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Volterra Integrodifferential Equations of Parabolic Type of Higher Order in Time in L^p Spaces.

DAVIDE GUIDETTI(*)

0. Introduction.

The aim of this paper is to construct and study evolution operators (or fundamental solutions) for quite general linear Volterra integrodifferential problems which are parabolic in the sense of Petrovskii in L^p spaces ($1 < p < +\infty$): we are looking for results applicable to mixed nonautonomous problems, even of higher order in time, with boundary conditions which can depend on time and we want to generalize the results of [6], where differential problems were considered. In our knowledge, the most general results concerning equations of this form in L^p spaces which are available in literature are due to Tanabe (see [12], [13]). We shall study the problem using methods which are inspired by the theory of analytic semigroups in Banach spaces and we shall reduce ourselves to a system of the form

$$(0.1) \quad \begin{cases} U'(t) = A(t)U(t) + \int_s^t \mathcal{C}(t, \sigma)U(\sigma) d\sigma + F(t), t \in]s, T], \\ U(s) = U_0. \end{cases}$$

Concerning the construction of an evolution operator for (0.1), we mention [11], where $\{A(t)\}$ is a family of closed linear densely defined operators in X which generate an evolution operator $U(t, s)$, $\{\mathcal{C}(t, s)\}$ is a family of closable linear operators with domain $D(\mathcal{C}(t, s))$ containing $D(A(s))$, satisfying some additional regularity assumptions, $U_0 \in X$,

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$F : [s, T] \rightarrow X$ is continuous. An evolution operator for (0.1) is constructed both in the hyperbolic and in the parabolic case. In this second case the author employs a series of assumptions introduced by Yagi, which do not require the independence of $D(A(t))$ of t , but some differentiability in t is needed.

Other authors study the equation (0.1) without constructing evolution operators. Limiting ourselves to the nonautonomous case, we mention [3] and [1], where the existence of strict and classical solutions (that is, solutions of (0.1) in a full sense for $t \geq s$ and for $t > s$ respectively) is studied, under assumptions of Kato-Tanabe type, again requiring some differentiability in t of $A(t)$. Other papers are devoted to the search of maximal regularity results and avoid problems of initial data with poor regularity: among them, [10], and [9]. We quote also [4], where solutions which are weakly differentiable in time in L^p sense are discussed. Finally, we consider the papers [12] and [13]; these papers are concerned with the following initial-boundary value problem for the linear equation of higher order in t :

$$(0.2) \quad \left\{ \begin{array}{l} \sum_{k=0}^l \mathcal{A}_{k-l}(t, x, \partial_x) \partial_t^k u(t, x) = \int_s^t \mathcal{C}(t, \sigma, x, \partial_\sigma, \partial_x) u(\sigma, x) d\sigma + f(t, x), \\ s < t \leq T, x \in \Omega, \\ \mathcal{B}_j(t, x', \partial_x) u(t, x') = 0, 1 \leq j \leq m, t \in]s, T], x' \in \partial\Omega, \\ \partial_t^j u(s, x) = u_j(x), \quad x \in \Omega, j = 0, \dots, l-1. \end{array} \right.$$

Here $\sum_{k=0}^l \mathcal{A}_{k-l}(t, x, \partial_x) \partial_t^k$ is parabolic in the sense of Petrovskii and, for $j = 1, \dots, m$, $\mathcal{B}_j(t, x', \partial_x)$ is a linear differential operator which does not contain derivatives with respect to t . In [12] only the case of $\mathcal{B}_j(t, x', \partial_x)$ independent of t is considered. In both papers (0.2) is studied directly, without reducing it to a system of the first order in time, but, in any case, following the ideas of [7] and [11]. This requires very strong assumptions of differentiability with respect to t of the coefficients and a very regular initial datum (u_0, \dots, u_{l-1}) . In [6] assumptions of differentiability in t of the coefficients were avoided and natural conditions on the initial data to get a strict or a classical solution were given (see in particular the fourth section). Moreover, we could treat cases where the operators \mathcal{B}_j contain derivatives with respect to t .

We pass to describe the content of this paper: in the first section we consider the problem in the purely differential case and recall the ab-

stract results of [5] in the particular concrete situation we are considering. These results were the basis of [6]. We complete them constructing an evolution operator $U(t, s)$ for the system of first order in time which is naturally associated to (0.2) in case $\mathcal{C} \equiv 0$ and deduce some of its properties. It could be shown that, in the case of homogeneous boundary conditions, part of the results of this section could be obtained also from the abstract theory of [2]. However in the mentioned paper an evolution operator is not explicitly constructed. Such operator and the variation of parameter formula are necessary for our approach to the integrodifferential case.

The second section is devoted to the construction of an evolution operator for an integrodifferential perturbation of the first order differential system. This perturbation has a kernel which is hölder continuous in (t, s) and a weak singularity on the diagonal is allowed. We observe that from this point of view our assumptions are more restrictive than, for example, the assumptions of [1], [3], [9], [10]. However, we continue to avoid any assumption of differentiability in time of the coefficients and are able to treat initial data which are in the basic space $E_0 = \prod_{k=0}^{l-1} W^{(l-k-1)d, p}(\Omega)$ (see (1.3)). We introduce a notion of classical solution (see 2.3) which is a modification of the notion given in [11], show the existence of classical and strict solutions under assumptions which are similar to those employed in the case of the purely differential problem (see 2.11 and 2.12) and construct an evolution operator $S(t, s)$.

In the third section we apply the results obtained for the first order system to general integrodifferential boundary value problems of higher order in time in L^p spaces ($1 < p < +\infty$) obtaining results which are analogous to those already available in the differential case. Finally, in 3.7 we discuss the representation of solutions in terms of evolution operators.

We conclude this introduction with some notations we shall use in the following: if E and F are Banach spaces, we shall indicate with $\mathcal{L}(E, F)$ the Banach space of linear bounded operators from E to F ; we shall write $\mathcal{L}(E)$ if $E = F$. If $\theta \in]0, 1[$, E, F, G are Banach spaces and $E \subseteq F \subseteq G$, we shall say that F is of type θ with respect to E and G if there exists $C > 0$ such that for every $x \in G$

$$\|x\|_F \leq C \|x\|_E^{1-\theta} \|x\|_G^\theta.$$

If $T > 0$, we set $\Delta_T := \{(t, s) \in \mathbf{R}^2 \mid 0 \leq s < t \leq T\}$.

If X is a topological space and $A \subseteq X$, we shall indicate with \bar{A} the topological closure of A .

If Ω is an open subset of \mathbf{R}^n , we shall indicate with $\partial\Omega$ its boundary. If $\partial\Omega$ is sufficiently regular and $x' \in \partial\Omega$, we shall indicate with $T_{x'}(\partial\Omega)$ the tangent space to $\partial\Omega$ in x' and with $\nu(x')$ the outer normal vector of length 1.

