k(\ln \eta_n)$ are constants and the coefficiens of $Q_k(\eta', \ln \eta_n)$, $\vec{U}_k'(\eta', \ln \eta_n)$, $U_{n,k}(\eta', \ln \eta_n)$ are smooth functions depending on η'. Degrees of these polynomials depend on the numbers κ_l (the degrees of the polynomials in (4.14)) and also of whether certain ν_k meet one of the conditions (3.11), (3.13), (3.20) or not. Notice that if

$$\mu_* < \lambda_0 ,$$

the numbers ν_k never meet (3.11), (3.13), (3.20). Let us put

$$\left\{\begin{aligned} &P^{[N]}(\eta', \eta_n) \sim \sum_{k=0}^{N} \eta_n^{\nu_k} P_k(\ln \eta_n) + \sum_{k=0}^{N} \eta_n^{\nu_k - 2\gamma} Q_k(\eta', \ln \eta_n), \\ &\vec{U}'^{[N]}(\eta', \eta_n) \sim \sum_{k=0}^{N} \eta_n^{\nu_k + 1 - 3\gamma} \vec{U}'_k(\eta', \ln \eta_n), \\ &U_n^{[N]}(\eta', \eta_n) \sim \sum_{k=0}^{N} \eta_n^{\nu_k + 1 - 2\gamma} B_k(\ln \eta_n)\, \Phi(\eta') + \\ &\qquad\qquad + \sum_{k=0}^{N} \eta_n^{\nu_k + 1 - 2\gamma} U_{n,k}(\eta', \ln \eta_n) \end{aligned}\right. \tag{4.19}$$

and

$$\vec{u}^{[N]}(x) = \vec{U}^{[N]}(x' x_n^{\gamma - 1}, x_n), \quad p^{[N]}(x) = P^{[N]}(x' x_n^{\gamma - 1}, x_n). \tag{4.20}$$

By using the formulas (3.34), it is easy to calculate that the discrepancies $\vec{H}'^{[N]}$, $H_n^{[N]}$ in the Stokes equations obey the estimates

$$|D_x^\alpha H_j^{[N]}(x)| \leq c|F_i| x_n^{\nu_N - 1 - |\alpha|(1-\gamma) - (3 + \alpha_n)\gamma + \varepsilon}, \qquad \varepsilon > 0, \qquad j = 1, \ldots, n-1, \tag{4.21}$$

$$|D_x^\alpha H_n^{[N]}(x)| \leq c|F_i| x_n^{\nu_N - 1 - |\alpha|(1-\gamma) - (2 + \alpha_n)\gamma + \varepsilon}, \qquad \varepsilon > 0. \tag{4.22}$$

REMARK 4.1. *If there is no dependence on* $\ln \eta_n$ *in* (4.14) *and all* $\nu \in \mathfrak{M}$ *satisfy* (3.7), *then also the coefficiens of the series* (4.18) *are independent of* $\ln \eta_n$.

4.3. *The Navier-Stokes problem.* We consider the problem (1.3), (1.4) with zero right-hand side $\vec{f}$. The main term of the asymptotic expansion of the solution $(\vec{u}, p)$ is the same as in the linear case (see Section 2). Let us consider the contribution of the nonlinear term $(\vec{u} \cdot \nabla)\, \vec{u}$. Passing to the coordinates $\{\eta\}$ we get

$$\vec{u}'(x) \cdot \nabla_{x'} + u_n(x)\, \partial_n = \\ = \eta_n^{\gamma - 1} \vec{u}'(\eta) \cdot \nabla' + u_n(\eta)(\partial_n + (\gamma - 1)\, \eta_n^{-1} \eta' \cdot \nabla'). \tag{4.23}$$

Let

$$P^{(\tau_i)}(\eta', \eta_n) = \eta_n^{\tau_i}(q_{\tau_i} + \eta_n^{-2\gamma} Q_{\tau_i}(\eta')),$$

$$\vec{U}'^{(\tau_i)}(\eta', \eta_n) = \eta_n^{\tau_i + 1 - 3\gamma} \vec{U}'_{\tau_i}(\eta'),\ U_n^{(\tau_i)}(\eta', \eta_n) = \eta_n^{\tau_i + 1 - 2\gamma} U_{n,\tau_i}(\eta'),$$

$$i = 1, 2.$$

Substituting these expressions into (4.23), we derive

$$(4.24)\quad \begin{cases} \eta_n^{\gamma - 1}(\vec{U}'^{(\tau_1)}(\eta)\cdot\nabla')\vec{U}'^{(\tau_2)}(\eta) + \\ \qquad + U_n^{(\tau_1)}(\eta)(\partial_n + (\gamma - 1)\eta_n^{-1}\eta'\cdot\nabla')\vec{U}'^{(\tau_2)}(\eta) \sim \\ \qquad\qquad \sim \eta_n^{\tau_1 + \tau_2 - 5\gamma + 1}\vec{\mathcal{F}}'(\eta') = \eta_n^{\Lambda - 1 - \gamma}\vec{\mathcal{F}}'(\eta'), \\ \eta_n^{\gamma - 1}(\vec{U}'^{(\tau_1)}(\eta)\cdot\nabla')\vec{U}_n^{(\tau_2)}(\eta) + \\ \qquad + U_n^{(\tau_1)}(\eta)(\partial_n + (\gamma - 1)\eta_n^{-1}\eta'\cdot\nabla')\vec{U}_n^{(\tau_2)}(\eta) \sim \\ \qquad\qquad \sim \eta_n^{\tau_1 + \tau_2 - 4\gamma + 1}F_n(\eta') = \eta_n^{\Lambda - 1}F_n(\eta'), \end{cases}$$

where

$$\Lambda = \tau_1 + \tau_2 + 2 - 4\gamma.$$

Let the solution $(\vec{u}, p)$ be represented in the form

$$P(\eta', \eta_n) \sim \sum_{\lambda \in \mathcal{M}} \eta_n^{\lambda}(q_\lambda + \eta_n^{-2\gamma} Q_\lambda(\eta')), \qquad q_\lambda = \text{const.},$$

$$\vec{U}'(\eta', \eta_n) \sim \sum_{\lambda \in \mathcal{M}} \eta_n^{\lambda + 1 - 3\gamma} \vec{U}'_\lambda(\eta'),$$

$$U_n(\eta', \eta_n) \sim \sum_{\lambda \in \mathcal{M}} \eta_n^{\lambda + 1 - 2\gamma} U_{n,\lambda}(\eta'),$$

where $\mathcal{M}$ is the certain set of numbers. From (4.24) we conclude, in addition to (3.10), the following rule for the elements of $\mathcal{M}$

$$(4.25)\qquad \tau_1, \tau_2 \in \mathcal{M} \Rightarrow \tau_1 + \tau_2 + 2 - 4\gamma \in \mathcal{M}.$$

Let us consider now separately the cases $n = 3$ and $n = 2$ and denote by $\mathcal{M}_3$ and $\mathcal{M}_2$ the corresponding most narrow sets of indices, satisfying (3.10), (4.25).

