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$PSp_4(3)$ as a Symmetric (36, 15, 6)-Design.

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ABSTRACT - In this paper we present a description of the symmetric design with parameters (36, 15, 6) on which the symplectic group $PSp_4(3)$ acts transitively. In particular we give a group theoretical approach to such a design.

1. Introduction and preliminary results.

Let G be the symplectic group $PSp_4(3)$ of order 25,920. It is our objective to prove that G can be viewed as a symmetric (36, 15, 6)-design.

We state some facts about G which can be found in [2] and [1]. With the notation in [2][Lemma 8] we have

LEMMA 1. (i) *The group G contains precisely four conjugacy classes of elements of order* 3 *with representatives* σ_1, σ_1^{-1}, $\varrho = \sigma_1 \cdot \sigma_2$ *and* $\sigma_1 \cdot \sigma_2^{-1}$. *We have* $|C_G(\sigma_1)| = |C_G(\sigma_1^{-1})| = 81 \cdot 8$, $|C_G(\varrho)| = 27 \cdot 4$, *and* $|C_G(\sigma_1 \cdot \sigma_2^{-1})| = 27 \cdot 2$. *A Sylow* 2*-subgroup of* $C_G(\sigma_1)$ *is a quaternion group, and a Sylow* 2*-subgroup of* $C_G(\varrho)$ *is a four-group.*

(ii) *Elements of order* 9 *in G are roots of* 3*-central elements of order* 3 *in G.*

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(iii) *A Sylow* 5*-normalizer in* G *is a Frobenius group of order* 20.

(iv) G *contains a maximal subgroup* S *isomorphic to* Σ_6. *The normalizer of* S *in* $\mathrm{Aut}(G)$ *is isomorphic to* $\Sigma_6 \times Z_2$.

Since we are interested in a transitive action of G on 36 objects we will have a closer look to subgroups of G which are isomorphic to Σ_6.

LEMMA 2. *Let* S *be a maximal subgroup of* G *isomorphic to* Σ_6. *Let* $R \neq S$ *be a subgroup of* G *which is conjugate to* S *in* G. *Then,*

(i) *the index of* S *in* G *is equal to* 36,

(ii) $R \cap S$ *is isomorphic to either* $Z_2 \times \Sigma_4$ *or* $\Sigma_3 \times \Sigma_3$,

(iii) S *acts on* $\Omega = \mathrm{ccl}_G(S)$ *in orbits of length* 1, 15 *and* 20.

PROOF. Obviously, (i) holds. From $|RS| = |S|^2 \cdot |R \cap S|^{-1} \leqslant |G|$ we get $|R \cap S| \geqslant 20$. If $|R \cap S| = 20$, we have $|S\colon R \cap S| = 36$. Thus, S acts transitively on Ω which is a contradiction to the fact that S and R can not be conjugate under the action of S. Thus, we have $|R \cap S| > 20$. We split the following argument into two cases.

Case 1. Here, 5 divides the order of $|R \cap S|$. Since both R and S contain a Sylow 5-normalizer of G, we get that a Frobenius group of order 20 lies in $R \cap S$. Now, $|R \cap S| > 20$ yields that $R \cap S$ is isomorphic to Σ_5. Note that the only proper subgroups of $S \cong \Sigma_6$ which contain a Frobenius group of order 20 as a proper subgroup are isomorphic to Σ_5. We get $|R \cap S| = 120$, i. e. $|S\colon R \cap S| = 6$ in this case.

Case 2. Here, 5 does not divide the order of $R \cap S$. Since $|R \cap S| > 20$, and since $S \cong \Sigma_6$ does not contain a subgroup of index 5, we have $|R \cap S| \in \{2^4 \cdot 3, 2^3 \cdot 3, 2^3 \cdot 3^2, 2^2 \cdot 3^2\}$, i.e. $|S\colon R \cap S| \in \{15, 30, 10, 20\}$. Furthermore, we get that $R \cap S \cong Z_2 \times \Sigma_4$ if $|R \cap S| = 2^4 \cdot 3$, and $R \cap S$ is 3-closed if and only if $|R \cap S| = 2^3 \cdot 3^2$ or $|R \cap S| = 2^2 \cdot 3^2$.

Case 1 and 2 yield that orbits of S on Ω are of length 1, 6, 10, 15, 20, or 30. Since $|\Omega| = 36$ an easy computation shows that S has precisely one orbit of length 1, precisely one orbit of length 15 and either 2 orbits of length 10 or one orbit of length 20. In particular, there are precisely 20 elements in Ω which intersect S in a 3-closed subgroup.