Let $\varrho \in]0, 1[$, $\mu \geq 0$, X a Banach space, $a, b \in \mathbf{R}$, $a < b$; we set

$$C_\mu^\varrho([a, b]; X) := \left\{ f: [a, b] \rightarrow X \mid \sup_{a < s < t \leq b} (t-s)^{-\varrho} (s-a)^\mu \|f(t) - f(s)\|_X < +\infty \right\}.$$

If A is a map, we shall indicate with $D(A)$ its domain.

We shall identify scalar functions of domain $[a, b] \times A$, with $a, b \in \mathbf{R}$, $a < b$, and $A \subseteq \mathbf{R}^n$ with corresponding functions of domain $[a, b]$ and values in functional spaces in A .

Finally, we shall indicate with C a positive constant, which may be different from time to time, even in the same sequence of calculations, and we are not interested to precise. If C depends on α, β, \dots , we shall write $C(\alpha, \beta, \dots)$.

1. Evolution operators for differential systems.

We start by recalling the definition of parabolic operator (in the sense of Petrovskii):

1.1 DEFINITION. Let $T > 0$, $\Omega \subseteq \mathbf{R}^n$, for $(t, x) \in [0, T] \times \bar{\Omega}$ $\mathcal{A}(t, x, \partial_t, \partial_x) = \sum_{k=0}^l \mathcal{A}_{l-k}(t, x, \partial_x) \partial_t^k$, with $l \in \mathbf{N}$, a linear partial differential operator; we shall say that it is d -parabolic ($d \in \mathbf{N}$) if

- (a) for $k = 0, \dots, l$ the order of $\mathcal{A}_k(t, x, \partial_x)$ is less or equal to dk ;
- (b) indicate with $\mathcal{A}_k^0(t, x, \partial_x)$ the part of order dk of $\mathcal{A}_k(t, x, \partial_x)$ and consider for every $(t, x) \in [0, T] \times \bar{\Omega}$ the polynomial $\mathcal{A}^0(t, x, \lambda, \xi) := \sum_{k=0}^l \mathcal{A}_{l-k}^0(t, x, \xi) \lambda^k$; then, $\mathcal{A}^0(t, x, \lambda, i\xi) \neq 0$ for every $(\lambda, \xi) \in \mathbf{C} \times \times \mathbf{R}^n$ with $\operatorname{Re} \lambda \geq 0$ and $(\lambda, \xi) \neq (0, 0)$.

1.2 One can show that d is necessarily even and $\mathcal{A}_0(t, x, \partial_x) = \mathcal{A}_0(t, x)$ never vanishes in $[0, T] \times \bar{\Omega}$ (see for this [6] 1.2); we put $2m := = dl$ and assume in the following that $\mathcal{A}_0(t, x) \equiv 1$.

We introduce now the following assumptions (k1)-(k4):

(k1) $m \in \mathbf{N}$, Ω is an open bounded subset of \mathbf{R}^n lying on one side of its boundary $\partial\Omega$, which is a submanifold of \mathbf{R}^n of class C^{2m} and dimension $n - 1$;

(k2) $\mathcal{A}(t, x, \partial_t, \partial_x) = \sum_{k=0}^l \mathcal{A}_{l-k}(t, x, \partial_x) \partial_t^k$ is a linear partial differential operator with coefficients in $C([0, T]; C(\overline{\Omega}))$, which is d -parabolic, with $ld = 2m$;

(k3) for $j = 1, \dots, m$, $\mathcal{B}_j(t, x, \lambda, \xi) = \sum_{k=0}^{l-1} \mathcal{B}_{jk}(t, x, \xi) \lambda^k$ is for every $(t, x) \in [0, T] \times \overline{\Omega}$ a polynomial in (λ, ξ) such that for $k = 0, \dots, l-1$ $\mathcal{B}_{jk}(t, x, \cdot)$ is a polynomial in ξ of degree at most $\sigma_j - dk$, with $\sigma_j \leq 2m - 1$ and coefficients of class $C([0, T]; C^{2m-\sigma_j}(\overline{\Omega}))$ (of course, $\mathcal{B}_{jk}(t, x, \cdot) \equiv 0$ if $\sigma_j - dk < 0$);

(k4) (complementing condition) indicate with $\mathcal{B}_{jk}^0(t, x, \cdot)$ the part of order $\sigma_j - dk$ of $\mathcal{B}_{jk}(t, x, \cdot)$ and set

$$\mathcal{B}_j^0(t, x, \lambda, \xi) = \sum_{k=0}^{l-1} \mathcal{B}_{jk}^0(t, x, \xi) \lambda^k;$$

consider the O. D. E. problem

$$(1.1) \quad \begin{cases} \mathcal{A}^0(t, x', \lambda, i\xi' + \nu(x') \partial_\tau) w(\tau) = 0, & \tau \in \mathbf{R}, \\ \mathcal{B}_j^0(t, x', \lambda, i\xi' + \nu(x') \partial_\tau) w(0) = g_j, & 1 \leq j \leq m, \\ w \text{ bounded in } \mathbf{R}^+, \end{cases}$$

with $t \in [0, T]$, $x' \in \partial\Omega$, $Re \lambda \geq 0$, $\xi' \in T_{x'}(\partial\Omega)$, $(\lambda, \xi') \neq (0, 0)$, $(g_1, \dots, g_m) \in \mathbf{C}^m$; then (1.1) has a unique solution.

Consider the problem

$$(1.2) \quad \begin{cases} \mathcal{A}(t, x, \partial_t, \partial_x) u(t, x) = f(t, x), \\ s < t \leq T, \quad x \in \Omega, \\ \mathcal{B}_j(t, x', \partial_t, \partial_x) u(t, x') = 0, \quad 1 \leq j \leq m, \quad t \in]s, T], \quad x' \in \partial\Omega, \\ \partial_t^j u(s, x) = u_j(x), \quad x \in \Omega, \quad j = 0, \dots, l-1. \end{cases}$$

with $s \in [0, T[$, $f \in C([s, T]; L^p(\Omega))$, for some $p \in]1, +\infty[$, and $u_0 \in W^{2m-d, p}(\Omega)$, \dots , $u_j \in W^{2m-(j+1)d, p}(\Omega)$, \dots , $u_{l-1} \in L^p(\Omega)$. In [6] the following notions of classical and strict solution of (1.2) were given:

1.3 DEFINITION. A classical solution u of (1.2) is a function $u \in \prod_{k=0}^l C^k([s, T]; W^{2m-kd, p}(\Omega)) \cap \prod_{k=0}^{l-1} C^k([s, T]; W^{2m-(k+1)d, p}(\Omega))$ satisfying the last condition in (1.2) and the two first for $t \in]s, T]$.

A strict solution u of (1.2) is a function $u \in \prod_{k=0}^l C^k([s, T]; W^{2m-kd, p}(\Omega))$, satisfying the last condition in (1.2) and the two first for $t \in [s, T]$.