LEMMA 4.1.

$$\mathcal{M}_3 = \{4\gamma - 3 - 2k\gamma - l:\ k, l = 0, 1, \dots\},$$

$$\mathcal{M}_2 = \{3\gamma - 2 - k\gamma:\ k = 0, 1, \dots\}.$$

PROOF. The main term of the asymptotic representation for the pressure P starts in the three-dimensional case from the power $\lambda_0 = 4\gamma - 3$. Thus, due to (4.25), (3.10)

$$\lambda_0 + \lambda_0 + 2 - 4\gamma = \lambda_0 - 1 \in \mathfrak{M}_3 , \qquad \lambda_0 - 2\gamma \in \mathfrak{M}_3 .$$

It suffices to mention that $\mathfrak{M}_3$ satisfies (4.25), (3.10), since for

$$\nu = 4\gamma - 3 - 2k\gamma - l, \qquad \tau = 4\gamma - 3 - 2m\gamma - s$$

we have

$$\nu + \tau = 4\gamma - 2(k+m)\gamma - (l+s+1),$$

$$\nu - 2\gamma = 4\gamma - 3 - 2(k+1)\gamma - l .$$

In the two-dimensional case $\lambda_0 = 3\gamma - 2$ and

$$\lambda_0 + \lambda_0 + 2 - 4\gamma = 3\gamma - 2 - \gamma .$$

Taking into account that for

$$\nu = 3\gamma - 2 - k\gamma , \qquad \tau = 3\gamma - 2 - m\gamma$$

there hold the formulas

$$\nu + \tau + 2 - 4\gamma = 3\gamma - 2 - (k+m+1)\gamma ,$$

$$\nu - 2\gamma = 3\gamma - 2 - (k+2)\gamma ,$$

we conclude $\mathfrak{M}_2$ to be the exponent set in the 2D-case.

It is evident that, excepting $\nu = \lambda_0 = n(n+1)^{-1}$, the elements $\nu \in \mathfrak{M}_n$ do not meet the conditions (3.11), (3.13), (3.20). Hence, if $n = 3$ and

$$\gamma \neq 3/4 ,$$

the asymptotic representation for the solution $(\vec{U}, P)$ of the nonlinear problem (1.3), (1.4) has the form

$$(4.26) \quad \begin{cases} P(\eta', \eta_3) \sim \eta_3^{4\gamma-3} \sum\limits_{k,l=0}^{\infty} \eta_3^{-2k\gamma - l}(q_*^{(k,l)} + \eta_3^{-2\gamma} Q_{k,l}(\eta')) , \\ U_3(\eta', \eta_3) \sim \eta_3^{4\gamma-3} \sum\limits_{k,l=0}^{\infty} \eta_3^{-2(k+1)\gamma - l + 1}(a_{k,l}\Phi(\eta') + U_{3,k,l}(\eta')) , \\ \vec{U}'(\eta', \eta_3) \sim \eta_3^{4\gamma-3} \sum\limits_{k,l=0}^{\infty} \eta_3^{-(2k+3)\gamma - l + 1} \vec{U}'_{k,l}(\eta') , \end{cases}$$

where $q_*^{(k,l)}$, $a_{k,l}$ are constants, $q_*^{(0,0)} = F_i \kappa_0^{-1}(4\gamma - 3)$. If

$$\gamma = 3/4\,,$$

the representation for the solution is the following

$$(4.27)\quad \begin{cases} P(\eta', \eta_3) \sim q_*^{(0,0)} \ln \eta_3 + \eta_3^{-3/2} Q_{0,0}(\eta') + \\ \qquad\qquad + \sum\limits_{k,l=1}^{\infty} \eta_3^{-3k/2 - l}(q_*^{(k,l)} + \eta_3^{-3/2} Q_{k,l}(\eta'))\,, \\ U_3(\eta', \eta_3) \sim \sum\limits_{k,l=0}^{\infty} \eta_3^{-3(k+1)/2 - l + 1}(a_{k,l}\Phi(\eta') + U_{3,k,l}(\eta'))\,, \\ \vec{U}'(\eta', \eta_3) \sim \sum\limits_{k,l=0}^{\infty} \eta_3^{-3(2k+3)/4 - l + 1} \vec{U}'_{k,l}(\eta')\,, \end{cases}$$

where $q_*^{(0,0)} = F_i \kappa_0^{-1}$.

Let $n = 2$ and

$$\gamma \neq 2/3\,.$$

Then

$$(4.28)\quad \begin{cases} P(\eta_1, \eta_2) \sim \eta_2^{3\gamma - 2} \sum\limits_{k=0}^{\infty} \eta_2^{-k\gamma}(q_*^{(k)} + \eta_2^{-2\gamma} Q_k(\eta'))\,, \\ U_2(\eta_1, \eta_2) \sim \eta_2^{3\gamma - 2} \sum\limits_{k=0}^{\infty} \eta_2^{-(k+2)\gamma + 1}(a_k \Phi(\eta_1) + U_{2,k}(\eta_1))\,, \\ U_1(\eta_1, \eta_2) \sim \eta_2^{3\gamma - 2} \sum\limits_{k=0}^{\infty} \eta_2^{-(k+3)\gamma + 1} U_{1,k}(\eta_1)\,, \end{cases}$$

where $q_*^{(0)} = F_i \kappa_0^{-1}(3\gamma - 2)$. If

$$\gamma = 2/3\,,$$

we take

$$(4.29)\quad \begin{cases} P(\eta_1, \eta_2) \sim q_*^{(0)} \ln \eta_2 + \eta_2^{-4/3} Q_0(\eta_1) + \\ \qquad\qquad + \sum\limits_{k=1}^{\infty} \eta_2^{-2k/3}(q_*^{(k)} + \eta_2^{-4/3} Q_k(\eta_1))\,, \\ U_2(\eta_1, \eta_2) \sim \sum\limits_{k=0}^{\infty} \eta_2^{-2(k+2)/3 + 1}(a_k \Phi(\eta_1) + U_{2,k}(\eta_1))\,, \\ U_1(\eta_1, \eta_2) \sim \sum\limits_{k=0}^{\infty} \eta_2^{-2(k+3)/3 + 1} U_{1,k,l}(\eta_1) \end{cases}$$