Let T be a subgroup of G which is conjugate to S in G. Assume that $T \cap S$ is 3-closed. Then, S is conjugate to T via the normalizer of

$D = O_3(S \cap T)$ in G. Thus, $|N_G(D): N_S(D)|$ is the number of conjugate subgroups of S in G containing D. Obviously, S contains precisely 10 Sylow 3-subgroups. Hence, D lies in precisely 3 elements of Ω. Thus, $|N_G(D)| = 3 \cdot |N_S(D)| = 2^3 \cdot 3^3$. Assume $C_G(D) = D$. Then $|N_G(D)/C_G(D)| = 2^3 \cdot 3$, and by the structure of $GL_2(3)$ we have $N_G(D)/C_G(D) \cong SL_2(3)$. But a Sylow 2-subgroup of $SL_2(3)$ is a quaternion group and $N_S(D)$ contains a subgroup isomorphic to D_8. Thus, we have $|C_G(D)| = 3^3$. Elements of $D^\#$ are not 3-central in G, since an involution of S acts invertingly on D. Since elements of order 9 are roots of 3-central elements of order 3 in G, we get that the centralizer of D in G is elementary abelian of order 27. Let P be a Sylow 2-subgroup of $N_S(D)$. Then, $P \cong D_8$. By the lemma of Maschke we have $C_G(D) = D \times X$ with $X^P = X$. By lemma 1 (i) we see that P does not centralize X. Hence, $X^\#$ does not contain any 3-central element of G. It follows that $|C_G(X)|$ is a group of order $27 \cdot 4$ which has a four-group as a Sylow 2-subgroup. Obviously the three conjugates of S, say S, S_1, S_2, containing D are conjugate via X. Thus, we have $S \cap S_1 \cong S \cap S_2 \cong \Sigma_3 \times \Sigma_3$ by the structure of $N_G(D)$. The assertion follows.

2. The design.

Let G, S, Ω be as in section 1, and $\Omega = \{S_0, S_1, \ldots, S_{15}, S_{16}, \ldots, S_{35}\}$ such that $S = S_0$, $S_i \cap S \cong Z_2 \times \Sigma_4$ for $1 \leq i \leq 15$, $S_i \cap S \cong \Sigma_3 \times \Sigma_3$ for $16 \leq i \leq 35$. Denote by $\overline{S}$ the set $\{S_1, \ldots, S_{15}\}$, and for $S_i = S^{g_i}$, $g_i \in G$, let $\overline{S}_i = \{S_1^{g_i}, \ldots, S_{15}^{g_i}\}$, $1 \leq i \leq 35$.

Define an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathfrak{J})$ by $\mathcal{P} = \Omega$, $\mathcal{B} = \{\overline{T} \mid T \in \Omega\}$, $\mathfrak{J} = \{(R, \overline{T}) \mid R, T \in \Omega, R \in \overline{T}\}$.

THEOREM 1. *The incidence structure $\mathcal{D}$ is a symmetric* (36, 15, 6)-*design on which* Aut(G) *acts as an automorphism group.*

REMARK. $\mathcal{D}$ is uniquely determined by $PSp_4(3)$. It is possible to show that Aut$(\mathcal{D})$ is isomorphic to Aut(G).

PROOF. Since Aut(G) acts on Ω we have that Aut(G) is an automorphism group of $\mathcal{D}$. Obviously, $|\mathcal{P}| = |\mathcal{B}| = 36$, and each block, i.e., element of $\mathcal{B}$, contains 15 points, i.e., elements of $\mathcal{P}$. Thus, we only have to show that the intersection of two different blocks contains precisely 6 points.

Consider $\overline{S}$. Since S contains precisely 15 subgroups isomorphic to $Z_2 \times \Sigma_4$, we get that $S_1, \ldots, S_{15}$ are uniquely determined by their intersection with S. Consider intersections of conjugate subgroups isomorphic to $Z_2 \times \Sigma_4$ in Σ_6. Such an intersection is isomorphic to $Z_2 \times \Sigma_3$ or E_8. Since $\Sigma_3 \times \Sigma_3$ does not contain an elementary abelian group of order 8, we get that there are $|Z_2 \times \Sigma_4|/|E_8| = 6$ elements of $\overline{S}$ which intersect S_1 in a subgroup isomorphic to $Z_2 \times \Sigma_4$. Thus, $|\overline{S} \cap \overline{S}_1| \geq 6$. Note that $S \cap S_1 \cong Z_2 \times \Sigma_4$ has precisely three orbits on $\overline{S}$ of length 1, 6, 8, respectively. If $|\overline{S} \cap \overline{S}_1| > 6$, then $|\overline{S} \cap \overline{S}_1| = 14$. Thus, $\langle S, S_1 \rangle$ stabilizes the set $\{S\} \cup \overline{S} = \{S_1\} \cup \overline{S}_1$ which is a contradiction to $\langle S, S_1 \rangle = G$. Thus, $|\overline{S} \cap \overline{S}_i| = 6$ for $1 \leq i \leq 15$.

For $1 \leq i \leq 15$ there are precisely 8 elements in $\{S_{16}, \ldots, S_{35}\}$ which lie in $\overline{S}_i$. Since S acts transitively on $\{S_{16}, \ldots, S_{35}\}$, we have that $|\overline{S}_j \cap \overline{S}| = |\overline{S}_k \cap \overline{S}|$ for any $j, k \in \{16, \ldots, 35\}$. Thus, we have $|\{S_{16}, \ldots, S_{35}\}| \cdot |\overline{S}_j \cap \overline{S}| = |\overline{S}| \cdot 8$, hence $|\overline{S}_j \cap \overline{S}| = 15 \cdot 8/20 = 6$, for $j \geq 16$. We have shown that $|\overline{S}_j \cap \overline{S}| = 15 \cdot 8/20 = 6$, for $1 \leq j \leq 35$. The transitivity of G on Ω completes the proof.

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