We shall employ the following assumption (L1)-(L4):

(L1) (k1)-(k4) are satisfied;

(L2) the coefficients of $\mathcal{A}_k(t, x, \partial_x)$ ($0 \leq k \leq l$) are of class $C^\beta([0, T]; C(\overline{\Omega}))$ with $\beta > 0$ such that:

(L3) if $2m - (r+1)d \leq \sigma_j < 2m - rd$ with $0 \leq r \leq l-1$, the coefficients of \mathcal{B}_{jk} ($0 \leq k \leq l-r-1$) belong to $\prod_{\alpha=0} C^{\alpha+\beta}([0, T]; C^{2m-\sigma_j-\alpha d}(\overline{\Omega}))$;

(L4) for every $j = 1, \dots, m$, if $2m - (r+1)d \leq \sigma_j < 2m - rd$ ($0 \leq r \leq l-1$),

$$\beta > \frac{2m - rd - \sigma_j - p^{-1}}{d}.$$

1.4 In [6] the problem (1.2) was previously considered under the further assumption

$$(k5) \quad \min_{1 \leq j \leq m} \sigma_j \geq 2m - d.$$

Observe that this condition is always satisfied in the particular (and most classical) case $l = 1$.

If the assumptions (L1)-(L4) and (k5) are satisfied, the most natural strategy to solve (1.2) is to write it in the form of a system of first order in time. To this aim, set

$$(1.3) \quad E_0 := \prod_{k=0}^{l-1} W^{(l-k-1)d, p}(\Omega) = W^{2m-d, p}(\Omega) \times \dots \times L^p(\Omega),$$

$$(1.4) \quad E_1 := \prod_{k=0}^{l-1} W^{(l-k)d, p}(\Omega) = W^{2m, p}(\Omega) \times \dots \times W^{d, p}(\Omega).$$

If $U \in E_1$, $U = (U_0, \dots, U_{l-1})$, we set, for $t \in [0, T]$,

$$(1.5) \quad \mathfrak{A}(t) U := \left(U_1, \dots, U_{l-1}, - \sum_{k=0}^{l-1} \mathfrak{A}_{l-k}(t, \dots, \partial_x) U_k \right).$$

Observe that for every $t \in [0, T]$ $\mathfrak{A}(t) \in \mathcal{L}(E_1, E_0)$. Next, for $t \in [0, T]$, $U \in E_1$, $j \in \{1, \dots, m\}$ we set

$$(1.6) \quad \mathfrak{B}_j(t) U := \sum_{k=0}^{l-1} \mathfrak{B}_{jk}(t, \dots, \partial_x) U_k.$$

$\mathfrak{B}_j(t) \in \mathcal{L}(E_1, E_{\mu_j})$ if we define

$$(1.7) \quad E_{\mu_j} := W^{2m - \sigma_j, p}(\Omega).$$

Finally, we indicate with γ the trace operator on $\partial\Omega$.

Then the fact that u is a classical (strict) solution of (1.2) is equivalent to the fact that

$$U(t) := (u(t), \dots, \dots, \partial_t^{l-1} u(t))$$

is a classical (strict) solution of the problem

$$(1.8) \quad \begin{cases} \partial_t U(t) = \mathfrak{A}(t) U(t) + F(t), t \in]s, T], \\ \gamma(\mathfrak{B}_j(t) U(t) - g_j(t)) = 0, 1 \leq j \leq m, t \in]s, T], \\ U(s) = U_0 \end{cases}$$

with $F(t) = (0, \dots, 0, f(t))$, $g_j(t) \equiv 0$, $U_0 = (u_0, \dots, u_{l-1})$, in the following sense:

1.5 DEFINITION. Let $F \in C([s, T]; E_0)$, for $1 \leq j \leq m$ let $g_j \in C([s, T]; E_{\mu_j})$, $U_0 \in E_0$. A classical solution of (1.8) is an element U in $C^1(]s, T]; E_0) \cap C(]s, T]; E_1) \cap C([s, T]; E_0)$ satisfying the last condition in (1.8) and the two first conditions for every $t \in]s, T]$.

If, moreover, $U_0 \in E_1$, a strict solution of (1.8) is an element U in $C^1([s, T]; E_0) \cap C([s, T]; E_1)$ satisfying the last condition in (1.8) and the two first conditions for every $t \in [s, T]$.

So in the remaining part of this section we shall consider the system (1.8) under the assumptions (L1)-(L4) and (k5). Let $\beta < 1$ such that (L4)

is satisfied for every $j = 1, \dots, m$. We fix $\sigma \in]0, p^{-1}[$ in such a way that

$$(1.9) \quad \beta > \frac{2m - \min_j \sigma_j - \sigma}{d}$$

and define, for $j = 1, \dots, m$,

$$(1.10) \quad F_{\nu_j} := W^{\sigma_j + \sigma, p}(\Omega) \times \dots \times W^{\sigma_j + \sigma - (l-1)d, p}(\Omega).$$

(Recall that $\sigma_j \geq 2m - d = (l-1)d$). Then F_{ν_j} is a space of type

$$(1.11) \quad \nu_j = \frac{\sigma_j + \sigma}{d} - (l-1)$$

between E_0 and E_1 . Finally, we set

$$(1.12) \quad F = W^{\sigma, p}(\Omega)$$

and

$$(1.13) \quad Z = L^p(\partial\Omega).$$

The following lemma is crucial:

1.6 LEMMA. (See [6], section 2) *Under the assumptions (k1)-(k4) and (k5), with the notations (1.3), (1.4), (1.5), (1.6), (1.7), (1.10), (1.11), (1.12), consider the problem*

$$(1.14) \quad \begin{cases} \lambda U - \mathcal{A}(t) U = F, \\ \gamma(\mathcal{B}_j(t) U - g_j) = 0, \quad j = 1, \dots, m, \end{cases}$$

with $\lambda \in \mathbb{C}$, $t \in [0, T]$, $F \in E_0$, $g_j \in E_{\mu_j}$ for $j = 1, \dots, m$.

Then, there exist $\Lambda > 0$ and $C > 0$ independent of λ , F , $(g_j)_{1 \leq j \leq m}$ such that, if $\operatorname{Re}(\lambda) \geq 0$ and $|\lambda| \geq \Lambda$, (1.14) has a unique solution $U \in E_1$ and

$$(1.15) \quad |\lambda| \|U\|_0 + \|U\|_1 \leq C \left[\|F\|_0 + \sum_{j=1}^m \|g_j\|_{E_{\mu_j}} + \sum_{j=1}^m |\lambda|^{1-\nu_j} \|g_j\|_F \right].$$

To summarize, if the assumption (L1)-(L4) and (k5) are all satisfied, with the notations (1.3), (1.4), (1.5), (1.6), (1.7), (1.10), (1.11), (1.12), (1.13) and the assumption (1.9), the following conditions are fulfilled:

(h1) E_0, E_1 are Banach spaces, with $E_1 \subseteq E_0$ with continuous and dense embedding and norms that we shall indicate with $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively;

(h2) for $j = 1, \dots, m$ ν_1, \dots, ν_m are real numbers in $]0, 1[$, $E_{\mu_1}, \dots, E_{\mu_m}, F, F_{\nu_1}, \dots, F_{\nu_m}, Z$ are Banach spaces with norms $\|\cdot\|_{\mu_1}, \dots, \|\cdot\|_{\mu_m}, \|\cdot\|_F, \|\cdot\|_{\nu_1}, \dots, \|\cdot\|_{\nu_m}, \|\cdot\|_Z$ respectively, such that for every $j = 1, \dots, m$ F_{ν_j} is of type ν_j between E_0 and E_1 and E_{μ_j} is continuously embedded into F ; we set $\nu := \min_{1 \leq j \leq m} \nu_j$.