and $q_*^{(0)} = F_i \kappa_0^{-1}$. Let

$$
(4.30)\quad \begin{cases} P^{[N,L]}(\eta', \eta_3) = \eta_3^{4\gamma-3} \sum_{k=0}^{N} \sum_{l=0}^{L} \eta_3^{-2k\gamma-l} (q_*^{(k,l)} + \eta_3^{-2\gamma} Q_{k,l}(\eta')), \\ U_3^{[N,L]}(\eta', \eta_3) = \\ \quad = \eta_3^{4\gamma-3} \sum_{k=0}^{N} \sum_{l=0}^{L} \eta_3^{-2(k+1)\gamma-l+1} (a_{k,l} \Phi(\eta') + U_{3,k,l}(\eta')), \\ \vec{U}'^{[N,L]}(\eta', \eta_3) = \eta_3^{4\gamma-3} \sum_{k=0}^{N} \sum_{l=0}^{L} \eta_3^{-(2k+3)n(n+1)^{-1}-l+1} \vec{U}'_{k,l}(\eta'), \end{cases}
$$

if $n = 3$, and

$$
(4.31)\quad \begin{cases} P^{[N]}(\eta_1, \eta_2) = \eta_2^{3\gamma-2} \sum_{k=0}^{N} \eta_2^{-k\gamma} (q_*^{(k)} + \eta_2^{-2\gamma} Q_k(\eta')), \\ U_2^{[N]}(\eta_1, \eta_2) = \eta_2^{3\gamma-2} \sum_{k=0}^{N} \eta_2^{-(k+2)\gamma+1} (a_k \Phi(\eta_1) + U_{2,k}(\eta_1)), \\ U_1^{[N]}(\eta_1, \eta_2) = \eta_2^{3\gamma-2} \sum_{k=0}^{N} \eta_2^{-(k+3)\gamma+1} U_{1,k}(\eta_1), \end{cases}
$$

if $n = 2$. In the cases $n = 3$, $\gamma = 3/4$ and $n = 2$, $\gamma = 2/3$, we take the corresponding partial sums from (4.27) and (4.29).

We put

$$
(4.32)\quad \begin{cases} \vec{u}^{[N,L]}(x) = \vec{U}^{[N,L]}(x' x_3^{\gamma-1}, x_3), \\ p^{[N,L]}(x) = P^{[N,L]}(x' x_3^{\gamma-1}, x_3). \end{cases}
$$

One can see that $\vec{u}^{[N,L]}$, $p^{[N,L]}$ satisfy inequalities (2.31)-(2.34) and their discrepancy $\vec{H}^{[N,L]}$ in the Navier-Stokes equations (1.3) obeys the estimates

$$
(4.33)\quad |D_x^\alpha H_j^{[N,L]}(x)| \leqslant c|F_i| x_3^{-(4+|\alpha|)(1-\gamma)-(2N+3+\alpha_n)\gamma-L}, \qquad j = 1, 2,
$$

$$
(4.34)\quad |D_x^\alpha H_3^{[N,L]}(x)| \leqslant c|F_i| x_n^{-(4+|\alpha|)(1-\gamma)-(2N+2+\alpha_n)\gamma-L}.
$$

Analogously, in the two-dimensional case we put

$$
(4.35)\quad \vec{u}^{[N]}(x) = \vec{U}^{[N]}(x_1 x_2^{\gamma-1}, x_2), \qquad p^{[N]}(x) = P^{[N]}(x_1 x_2^{\gamma-1}, x_2)
$$

and for the discrepancy $\vec{H}^{[N]}$ we derive the estimates

$$|D_x^\alpha H_1^{[N]}(x)| \leq c|F_i|x_2^{-(3+|\alpha|)(1-\gamma)-(N+1+\alpha_2)\gamma}, \tag{4.36}$$

$$|D_x^\alpha H_2^{[N]}(x)| \leq c|F_i|x_2^{-(3+|\alpha|)(1-\gamma)-(N+\alpha_2)\gamma}. \tag{4.37}$$

REMARK 4.2. *Using the above considerations one can construct also the asymptotic decomposition of the solution to the Navier-Stokes problem with the right-hand side $\vec{f}$, having the series representation* (4.14).

5. – Justification of asymptotic decompositions.

5.1. *Stokes and Navier–Stokes problems in weighted Hölder spaces.* For an arbitrary domain $\Omega \subset \mathbb{R}^n$ we denote by $C^{l,\delta}(\Omega)$, l being an integer, $0 < \delta < 1$, a Hölder space of continuous in $\overline{\Omega}$ functions u which have continuous derivatives $D^\alpha u = \partial^{|\alpha|} u/\partial x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, up to the order l and the finite norm

$$\|u;\ C^{l,\delta}(\Omega)\| = \sum_{|\alpha| \leq l} \sup\{|D^\alpha u(x)|\} + \sum_{|\alpha| = l} \sup\{[D^\alpha u]_\delta(x)\},$$

where the supremum is taken over $x \in \Omega$ and

$$[u]_\delta(x) = \sup_{\substack{0<|x-y|<|x|/2 \\ y\in\Omega}} \frac{|u(x)-u(y)|}{|x-y|^\delta}.$$

Let us consider now a domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, having m outlets to infinity, i.e. outside the sphere $|x| = R_0$ the domain Ω splits into m connected components Ω_i (outlets to infinity) which in some coordinate systems $x^{(i)}$ are given by the relation (1.7) with the function g_i satisfying (1.8), (1.9). Below we omit the index i in the notations for local coordinates.

In the domain Ω we introduce the weighted Hölder space $C^{l,\delta}_{\vec{æ}}(\Omega)$, consisting of functions u, continuously differentiable up to the order l in $\overline{\Omega}$, and having the finite norm

$$\|u;\ C^{l,\delta}_{\vec{æ}}(\Omega)\| =$$

$$= \|u;\ C^{l,\delta}(\Omega_{(k_0)})\| + \sum_{i=1}^m \sum_{|\alpha| \leq l} \sup_{x\in\Omega_i}\{g_i(x_n)^{æ_i - l - \delta + |\alpha|}\,|D^\alpha u(x)|\} +$$

$$+ \sum_{i=1}^m \sum_{|\alpha| = l} \sup_{x\in\Omega_i}\{g_i(x_n)^{æ_i}[D^\alpha u]_\delta(x)\}.$$

Here $\vec{æ} = (æ_1, \dots, æ_m)$ and $\Omega_{(k_0)} = \{x \in \Omega\colon\ |x| < k_0\}$.

The solvability of the Stokes (1.2), (1.4) and Navier-Stokes (1.3), (1.4) problems in weighted function spaces has been studied in [23], [24], [25]. Here, for the justification of the obtained asymptotic decompositions, we need the following theorems.

THEOREM 5.1 [24]. *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a domain with $m \geqslant 1$ outlets to infinity, $\partial\Omega \in C^{l+2,\delta}$, $\vec{f} \in C^{l,\delta}_{\vec{æ}}(\Omega)$, where $l \geqslant 0$, $\delta \in (0, 1)$ and $\vec{æ}$ is an arbitrary vector. Then there exists a unique solution $(\vec{u}, p)$ of the linear Stokes problem* (1.2), (1.4) *with zero fluxes* ($F_i = 0$, $i = 1, \dots, m$) *such that $\vec{u} \in C^{l+2,\delta}_{\vec{æ}}(\Omega)$, $\nabla p \in C^{l,\delta}_{\vec{æ}}(\Omega)$ and there holds the estimate*

$$\|\vec{u};\ C^{l+2,\delta}_{\vec{æ}}(\Omega)\| + \|\nabla p;\ C^{l,\delta}_{\vec{æ}}(\Omega)\| \leqslant c\|\vec{f};\ C^{l,\delta}_{\vec{æ}}(\Omega)\|. \tag{5.1}$$

In particular, from (5.1) *it follows that*

$$|D^\alpha \vec{u}(x)| \leqslant c\|\vec{f};\ C^{l,\delta}_{\vec{æ}}(\Omega)\| g_i(x_n)^{-æ_i+l+2+\delta-|\alpha|}, \qquad x \in \Omega_i, \tag{5.2}$$

$$|D^\alpha \nabla p(x)| \leqslant c\|\vec{f};\ C^{l,\delta}_{\vec{æ}}(\Omega)\|\ \ g_i(x_n)^{-æ_i+l+\delta-|\alpha|}, \qquad x \in \Omega_i, \tag{5.3}$$

while $0 \leqslant |\alpha| \leqslant l+2$ *in* (5.2) *and* $0 \leqslant |\alpha| \leqslant l$ *in* (5.3).

THEOREM 5.2 [25]. *Let $\Omega \subset \mathbb{R}^3$ be a domain with $m \geqslant 1$ outlets to infinity. Assume that, in addition to* (1.8), (1.9), *the functions g_i satisfy the conditions* (1.10), (1.11). *Let $\partial\Omega \in C^{l+2,\delta}$, $l \geqslant 0$, $0 < \delta < 1$, $\vec{f} = 0$,*

$$æ_i = n + 1 + l + \delta\,, \qquad i = 1, \dots, m\,. \tag{5.4}$$

Then for arbitrary fluxes F_i, $i = 1, \dots, m$, there exists a solution $(\vec{u}, p)$ of the Navier-Stokes problem (1.3), (1.4), *admiting the estimates*

$$\|\vec{u};\ C^{l+2,\delta}_{\vec{æ}}(\Omega)\| + \|\nabla p;\ C^{l,\delta}_{\vec{æ}}(\Omega)\| \leqslant C(|\vec{F}|), \tag{5.5}$$

$$|p(x)| \leqslant C(|\vec{F}|)\int_0^{x_3} g_i(t)^{-4}\,dt + c_1, \qquad x \in \Omega_i\,, \tag{5.6}$$

$|\vec{F}| = (\sum_i^m F_i^2)^{1/2}$. *For small $|\vec{F}|$ the solution $(\vec{u}, p)$ is unique.*

REMARK 5.1. *Theorem 5.2 is also valid for nonzero right-hand sides $\vec{f}$ having an appropriate decay at infinity.*

5.2. *Estimates of the remainder in asymptotic formulas; Stokes problem.* Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a domain with $m \geqslant 1$ outlets to infinity Ω_i of the form (1.1). Assume that $\partial\Omega \in C^{l+2,\delta}$, $l \geqslant 0$, $\delta \in (0, 1)$, and denote by ζ_j the smooth cut-off functions equal to 1 in $\Omega_j \setminus \Omega_{k_0+1}$ and equal to 0 in $\Omega \setminus (\Omega_j \setminus \Omega_{k_0})$. We specify the spaces $C^{l,\delta}_{\vec{æ}}(\Omega)$, by taking $g_i(t) = t^{1-\gamma_i}$ in the definition of the norm $\|\cdot;\ C^{l,\delta}_{\vec{æ}}(\Omega)\|$.

THEOREM 5.3. (i) *Let* $\vec{f} \in C^{l,\delta}_{\vec{æ}}(\Omega)$ *with*

$$\text{(5.7)} \qquad æ_i = n + 1 + l + \delta + 2(N+1)\gamma_i(1-\gamma_i)^{-1}, \qquad i = 1, \dots, m.$$

Then there exists a unique solution $(\vec{u}, p)$ *of the Stokes problem* (1.2), (1.4) *with*

$$\text{(5.8)} \qquad \vec{u} \in C^{l+2,\delta}_{\vec{æ}}(\Omega), \qquad \nabla p \in C^{l,\delta}_{\vec{æ}}(\Omega),$$

$$æ_i = n + 1 + l + \delta, \quad i = 1, \dots, m.$$

The solution $(\vec{u}, p)$ *admits the asymptotic representation*

$$\text{(5.9)} \qquad \vec{u} = \sum_{i=1}^{m} \zeta_i \vec{u}_i^{[N]} + \vec{v}, \qquad p = \sum_{i=1}^{m} \zeta_i p_i^{[N]} + q,$$

where $(\vec{u}_i^{[N]}, p_i^{[N]})$ *are the partial sums* (4.11) *constructed for the outlet to infinity* Ω_i, $\vec{v} \in C^{l+2,\delta}_{\vec{æ}}(\Omega)$, $\nabla q \in C^{l,\delta}_{\vec{æ}}(\Omega)$ *and there holds the estimate*

$$\text{(5.10)} \quad \|\vec{v};\ C^{l+2,\delta}_{\vec{æ}}(\Omega)\| + \|\nabla q;\ C^{l,\delta}_{\vec{æ}}(\Omega)\| \leqslant c\left(\sum_{i=1}^{m} |F_i| + \|\vec{f};\ C^{l,\delta}_{\vec{æ}}(\Omega)\|\right).$$