(h3) $T \in]0, +\infty[$, $\mathcal{A}: [0, T] \rightarrow \mathcal{L}(E_1, E_0)$;

(h4) for $j = 1, \dots, m$ $\mathcal{B}_j: [0, T] \rightarrow \mathcal{L}(E_1, E_{\mu_j}) \cap \mathcal{L}(F_{\nu_j}, F)$;

(h5) there exists $\beta > 0$ such that $\beta + \nu_j > 1$ for every j and, for $0 \leq s \leq t \leq T$

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(E_1, E_0)} + \sum_{j=1}^m \|\mathcal{B}_j(t) - \mathcal{B}_j(s)\|_{\mathcal{L}(E_1, E_{\mu_j}) \cap \mathcal{L}(F_{\nu_j}, F)} \leq C(t-s)^\beta;$$

(h6) γ is a linear operator from $\sum_{j=1}^m E_{\mu_j}$ to Z and for every $j = 1, \dots, m$ $\gamma|_{E_{\mu_j}} \in \mathcal{L}(E_{\mu_j}, Z)$;

(h7) there exists $\Lambda > 0$ such that for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) \geq 0$, $|\lambda| \geq \Lambda$, for every $t \in [0, T]$ the problem (1.14) has a unique solution $U \in E_1$ for every $F \in E_0$, $(g_1, \dots, g_m) \in \prod_{j=1}^m E_{\mu_j}$ and the estimate (1.15) is available.

Now, (h1)-(h7) were the basic assumptions in [5], where the abstract problem (1.8) was considered. In fact, in [5] slightly different notations were used: we wrote $\theta_0 + \mu_j$ instead of ν_j ; we wrote τ instead of γ ; moreover, it was assumed that for $j = 1, \dots, m$ E_{μ_j} were intermediate between E_0 and E_1 , which is not necessary for the conclusions we aim to. See also [6] for other remarks of this type. We observe that in our concrete situation we have (as already declared in (h1)) that E_1 is dense in E_0 , which was not assumed in [5].

Now we revise the results of [5] and use them to construct an evolution operator for the problem (1.8) using (h1)-(h7).

Given $F \in E_0$ and $(g_j)_{j=1}^m \in \prod_{j=1}^m E_{\mu_j}$, we indicate the solution U of (1.14) with the notation $R(\lambda, t)F + \sum_{j=1}^m N_j(\lambda, t)g_j$. Also we set, for

$t \in [0, T]$,

$$(1.16) \quad D(A(t)) := \{U \in E_1 \mid \gamma \mathcal{B}_j(t) U = 0 \text{ for } 1 \leq j \leq m\}, \quad A(t) U := \mathcal{A}(t) u.$$

By a simple perturbation argument, one can verify that the problem (1.14) is solvable for $\lambda \in \mathbf{C}$ such that $|\lambda| \geq A'$, $|\text{Arg}(\lambda)| \leq (\pi/2) + \eta$, for some A' , η positive depending on C and A and the estimate (1.15) holds in this larger set, modifying (if necessary) C .

Assumption (h7) together with the density of E_1 in E_0 implies that $A(t)$ is the infinitesimal generator of an analytic semigroup in E_0 (see [5], corollary 4.5).

We introduce now the following notations: let $s \in [0, T]$, $t \geq 0$; we set:

$$(1.17) \quad T(t, s) = \begin{cases} (2\pi i)^{-1} \int_{\Gamma} \exp(\lambda t) R(\lambda, s) d\lambda & \text{if } t \geq 0, \\ I_{E_0} & \text{if } t = 0, \end{cases}$$

where we have indicated with Γ the clockwise oriented boundary of $\{\lambda \in \mathbf{C}; |\lambda| \geq A', |\text{Arg}(\lambda)| \leq (\pi/2) + \eta\}$ and, more generally, for $k \in \mathbf{Z}$, $t > 0$, $s \in [0, T]$,

$$(1.18) \quad T^{(k)}(t, s) = (2\pi i)^{-1} \int_{\Gamma} \exp(\lambda t) \lambda^k R(\lambda, s) d\lambda.$$

One can verify that, if $k < 0$,

$$T^{(k)}(t, s) = \frac{1}{(-k-1)!} \int_0^t (t-\tau)^{-k-1} T(\tau, s) d\tau.$$

Let now $j \in \{1, \dots, m\}$, $t > 0$, $s \in [0, T]$; we set

$$(1.19) \quad K_j(t, s) := (2\pi i)^{-1} \int_{\Gamma} \exp(\lambda t) N_j(\lambda, s) d\lambda,$$

and, for $k \in \mathbf{Z}$,

$$(1.20) \quad K_j^{(k)}(t, s) := (2\pi i)^{-1} \int_{\Gamma} \exp(\lambda t) \lambda^k N_j(\lambda, s) d\lambda.$$

It is not difficult to verify, using Cauchy's theorem, that if $k < 0$,

$$(1.21) \quad K_j^{(k)}(t, s) = \frac{1}{(-k-1)!} \int_0^t (t-\tau)^{-k-1} K_j(\tau, s) d\tau.$$

Moreover,

1.7 LEMMA. (I) For every $t > 0$, $s \in [0, T]$, $k \in \mathbf{Z}$, $j \in \mathbf{N}$ $T^{(k)}(t, s) \in \mathcal{L}(E_0, E_1)$, $t \rightarrow T^{(k)}(t, s) \in C^\infty(]0, +\infty[; \mathcal{L}(E_0, E_1))$ and $(d/dt)^j T^{(k)}(\cdot, s) = T^{(k+j)}(\cdot, s)$;

(II) if $\omega > A'$,

$$\|T^{(k)}(t, s)\|_{\mathcal{L}(E_0)} \leq c(k, \omega) t^{-k} \exp(\omega t), \|T^{(k)}(t, s)\|_{\mathcal{L}(E_0, E_1)} \leq c(k, \omega) t^{-k-1} \exp(\omega t);$$

(III)

$$\lim_{t \rightarrow 0} \|T(t, s)F - F\|_0 = 0$$

for every $F \in E_0$;

(IV) for every $F \in E_0$

$$\lim_{t \rightarrow 0} \|T^{(-1)}(t, s)F\|_1 = 0;$$

(V) for every $t > 0$, $s \in [0, T]$ $T^{(1)}(t, s) - \mathcal{C}(s)T(t, s) = 0$;