(ii) *Assume that in each* Ω_i *the right-hand side* $\vec{f} = (\vec{f}', f_n)$ *can be represented as a sum*

$$\text{(5.11)} \quad \begin{cases} \vec{f}'(x) = \\ = \zeta_i(x) \displaystyle\sum_{l=0}^{N} x_n^{\mu_*^{(i)} - \mu_l^{(i)} - 1 - \gamma_i} \sum_{j=0}^{\kappa_l^{(i)}} \vec{f}_l^{(j)\prime}(x' x_n^{\gamma_i - 1})(\ln x_n)^j + \vec{f}^{(*)\prime}(x), \\ \hfill x \in \Omega_i, \\ f_n(x) = \\ = \zeta_i(x) \displaystyle\sum_{l=0}^{N} x_n^{\mu_*^{(i)} - \mu_l^{(i)} - 1} \sum_{j=0}^{\kappa_l^{(i)}} f_{n,l}^{(j)}(x' x_n^{\gamma_i - 1})(\ln x_n)^j + f_n^{(*)}(x), \\ \hfill x \in \Omega_i, \end{cases}$$

where

$$(5.12)\qquad \mu^{*(i)} < 1-(n+1)(1-\gamma_i), \qquad i=1,\dots,m,$$

$\{\mu_l^{(i)}\}_{l=0}^{\infty}$ *is an increasing sequence of nonnegative numbers,* $\mu_0^{(i)}=0$, $\mu_l^{(i)} \to \infty$ *as* $l\to\infty$, $\vec{f}_l^{(j)}$ *are smooth functions and* $\vec{f}^{(*)} = (\vec{f}^{(*)\prime}, f_n^{(*)}) \in C_{æ^*}^{l,\delta}(\Omega)$ *with*

$$(5.13)\qquad æ_i^* = l+\delta+(1+2\gamma_i-\varepsilon-\nu_N^{(i)})(1-\gamma_i)^{-1},$$

$$\varepsilon>0, \quad i=1,\dots,m,$$

where $\nu_l^{(i)} \to -\infty$ *as* $l\to\infty$ *are the numbers defined by* (4.15), (4.17). *Then there exists a unique solution* $(\vec{u}, p)$ *of the Stokes problem* (1.2), (1.4), *satisfying the inclusions* (5.8) *and the representation* (5.9) *with* $(\vec{u}_i^{[N]}, p_i^{[N]})$ *being the partial sums* (4.20) *and* $\vec{v}\in C_{æ^*}^{l+2,\delta}(\Omega)$, $\nabla q \in C_{æ^*}^{l,\delta}(\Omega)$. *There holds the estimate*

$$(5.14)\qquad \|\vec{v}; C_{æ^*}^{l+2,\delta}(\Omega)\| + \|\nabla q; C_{æ^*}^{l,\delta}(\Omega)\| \le c\left(\sum_{i=1}^m |F_i| + \|\vec{f}; C_{æ^*}^{l,\delta}(\Omega)\|\right).$$

PROOF. The solvability of the problem (1.2), (1.4) follows from Theorem 5.1. In fact, if we represent the velocity field $\vec{u}$ in the form $\vec{u} = \vec{A}+\vec{w}$, where $\vec{A}$ is the divergence free vector field satisfying the inequalities (1.6), we get for $(\vec{w}, p)$ the same problem with zero fluxes and the new right-hand side equal to $\vec{f}+\nu\Delta\vec{A}$. It is easy to verify that $\vec{f}+\nu\Delta\vec{A}\in C_{æ}^{l,\delta}(\Omega)$ and, thus, according to Theorem 5.1 there exists a solution $(\vec{w}, p)$ with $\vec{w}\in C_{æ}^{l+2,\delta}(\Omega)$, $\nabla p\in C_{æ}^{l,\delta}(\Omega)$. Since $\vec{A}\in C_{æ}^{l+2,\delta}(\Omega)$, we also have $\vec{u}\in C_{æ}^{l+2,\delta}(\Omega)$.

Let us represent the solution $(\vec{u}, p)$ in the form

$$(5.15)\qquad \vec{u} = \sum_{i=1}^m \zeta_i \vec{u}_i^{[N]} + \vec{W}^{[N]} + \vec{V}, \qquad p = \sum_{i=1}^m \zeta_i p_i^{[N]} + q,$$

where $(\vec{u}_i^{[N]}, p_i^{[N]})$ are either the functions (4.11) in the case (i), or the functions (4.20) in the case (ii), and $\vec{W}^{[N]}$ is a solution of the equation

$$(5.16)\qquad \begin{cases} \operatorname{div} \vec{W}^{[N]} = -\sum_{i=1}^m \nabla\zeta_i\cdot\vec{u}_i^{[N]} & \text{in } \Omega_{(k_0+2)}, \\ \vec{W}^{[N]} = 0 \quad \text{on } \partial\Omega_{(k_0+2)}. \end{cases}$$

We have

$$\operatorname{supp}\left(\sum_{i=1}^{m} \nabla\zeta_i \cdot \vec{u}_i^{[N]}\right) \subset \Omega_{(k_0+1)}$$

and the condition

$$\sum_{i=1}^{m} F_i = 0$$

yields

$$\int_{\Omega_{(k_0+1)}} \sum_{i=1}^{m} \nabla\zeta_i \cdot \vec{u}_i^{[N]} dx = 0 .$$

Thus (see [6]), (5.16) has a solution $\vec{W}^{[N]} \in C^{l+2,\delta}(\Omega_{(k_0+2)})$ with supp $\vec{W}^{[N]} \subset \Omega_{(k_0+3/2)}$. Without loss of generality we assume that $\vec{W}^{[N]}$ is extended by zero to $\Omega \backslash \Omega_{(k_0+2)}$. The function $\vec{V}$ is solenoidal and satisfies together with q equations (1.2), (1.4) with $F_i = 0$, $i = 1, \ldots, m$ and the right-hand side

$$\vec{\tilde{f}} = \begin{cases} \nu\Delta\left(\sum_{i=1}^{m} \zeta_i \vec{u}_i^{[N]}\right) - \nabla\left(\sum_{i=1}^{m} \zeta_i p_i^{[N]}\right) + \nu\Delta \vec{W}^{[N]} + \vec{f}, & \text{case (i)}, \\ \nu\Delta\left(\sum_{i=1}^{m} \zeta_i \vec{u}_i^{[N]}\right) - \nabla\left(\sum_{i=1}^{m} \zeta_i p_i^{[N]}\right) + \nu\Delta \vec{W}^{[N]} + \vec{f}^{(*)}, & \text{case (ii)}, \end{cases}$$

which belongs in the case (i) to the space $C^{l,\delta}_{\vec{æ}}(\Omega)$ (see (4.12), (4.13), (5.7)) and in the case (ii) to the space $C^{l,\delta}_{\vec{æ}^*}(\Omega)$ (see (4.21), (4.22), (5.13)). Applying Theorem 5.1 and taking $\vec{v} = \vec{W}^{[N]} + \vec{V}$, we conclude the proof of the theorem.

REMARK. 5.2. *In particular, from* (5.10), (5.14) *there follow the pointwise estimates for the remainder* $(\vec{v}, q)$:

$$|D_x^{\alpha} \vec{v}(x)| \leq c\left(\sum_{i=1}^{m} |F_i| + \|\vec{f};\ C^{l,\delta}_{\vec{æ}}(\Omega)\|\right) x_n^{-(n-1+|\alpha|)(1-\gamma_i)-(2N+2)\gamma_i},$$

$$x \in \Omega_i ,$$

$$|D_x^{\alpha} \nabla q(x)| \leq c\left(\sum_{i=1}^{m} |F_i| + \|\vec{f};\ C^{l,\delta}_{\vec{æ}}(\Omega)\|\right) x_n^{-(n+1+|\alpha|)(1-\gamma_i)-(2N+2)\gamma_i},$$

$$x \in \Omega_i ,$$

in the case (i), *and*

$$|D_x^\alpha \vec{v}(x)| \leqslant c\left(\sum_{i=1}^{m} |F_i| + \|\vec{f}^{(*)};\ C^{l,\delta}_{\vec{æ}^*}(\Omega)\|\right) x_n^{-(|\alpha|-2)(1-\gamma_i)+\nu_N^{(i)}-2\gamma_i-1+\varepsilon},$$

$$x \in \Omega_i\,,$$

$$|D_x^\alpha \nabla q(x)| \leqslant c\left(\sum_{i=1}^{m} |F_i| + \|\vec{f}^{(*)};\ C^{l,\delta}_{\vec{æ}^*}(\Omega)\|\right) x_n^{-|\alpha|(1-\gamma_i)+\nu_N^{(i)}-2\gamma_i-1+\varepsilon},$$

$$x \in \Omega_i\,,$$

in the case (ii). *Notice that the condition* (5.12) *implies* $2(1-\gamma_i) + \nu_0^{(i)} - 2\gamma_i - 1 + \varepsilon < -(n-1)(1-\gamma_i)$ *and, therefore, also in the case* (ii) *we have got the improved decay estimates for the remainder* $(\vec{v}, q)$ (*comparing with the estimates for* $(\vec{u}, p)$).

REMARK 5.3. *Applying the results from* [23], *it is also possible to obtain the estimates of the remainder* $(\vec{v}, q)$ *in weighted Sobolev spaces* $V^{l,\vec{s}}_{æ}(\Omega)$ *with the norm*

$$\|\vec{u};\ V^{l,\vec{s}}_{æ}(\Omega)\| = \sum_{|\alpha|=0}^{l} \|D^\alpha \vec{u};\ L^{\vec{s}}_{æ+|\alpha|-l}(\Omega)\|,$$

where

$$\|\vec{u};\ L^{\vec{s}}_{\vec{æ}}(\Omega)\| = \left(\int_{\Omega_{(k_0+1)}} |\vec{u}|^{s_0} dx\right)^{1/s_0} + \sum_{i=1}^{m}\left(\int_{\Omega_i\setminus\Omega_{(k_0)}} x_n^{s_i æ_i(1-\gamma_i)} |\vec{u}|^{s_i} dx\right)^{1/s_i}.$$

For example, let there exist numbers $\tilde{s}_i^* = \tilde{s}_i^*(N) > 1$, $i = 1, \ldots, m$, *such that*

$$\int_1^\infty t^{-\tilde{s}_i^*[n(1-\gamma_i)+2\gamma_i(N+1)]+(n-1)(1-\gamma_i)} dt < \infty\,.$$

Suppose that in the case (i) $\vec{f} \in V^{l,\vec{s}}_{\vec{æ}^*}(\Omega)$ *with* $l \geqslant -1$, $s_i > 1$ *and* $\vec{æ}^*$ *is defined by*

$$\tilde{æ}_i^* = \tilde{æ}_i^*(N) = l + n + 1 - \frac{n\tilde{s}_i^*}{s_i} - \frac{(\tilde{s}_i^* - s_i)2(N+1)\gamma_i}{s_i(1-\gamma_i)},$$

then

$$\vec{v} \in V^{l+2,\vec{s}}_{\tilde{æ}^*}(\Omega), \qquad \nabla q \in V^{l,\vec{s}}_{\vec{æ}}(\Omega).$$

5.3. *Estimates of the remainder in asymptotic formulas; Navier-Stokes problem.* According to Theorem 5.2, the solvability of the Navier-Stokes problem (1.3), (1.4) is proved for arbitrary large data only for three-dimensional domains Ω under the additional conditions (1.10), (1.11). For $g_i(t) = g_0 t^{1-\gamma_i}$ (1.10), (1.11) mean $1/4 < \gamma_i < 1$. If $\Omega \subset \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$ and $0 < \gamma_i \leqslant 1/4$, the existens results are known only for small data (see [25]). We start with the justification of the asymptotic representation for the solution $(\vec{u}, p)$ of (1.3), (1.4) in the case of small data without any additional assumptions on γ_i.