(VI) for every $t > 0$, $s \in [0, T]$ $T(t, s) - \mathcal{C}(s)T^{(-1)}(t, s) = I_{E_0}$;

(VII) for every $t > 0$, $s \in [0, T]$, $j \in \{1, \dots, m\}$, $k \in \mathbf{Z}$ $\gamma \mathcal{B}_j(s) T^{(k)}(t, s) = 0$;

(VIII) for every $k \in \mathbf{Z}$, $t > 0$, $s, \sigma \in [0, T]$, $\omega > A'$

$$\|T^{(k)}(t, s) - T^{(k)}(t, \sigma)\|_{\mathcal{L}(E_0)} \leq c(k, \omega) \exp(\omega t) t^{-k} |s - \sigma|^\beta,$$

$$\|T^{(k)}(t, s) - T^{(k)}(t, \sigma)\|_{\mathcal{L}(E_0, E_1)} \leq c(k, \omega) \exp(\omega t) t^{-k-1} |s - \sigma|^\beta;$$

(IX) for every $t > 0$, $s \in [0, T]$, $k \in \mathbf{Z}$, $j \in \{1, \dots, m\}$, $r \in \mathbf{N}_0$ $K_j^{(k)}(t, s) \in \mathcal{L}(E_{\mu_j}, E_1)$, $t \rightarrow K_j^{(k)}(t, s) \in C^\infty(]0, +\infty[; \mathcal{L}(E_{\mu_j}, E_1))$ and

$$\left(\frac{d}{dt}\right)^r K_j^{(k)}(\cdot, s) = K_j^{(k+r)}(\cdot, s);$$

(X) if $\omega > \Lambda'$, $g \in E_{\mu_j}$,

$$\|K_j^{(k)}(t, s) g\|_0 \leq c(k, \omega) t^{-k} \exp(\omega t) [\|g\|_{\mu_j} + t^{\nu_j-1} \|g\|_F],$$

$$\|K_j^{(k)}(t, s) g\|_1 \leq c(k, \omega) t^{-k-1} \exp(\omega t) [\|g\|_{\mu_j} + t^{\nu_j-1} \|g\|_F];$$

(XI) for every $t > 0$, $s \in [0, T]$

$$K_j^{(1)}(t, s) - \mathcal{C}(s) K_j(t, s) = K_j(t, s) - \mathcal{C}(s) K_j^{(-1)}(t, s) = 0;$$

(XII) for every $t > 0$, $s \in [0, T]$, $j, l \in \{1, \dots, m\}$
 $\gamma \mathcal{B}_l(s) K_j(t, s) = 0;$

(XIII) for every $t > 0$, $s \in [0, T]$, $j, l \in \{1, \dots, m\}$, $g \in E_{\mu_j}$
 $\gamma \mathcal{B}_l(s) K_j^{(-1)}(t, s) g = \delta_{lj} \gamma g;$

(XIV) for every $j \in \{1, \dots, m\}$, $k \in \mathbf{Z}$, $t > 0$, $s, \sigma \in [0, T]$, $\omega > \Lambda'$,
 $g \in E_{\mu_j}$

$$\|K_j^{(k)}(t, s) g - K_j^{(k)}(t, \sigma) g\|_0 \leq c(k, \omega) \exp(\omega t) t^{-k} |s - \sigma|^\beta (\|g\|_{\mu_j} + t^{\nu_j-1} \|g\|_F),$$

$$\|K_j^{(k)}(t, s) g - K_j^{(k)}(t, \sigma) g\|_1 \leq c(k, \omega) \exp(\omega t) t^{-k-1} |s - \sigma|^\beta (\|g\|_{\mu_j} + t^{\nu_j-1} \|g\|_F).$$

PROOF. Standard, using also the expression of $T^{(k)}(t, s)$ for $k < 0$. Concerning (III) and (IV), use [5], corollary 4.5.

1.8 We consider now $R \in C_\mu^\varrho(\cdot, T]; E_0)$ for certain ϱ and $\mu \in [0, 1[$ and, for every $k \in \{1, \dots, m\}$, a function $S_k \in C_\mu^\varrho(\cdot, T]; E_{\mu_k}) \cap C_{\mu_k}^{\varrho'_k}(\cdot, T]; F)$ for $\varrho'_k, \mu'_k \in [0, 1[$, $\varrho'_k > 1 - \nu_k$, and set, for $s < t \leq T$,

$$(1.22) \quad U(t) := \int_s^t T(t - \sigma, \sigma) R(\sigma) d\sigma + \sum_{k=1}^m \int_s^t K_k(t - \tau, \tau) S_k(\tau) d\tau.$$

We have:

1.9 LEMMA. Let U be defined as in (1.22) with the properties declared in 1.8. Then,

(I) $U \in C(\cdot, T]; E_1) \cap C^1(\cdot, T]; E_0)$ and $\|U(t)\|_1 \leq C(t - s)^{\delta' - 1}$ for some $\delta' > 0$;

- (II) if for every $k \in \{1, \dots, m\}$ $\mu'_k < \nu_k$, $U \in C([s, T]; E_0)$;
 (III) for every $t \in]s, T]$

$$U'(t) - \mathcal{A}(t) U(t) = R(t) - \int_s^t N_{00}(t, \sigma) R(\sigma) d\sigma - \sum_{k=1}^m \int_s^t N_{0k}(t, \sigma) S_k(\sigma) d\sigma,$$

where, for $0 \leq s < t \leq T$, $k \in \{1, \dots, m\}$,

(1.23) $N_{00}(t, s) = [\mathcal{A}(t) - \mathcal{A}(s)] T(t - s, s),$

(1.24) $N_{0k}(t, s) = [\mathcal{A}(t) - \mathcal{A}(s)] K_k(t - s, s);$

- (IV) if $1 \leq j \leq m$,

$$\gamma(\mathcal{B}_j(t) U(t)) = \gamma \left(S_j(t) - \int_s^t N_{j0}(t, \sigma) R(\sigma) d\sigma - \sum_{k=1}^m \int_s^t N_{jk}(t, \sigma) S_k(\sigma) d\sigma \right),$$

where, for $j, k \in \{1, \dots, m\}$,

(1.25) $N_{j0}(t, s) = [\mathcal{B}_j(s) - \mathcal{B}_j(t)] T(t - s, s),$

(1.26) $N_{jk}(t, s) = [\mathcal{B}_j(s) - \mathcal{B}_j(t)] K_k(t - s, s);$

- (V) if $\mu = 0$ and, for every $k \in \{1, \dots, m\}$, $\mu'_k = 0$ and $\gamma S_k(s) = 0$, then $U \in C([s, T]; E_1) \cap C^1([s, T]; E_0)$.

PROOF. It follows from [5], lemmata 2.2 and 2.5.