THEOREM 5.4. *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a domain with $m \geqslant 1$ outlets to infinity Ω_i of the form* (1.1) *and let $\vec{f} \in C^{l,\delta}_{\vec{æ}^*}(\Omega)$ with*

$$(5.17)\qquad æ_i^* = æ_i^*(N, L) =$$

$$= 4 + l + \delta + 2(N+1)\gamma_i(1-\gamma_i)^{-1} - L(1-\gamma_i)^{-1}, \qquad n = 3,$$

$$(5.18)\qquad æ_i = æ_i^*(N) = 3 + l + \delta + N\gamma_i(1-\gamma_i)^{-1}, \qquad n = 2,$$

$i = 1, \dots, m$. Then for sufficiently small $|\vec{F}|$ and $\|\vec{f};\, C^{l,\delta}_{\vec{æ}^*}(\Omega)\|$ the Navier-Stokes problem (1.3), (1.4) *has a unique solution $(\vec{u}, p)$ satisfying the asymptotic representation*

$$(5.19)\qquad \vec{u} = \sum_{i=1}^{m} \zeta_i \vec{u}_i^{[N,L]} + \vec{v}, \qquad p = \sum_{i=1}^{m} \zeta_i p_i^{[N,L]} + q, \qquad n = 3,$$

or

$$(5.20)\qquad \vec{u} = \sum_{i=1}^{m} \zeta_i \vec{u}_i^{[N]} + \vec{v}, \quad p = \sum_{i=1}^{m} \zeta_i p_i^{[N]} + q, \quad n = 2,$$

where $(\vec{u}_i^{[N,L]}, p_i^{[N,L]})$ are defined by (4.32), *$(\vec{u}_i^{[N]}, p_i^{[N]})$ by* (4.35), *$\vec{v} \in \in C^{l+2,\delta}_{\vec{æ}^*}(\Omega)$, $\nabla q \in C^{l,\delta}_{\vec{æ}^*}(\Omega)$ and there holds the estimate*

$$(5.21)\qquad \|\vec{v};\, C^{l+2,\delta}_{\vec{æ}^*}(\Omega)\| + \|\nabla q;\, C^{l,\delta}_{\vec{æ}^*}(\Omega)\| \leqslant c(|\vec{F}|) + c\|\vec{f};\, C^{l,\delta}_{\vec{æ}^*}(\Omega)\|.$$

PROOF. We prove the theorem in the case $n = 3$. For the two-dimensional case the proof is completely analogous. We look for the solution $(\vec{u}, p)$ in the form

$$(5.22)\quad \vec{u} = \sum_{i=1}^{m} \zeta_i \vec{u}_i^{[N,L]} + \vec{W}^{[N,L]} + \vec{V}, \qquad p = \sum_{i=1}^{m} \zeta_i p_i^{[N,L]} + q,$$

where $\vec{W}^{[N,L]}$ is the solution of the divergence equation (5.16). Then for

$(\vec{V}, q)$ we derive the problem

$$(5.23)\quad \begin{cases} -\nu\Delta \vec{V} + \nabla q = \vec{f} + \nu\Delta \vec{w}^{[N,L]} - (\vec{w}^{[N,L]}\cdot\nabla)\,\vec{w}^{[N,L]} - \nabla Q^{[N,L]} - \\ \qquad -(\vec{V}\cdot\nabla)\,\vec{V} - (\vec{w}^{[N,L]}\cdot\nabla)\,\vec{V} - (\vec{V}\cdot\nabla)\,\vec{w}^{[N,L]} \quad \text{in } \Omega, \\ \operatorname{div} \vec{V} = 0, \quad \text{in } \Omega, \\ \vec{V} = 0 \quad \text{on } \partial\Omega, \\ \int_{\sigma_i} \vec{V}\cdot\vec{n}\, ds = 0, \quad i = 1, \dots, m, \end{cases}$$

where

$$\vec{w}^{[N,L]} = \sum_{i=1}^{m} \zeta_i \vec{u}_i^{[N,L]} + \vec{W}^{[N,L]}, \qquad Q^{[N,L]} = \sum_{i=1}^{m} \zeta_i p_i^{[N,L]}.$$

Denote

$$M\vec{V} := \vec{f} + \nu\Delta \vec{w}^{[N,L]} - (\vec{w}^{[N,L]}\cdot\nabla)\,\vec{w}^{[N,L]} - \nabla Q^{[N,L]} - \\ -(\vec{V}\cdot\nabla)\,\vec{V} - (\vec{w}^{[N,L]}\cdot\nabla)\,\vec{V} - (\vec{V}\cdot\nabla)\,\vec{w}^{[N,L]}.$$

Let $\vec{V} \in C^{l+2,\delta}_{æ^*}(\Omega)$ with $æ^*$ defined by (5.17). By using the estimates (4.33), (4.34) it is easy to verify that $M\vec{V} \in C^{l,\delta}_{æ^*}(\Omega)$. Thus, the problem (5.23) is equivalent to an operator equation in the space $C^{l+2,\delta}_{æ^*}(\Omega)$:

$$\vec{V} = \mathcal{A}\vec{V},$$

where $\mathcal{A}\vec{V} = \mathcal{L}^{-1}M\vec{V}$ and $\mathcal{L}$ is the operator of the linear Stokes problem (1.2), (1.4) with zero fluxes. In virtue of Theorem 5.1 the inverse operator $\mathcal{L}^{-1}: C^{l,\delta}_{æ^*}(\Omega) \to C^{l+2,\delta}_{æ^*}(\Omega)$ is bounded. The direct computations show that

$$\|\mathcal{A}\vec{V};\ C^{l+2,\delta}_{æ^*}(\Omega)\| \leq c_\star\Big(C(|\vec{F}|) + \|\vec{f};\ C^{l,\delta}_{æ^*}(\Omega)\| + \|\vec{V};\ C^{l+2,\delta}_{æ^*}(\Omega)\|^2 + \\ + C(|\vec{F}|)\|\vec{V};\ C^{l+2,\delta}_{æ^*}(\Omega)\|\Big),$$

$$\|\mathcal{A}\vec{V}^{(1)} - \mathcal{A}\vec{V}^{(1)};\ C^{l+2,\delta}_{æ^*}(\Omega)\| \leq c_{\star\star}\big((\|\vec{V}^{(1)};\ C^{l+2,\delta}_{æ^*}(\Omega)\| + \\ + \|\vec{V}^{(2)};\ C^{l+2,\delta}_{æ^*}(\Omega)\|)\|\vec{V}^{(1)} - \vec{V}^{(2)};\ C^{l+2,\delta}_{æ^*}(\Omega)\| + \\ + C(|\vec{F}|)\|\vec{V}^{(1)} - \vec{V}^{(2)};\ C^{l+2,\delta}_{æ^*}(\Omega)\|\big),$$

where $C(|\vec{F}|) \to 0$ as $|\vec{F}| \to 0$. Hence, for sufficiently small $|F_i|$, $i = 1, \ldots, m$, and $\|\vec{f}; C^{l,\delta}_{æ^*}(\Omega)\|$ the operator $\mathfrak{A}$ is a contraction in a small ball of the space $C^{l+2,\delta}_{æ^*}(\Omega)$ and the theorem follows from the Banach contraction principle.