Now we come back to the system (1.8) and look for a solution U in the form

(1.27) $U(t) = T(t - s, s) U_0 + \int_s^t T(t - \sigma, \sigma) R(\sigma) d\sigma + \sum_{k=1}^m \int_s^t K_k(t - \sigma, \sigma) S_k(\sigma) d\sigma.$

(IV) if $j \geq 1$ and $\nu_j < \nu_k$,

$$\|N_{jk}(t, s) - N_{jk}(\tau, s)\|_{\mathcal{L}(E_{\mu_k}, F)} \leq C(t - \tau)^\beta (\tau - s)^{\nu_k - \nu_j - 1};$$

(V) if $j \geq 1$ and $\nu_k \leq \nu_j$, for every $\theta \in [0, \beta + \nu_k - \nu_j[$ there exist $\delta(\theta) < 1$, $C(\theta) > 0$ such that

$$\|N_{jk}(t, s) - N_{jk}(\tau, s)\|_{\mathcal{L}(E_{\mu_k}, F)} \leq C(\theta)(t - \tau)^\theta (\tau - s)^{-\delta(\theta)}.$$

We come back to the system (1.29); it is useful to introduce the notion of mild solution:

1.11 DEFINITION. Let $U_0 \in E_0$, $F \in C([s, T]; E_0)$, $g_j \in C([s, T]; E_{\mu_j})$ for $j \in \{1, \dots, m\}$. A mild solution of (1.8) is a function U of the form (1.27), with R, S_k ($k \in \{1, \dots, m\}$) solutions of (1.29), that is, with $R \in L^1(]s, T[; E_0)$ and, for $k = 1, \dots, m$, $S_k \in L^1(]s, T[; E_{\mu_k})$ such that (1.29) is satisfied for almost every t in $]s, T[$.

Concerning the solution of (1.29), we have

1.12 THEOREM. Assume that X_0, \dots, X_m are Banach spaces and, for $(t, s) \in \Delta_T$, $N(t, s) = (N_{jk}(t, s))_{0 \leq j, k \leq m}$, with $N_{jk} \in C(\Delta_T; \mathcal{L}(X_k, X_j))$. Let

$$\|N_{jk}(t, s)\|_{\mathcal{L}(X_k, X_j)} \leq C(t - s)^{-\gamma_{jk}},$$

for $0 \leq s < t \leq T$, with $\gamma_{jk} < 1$. Let $(\sigma_0, \dots, \sigma_m) \in \mathbf{R}^{m+1}$, with $\sigma_k < 1$ for every k ,

$$C_\sigma(X) := \prod_{k=0}^m C_{\sigma_k}(X_k),$$

with

$$C_{\sigma_k}(X_k) := \{v \in C(]0, T[; X_k) \mid t^{\sigma_k} v \text{ is bounded in }]0, T[\}.$$

Let, for every j, k , $\sigma_k - \sigma_j + \gamma_{jk} \leq 1$.

Then, the integral equation

$$u(t) = \phi(t) + \int_0^t N(t, \tau) u(\tau) d\tau$$

has for every $\phi \in C_\sigma(X)$ a unique solution in $L^1(]0, T[; X) :=$

$:= \prod_{k=0}^m L^1(]0, T[; X_k)$. Such solution u belongs to $C_o(X)$ and for every $t \in]0, T]$

$$u(t) = \lim_{n \rightarrow \infty} u_n(t),$$

where $u_1(t) = \phi(t)$ and, for every $n \in \mathbb{N}$,

$$u_{n+1}(t) = \phi(t) + \int_0^t N(t, \tau) u_n(\tau) d\tau.$$

PROOF. See [5], proposition 3.2.

1.13 COROLLARY. For every U_0 in E_0 , $F \in C([s, T]; E_0)$, $(g_k)_{1 \leq k \leq m} \in \prod_{k=1}^m C([s, T]; E_{\mu_k})$ the system (1.8) has a unique mild solution. Moreover,

$$\|R(t)\|_0 + \sum_{j=1}^m \|S_j(t)\|_{\mu_j} \leq C(t-s)^{\beta-1}$$

if $0 \leq s < t \leq T$.

PROOF. We set $X_0 := E_0$, for $k = 1, \dots, m$ $X_k := E_{\mu_k}$. From 1.10(I) we have $\gamma_{jk} = 2 - \beta - \nu_k$ for every j, k and, $\phi_j(t) = g_j(t) + N_{j0}(t, s) U_0$ (putting $g_0 := F$). Again from 1.10(I) we can take $\sigma_k = 1 - \beta$ for every k . So for every j and k $\sigma_k - \sigma_j + \gamma_{jk} = 2 - \beta - \nu_k < 1$. Therefore the result follows from 1.12.

1.14 PROPOSITION. Take $U_0 = 0$, $F \equiv 0$, $\gamma g_j \equiv 0$ for every $j = 1, \dots, m$. Then, the mild solution of (1.8) vanishes identically.

PROOF. We have $\phi(t) = (0, g_1(t), \dots, g_m(t))$. Evidently $K_k(t, \sigma) \phi_k(\tau) \equiv 0$ for $(t, \sigma) \in \Delta_T$, $\tau \in]s, T]$ and $j, k \in \{0, \dots, m\}$. It follows that $N_{jk}(t, \tau) \phi_k(\tau) = 0$ for every $(t, \tau) \in \Delta_T$, which implies $R(t) \equiv 0$ and $S_k = g_k$ for $k = 1, \dots, m$. So we have $U \equiv 0$.

The following statement collects together the main results of [5], section IV in the particular case that E_1 is dense in E_0 :

1.15 THEOREM. (I) Every mild solution of (1.8) is continuous in $[s, T]$ with values in E_0 ;

(II) if $F \in C^\varepsilon([s, T]; E_0)$, for $j = 1, \dots, m$ $g_j \in C^\varepsilon([s, T]; E_{\mu_j}) \cap \cap C^{1-\nu_j+\varepsilon}([s, T]; F)$ for some $\varepsilon > 0$, then the mild solution of (1.8) is a classical solution;

(III) if, moreover, $U_0 \in E_1$ and for $j = 1, \dots, m$ $\gamma(\mathcal{B}_j(s) U_0 - g_j(s)) = 0$, the mild solution of (1.8) is a strict solution;

(IV) the classical solution of (1.8), if existing, is unique;

(V) the strict solution of (1.8), if existing, coincides with the mild solution.

PROOF. See [5], section IV.

Now we shall try to represent by explicit formulas the mild solution of (1.8) in case $g_j \equiv 0$ for every $j = 1, \dots, m$; we come back to the system (1.29), set $N(t, s) := (N_{jk}(t, s))_{0 \leq j, k \leq m}$ and $E := E_0 \times E_{\mu_1} \times \dots \times E_{\mu_r}$. We consider again the Volterra integral equation

$$\phi(t) = h(t) + \int_s^t N(t, \sigma) \phi(\sigma) d\sigma,$$

with $h \in C([s, T]; E)$, $\|h(t)\|_E \leq C(t-s)^{-\delta}$ for some $\delta < 1$. With the method of [8] II.4.2, the solution ϕ can be represented in the form

$$(1.30) \quad \phi(t) = h(t) + \int_s^t R(t, \sigma) h(\sigma) d\sigma,$$

with

$$(1.31) \quad R(t, s) = \sum_{k=1}^{\infty} N^{(k)}(t, s),$$

$$(1.32) \quad N^{(1)}(t, s) = N(t, s),$$

$$(1.34) \quad N^{(k+1)}(t, s) = \int_s^t N(t, \sigma) N^{(k)}(\sigma, s) d\sigma = \int_s^t N^{(k)}(t, \sigma) N(\sigma, s) d\sigma,$$

$$(1.34) \quad R(t, s) = N(t, s) + \int_s^t N(t, \sigma) R(\sigma, s) d\sigma = N(t, s) + \int_s^t R(t, \sigma) N(\sigma, s) d\sigma.$$

If we put $R(t, s) = (R_{jk}(t, s))_{0 \leq j, k \leq m}$, we have, continuing to set $\mu_0 = 0, \nu_0 = 1$:

1.16 PROPOSITION. *Let $0 \leq s < \tau \leq t \leq T$, $0 \leq j, k \leq m$; then, $R_{jk} \in \mathcal{L}(E_{\mu_k}, E_{\mu_j})$ and*

$$(I) \ \|R_{jk}(t, s)\|_{\mathcal{L}(E_{\mu_k}, E_{\mu_j})} \leq C(t-s)^{\beta + \nu_k - 2};$$

(II) *for every $\theta \in]0, \beta + \nu - 1[$ there exist $\delta(\theta) < 1$, $C(\theta) > 0$ such that*

$$\|R_{jk}(t, s) - R_{jk}(\tau, s)\|_{\mathcal{L}(E_{\mu_k}, E_{\mu_j})} \leq C(\theta)(t-\tau)^\theta (\tau-s)^{-\delta(\theta)};$$

(III) *if $j \geq 1$,*

$$\|R_{jk}(t, s)\|_{\mathcal{L}(E_{\mu_k}, F)} \leq C(t-s)^{\beta + \nu_k - \nu_j - 1};$$

(IV) *if $j \geq 1$, for every $\theta \in]0, \beta + \nu - \nu_j[$ there exist $\delta(\theta) < 1$, $C(\theta) > 0$ such that*

$$\|R_{jk}(t, s) - R_{jk}(\tau, s)\|_{\mathcal{L}(E_{\mu_k}, F)} \leq C(\theta)(t-\tau)^\theta (\tau-s)^{-\delta(\theta)}.$$

PROOF. (I) We have

$$R_{jk}(t, s) = N_{jk}(t, s) + \sum_{r=0}^m \int_s^t N_{jr}(t, \sigma) R_{rk}(\sigma, s) d\sigma.$$

Fix $k \in \{0, \dots, m\}$ and set $\phi(t) := (N_{0k}(t, s), \dots, N_{mk}(t, s))$. Then, we can apply 1.12 with $\gamma_{jr} = 2 - \beta - \nu_r$, $\sigma_j = 2 - \beta - \nu_k$ ($0 \leq j, r \leq m$), $\sigma_r - \sigma_j + \gamma_{jr} = 2 - \beta - \nu_r < 1$.

(II) We have

$$\begin{aligned} \|R_{jk}(t, s) - R_{jk}(\tau, s)\|_{\mathcal{L}(E_{\mu_k}, E_{\mu_j})} &\leq \|N_{jk}(t, s) - N_{jk}(\tau, s)\|_{\mathcal{L}(E_{\mu_k}, E_{\mu_j})} + \\ &+ \sum_{r=0}^m \left\| \int_\tau^t N_{jr}(t, \sigma) R_{rk}(\sigma, s) d\sigma \right\|_{\mathcal{L}(E_{\mu_k}, E_{\mu_j})} + \\ &+ \sum_{r=0}^m \left\| \int_s^\tau [N_{jr}(t, \sigma) - N_{jr}(\tau, \sigma)] R_{rk}(\sigma, s) d\sigma \right\|_{\mathcal{L}(E_{\mu_k}, E_{\mu_j})} = I_1 + I_2 + I_3. \end{aligned}$$

By 1.10 (II) $I_1 \leq C(\theta)(t-\tau)^\theta (\tau-s)^{-\delta(\theta)}$ for every $\theta \in]0, \beta + \nu_k - 1[$ with $\delta(\theta) < 1$. The r -th summand in I_2 can be majorized with

$$C \int_\tau^t (t-\sigma)^{\beta + \nu_r - 2} (\sigma-s)^{\beta + \nu_k - 2} d\sigma \leq C(t-\tau)^{\beta + \nu_r - 1} (\tau-s)^{\beta + \nu_k - 2}.$$

Finally, the r -th summand in I_3 is majorized by

$$C \int_s^\tau (t - \tau)^\theta (\tau - \sigma)^{-\delta(\theta)} (\sigma - s)^{\beta + \nu_k - 2} d\sigma$$

for every $\theta < \beta + \nu_r - 1$, with $\delta(\theta) < 1$, majorized by

$$C(t - \tau)^\theta (\tau - s)^{\beta + \nu_k - 1 - \delta(\theta)}.$$

So, (II) is proved.

Concerning (III), using 1.10(III), with the same method of (I), we obtain, if $j \geq 1$,

$$\begin{aligned} \|R_{jk}(t, s)\|_{\mathcal{L}(E_{\mu_k}, F)} &\leq C \left[(t - s)^{\beta + \nu_k - \nu_j - 1} + \sum_{r=0}^m (t - s)^{2\beta + \nu_r + \nu_k - \nu_j - 2} \right] \leq \\ &\leq C(t - s)^{\beta + \nu_k - \nu_j - 1}. \end{aligned}$$

Concerning (IV), we have, for $0 \leq r \leq m$,

$$\left\| \int_\tau^t N_{jr}(t, \sigma) R_{rk}(\sigma, s) d\sigma \right\|_{\mathcal{L}(E_{\mu_k}, F)} \leq C(t - \tau)^{\beta + \nu_l - \nu_j} (\tau - s)^{\beta + \nu_k - 2}.$$

Next, for every $\theta < \min \{ \beta, \beta + \nu_r - \nu_j \}$, for a suitable $\delta(\theta) < 1$,

$$\begin{aligned} \left\| \int_s^\tau [N_{jr}(t, \sigma) - N_{jr}(\tau, \sigma)] R_{rk}(\sigma, s) d\sigma \right\|_{\mathcal{L}(E_{\mu_k}, F)} &\leq \\ &\leq C \int_s^\tau (t - \tau)^\theta (\tau - \sigma)^{-\delta(\theta)} (\sigma - s)^{\beta + \nu_k - 2} d\sigma \leq C(t - \tau)^\theta (\tau - s)^{\beta + \nu_k - \delta(\theta) - 1}. \end{aligned}$$

Putting together all the estimates and using 1.10(IV)-(V), we obtain (IV).