Let us consider now the Navier-Stokes problem (1.3), (1.4) for arbitrary large data in the case of three–dimensional domains Ω, satisfying the additional condition

$$(5.24) \qquad 1/4 < \gamma_i < 1, \qquad i = 1, \ldots, m.$$

THEOREM 5.5. *Let $\Omega \subset \mathbb{R}^3$ be a domain with $m \geqslant 1$ outlets to infinity Ω_i of the form* (1.1) *and let $\vec{f} = 0$. Assume additionaly that* (5.24) *holds and let $(\vec{u}, p)$ be the solution to* (1.3), (1.4) *from Theorem* 5.2. *Then in each outlet to infinity Ω_i the solution $(\vec{u}, p)$ admits the asymptotic expansion* (5.19) *with $\vec{v} \in C^{l+2,\delta}_{æ^*}(\Omega)$, $\nabla q \in C^{l,\delta}_{æ^*}(\Omega)$, where $æ^*$ is defined by* (5. 17)). *Moreover, there holds the estimate*

$$(5.25) \qquad \|\vec{v};\ C^{l+2,\delta}_{æ^*}(\Omega)\| + \|\nabla q;\ C^{l,\delta}_{æ^*}(\Omega)\| \leqslant c(|\vec{F}|).$$

PROOF. Because of (5.24) the conditions of Theorem 5.2 are satisfied and there exists a solution $(\vec{u}, p)$ of (1.3), (1.4) with $\vec{u} \in C^{l+2,\delta}_{æ}(\Omega)$, $\nabla p \in C^{l,\delta}_{æ^*}(\Omega)$ $(æ_i = 4 + l + \delta)$. Moreover, $(\vec{u}, p)$ satisfies the estimate (5.5). In particular, from (5.5) follows that

$$(5.26) \qquad |D^\alpha \vec{u}(x)| \leqslant c(|\vec{F}|)\, x_3^{-(2+|\alpha|)(1-\gamma_i)}, \qquad x \in \Omega_i, \quad |\alpha| \geqslant 0.$$

By the construction (see Section 4.3) the same estimate is true for the function $\vec{u}_i^{[N,L]}$. Let us represent the solution $(\vec{u}, p)$ in the form (5.22). For the remainder $(\vec{V}, q)$ we obtain the problem (5.23). By using (5.26), it is easy to verify that

$$(5.27) \qquad |D^\alpha((\vec{w}^{[N,L]}\cdot\nabla)\,\vec{w}^{[N,L]} +$$

$$+(\vec{V}\cdot\nabla)\,\vec{V} + (\vec{w}^{[N,L]}\cdot\nabla)\,\vec{V} + (\vec{V}\cdot\nabla)\,\vec{w}^{[N,L]})| \leqslant c(|\vec{F}|)x_3^{-(5+|\alpha|)(1-\gamma_i)},$$

$$x \in \Omega_i, \qquad |\alpha| \geqslant 0.$$

In Section 4.3 we have proved that the discrepency $\vec{H}_i^{[N,L]}(x) = \nu\Delta\,\vec{u}_i^{[N,L]} - (\vec{u}_i^{[N,L]}\cdot\nabla)\,\vec{u}_i^{[N,L]} - \nabla p_i^{[N,L]}$ satisfies the relations (4.33), (4.34), i.e.

$$(5.28) \qquad |D_x^\alpha \vec{H}_i^{[N,L]}(x)| \leq c(|\vec{F}|)\, x_3^{-(4+|\alpha|)(1-\gamma_i)-2(N+1)\gamma_i - L},$$

$$x \in \Omega_i, \quad |\alpha| \geq 0.$$

From (5.27), (5.28) it follows that the right-hand side $M\vec{V}$ of the problem (5.23) belongs to the space $C^{l,\delta}_{æ^{(1)}}(\Omega)$ with

$$æ_i^{(1)} = 4 + l + \delta + \min\{1, (2(N+1)\gamma_i + L)(1-\gamma_i)^{-1}\}, \qquad i = 1, \ldots, m.$$

We consider the solution $(\vec{V}, q)$ of (5.23) as a solution of the linear Stokes problem (1.2), (1.4). Applying to $(\vec{V}, q)$ Theorem 5.1, we obtain $\vec{V} \in C^{l+2,\delta}_{æ^{(1)}}(\Omega)$, $\nabla q \in C^{l,\delta}_{æ^{(1)}}(\Omega)$ and the estimate (5.25) with $æ^*$ changed to $æ^{(1)}$. Since $æ_i^{(1)} > æ_i$, we can repeate the above arguments. After the finite number of steps we derive $\vec{V} \in C^{l+2,\delta}_{æ^*}(\Omega)$, $\nabla q \in C^{l,\delta}_{æ^*}(\Omega)$ and the estimate (5.25). The theorem is proved.

REMARK 5.4. *Theorem* 5.5 *remains valid if the right-hand side* $\vec{f} \in C^{l,\delta}_{æ^*}(\Omega)$ *with* $æ^*$ *defined by* (5.17).

REMARK 5.5. *In the same way the asymptotics of the solutions to the Stokes problem can be investigated near the singularity point of the boundary of the peak type, i.e. if* $0 \in \partial\Omega$ *and in the neibourhood of* 0 *the boundary* $\partial\Omega$ *can be represented in the form* $\{x: |x'| < g(x_n), x_n \in (0, \delta)\}$ *with* $\lim_{x_n \to 0} g(x_n) = 0$ *and* $\lim_{x_n \to 0} g'(x_n) = 0$. *Assuming that the right-hand side* $\vec{f}$ *has the series representation in powers of* $x_n = \eta_n$, *one can construct and justify the asymptotics of the solutions just by repeating word by word the above arguments* (*even with some simplifications*).

REMARK 5.6. *Finally, we mention that, of course, all results of the paper remain valid in domains* Ω *having the outlets to infinity* Ω_i *with noncircular sections, i.e. for* Ω_i *given by the relations*

$$\Omega_i = \{x \in \mathbb{R}^n: x_n^{\gamma-1} x' \in S_i, x_n > 0\},$$

where S_i *is an arbitrary bounded domain in* $\mathbb{R}^{n-1}$. *One can see that we did not use in the proofs the assumption that* Ω_i *has a circular cross-section. The same is true for the context of Remark* 5.5. *All the formal calculations in these cases can be taken from* [20].

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