1.17 Let now $s \in [0, T]$, $U_0 \in E_0$ and consider the mild solution U in $[s, T]$ with data $(U_0, 0, (O)_{1 \leq j \leq m})$. In this case we have, referring to (1.30), $h(t) = (N_{j0}(t, s) U_0)_{0 \leq j \leq m}$, so that, for $0 \leq j \leq m$,

$$\phi_j(t) = N_{j0}(t, s) U_0 + \sum_{k=0}^m \int_s^t R_{jk}(t, \sigma) N_{k0}(\sigma, s) U_0 d\sigma = R_{j0}(t, s) U_0,$$

owing to (1.34). It follows that, if $t \in [s, T]$,

$$U(t) = T(t-s, s) U_0 + \int_s^t T(t-\sigma, \sigma) R_{00}(\sigma, s) U_0 d\sigma + \\ + \sum_{j=1}^m \int_s^t K_j(t-\sigma, \sigma) R_{j0}(\sigma, s) U_0 d\sigma.$$

So, we set, for $(s, t) \in \overline{\mathcal{A}_T}$,

$$(1.35) \quad U(t, s) = T(t-s, s) + \int_s^t T(t-\sigma, \sigma) R_{00}(\sigma, s) d\sigma + \\ + \sum_{j=1}^m \int_s^t K_j(t-\sigma, \sigma) R_{j0}(\sigma, s) d\sigma.$$

We examine now certain properties of the family of operators $(U(t, s))_{(t, s) \in \overline{\mathcal{A}_T}}$:

1.18 LEMMA. *We have, for $0 \leq s < t \leq T$,*

$$(I) \quad \|U(t, s) - T(t-s, s)\|_{\mathcal{L}(E_0)} \leq C(t-s)^\beta;$$

(II) *for every $\theta < \beta + \nu - 2$ there exists $C(\theta) > 0$ such that*

$$\|U(t, s) - T(t-s, s)\|_{\mathcal{L}(E_0, E_1)} \leq C(\theta)(t-s)^\theta.$$

PROOF. (I) follows almost immediately from (1.35), 1.7(II), 1.7(X), 1.16(I), 1.16(III).

We show (II) (as usual, we set $K_0(t, s) := T(t-s, s)$); for $j = 0, \dots, m$ we have

$$\int_s^t K_j(t-\sigma, \sigma) R_{j0}(\sigma, s) d\sigma = \int_s^t K_j(t-\sigma, \sigma) [R_{j0}(\sigma, s) - R_{j0}(t, s)] d\sigma + \\ + \int_s^t [K_j(t-\sigma, \sigma) - K_j(t-\sigma, t)] R_{j0}(t, s) d\sigma + K_j^{(-1)}(t-s, t) R_{j0}(t, s).$$

From 1.7(II),1.7(X), 1.16(II), 1.16(IV) we have

$$\begin{aligned} & \left\| \int_s^t K_j(t-\sigma, \sigma)[R_{j0}(\sigma, s) - R_{j0}(t, s)] d\sigma \right\|_{\mathcal{L}(E_0, E_1)} \leq \\ & \leq C \int_s^t [(t-\sigma)^{\theta-1}(\sigma-s)^{-\delta(\theta)} + (t-\sigma)^{\varrho+\nu_j-2}(\sigma-s)^{-\delta(\varrho)}] d\sigma \leq \\ & \leq C[(t-s)^{\theta-\delta(\theta)} + (t-s)^{\varrho+\nu_j-\delta(\varrho)-1}] \end{aligned}$$

for every $\theta \in]0, \beta + \nu - 1[$, $\varrho \in]1 - \nu_j, \beta + \nu - \nu_j[$ for certain $\delta(\theta)$ and $\delta(\varrho)$ less than 1. Again from 1.7(VIII), 1.7(XIV), 1.16(I), 1.16(III)

$$\begin{aligned} & \left\| \int_s^t [K_j(t-\sigma, \sigma) - K_j(t-\sigma, t)] R_{j0}(t, s) d\sigma \right\|_{\mathcal{L}(E_0, E_1)} \leq \\ & \leq C \int_s^t [(t-\sigma)^{\beta-1}(t-s)^{\beta-1} + (t-\sigma)^{\beta+\nu_j-2}(t-s)^{\beta-\nu_j}] d\sigma \leq C(t-s)^{2\beta-1}. \end{aligned}$$

Finally, from 1.7(II), 1.7(X),1.16(I), 1.16(III)

$$\|K_j^{(-1)}(t-s, s) R_{j0}(t, s)\|_{\mathcal{L}(E_0, E_1)} \leq C(t-s)^{\beta-1}.$$

So (II) is proved.

1.19 LEMMA. *Let $T > 0$, X a Banach space, $\Phi : A := \{(t, \sigma, s) \in \mathbf{R}^3 \mid 0 \leq s < \sigma < t \leq T\} \rightarrow X$, continuous and such that, for certain α, β less than 1, $C > 0$,*

$$\|\Phi(t, \sigma, s)\|_X \leq C(t-\sigma)^{-\alpha}(\sigma-s)^{-\beta}.$$

Then, $(t, s) \rightarrow \int_s^t \Phi(t, \sigma, s) d\sigma$ is continuous from Δ_T to X .

If $\alpha + \beta < 1$, it is extensible with 0 to a continuous function of domain $\overline{\Delta_T}$.

PROOF. Let (t_k, s_k) be a sequence in Δ_T converging to $(t, s) \in \Delta_T$. If $0 < \delta < ((t-s)/2)$, for k sufficiently large one has $s_k \leq s + \delta$ and $t_k \geq$

$\geq t - \delta$; so

$$\begin{aligned} \left\| \int_{s_k}^{t_k} \Phi(t_k, \sigma, s_k) d\sigma - \int_s^t \Phi(t, \sigma, s) d\sigma \right\|_X &\leq \\ &\leq \int_{s_k}^{s+\delta} \|\Phi(t_k, \sigma, s_k)\|_X d\sigma + \int_{t-\delta}^{t_k} \|\Phi(t_k, \sigma, s_k)\|_X d\sigma + \\ &+ \int_s^{s+\delta} \|\Phi(t, \sigma, s)\|_X d\sigma + \int_{t-\delta}^t \|\Phi(t, \sigma, s)\|_X d\sigma + \\ &+ \int_{s+\delta}^{t-\delta} \|\Phi(t_k, \sigma, s_k) - \Phi(t, \sigma, s)\|_X d\sigma. \end{aligned}$$

It is easily seen that δ can be chosen in such a way that, for k sufficiently large, the sum of the first four integrals is less than a fixed ε , while the last integral converges to 0 by (for example) the dominated convergence theorem as $k \rightarrow \infty$.

The second statement is immediate.

Next,

1.20 PROPOSITION. (I) *there exists $C > 0$ such that $\|U(t, s)\|_{\mathcal{L}(E_0)} \leq C$ for every $(t, s) \in \overline{\Delta_T}$;*

(II) *if $(t, s) \in \Delta_T$, $U(t, s) \in \mathcal{L}(E_0, E_1)$ and $\|U(t, s)\|_{\mathcal{L}(E_0, E_1)} \leq C(t-s)^{-1}$;*

(III) *for every $U_0 \in E_0$, $0 \leq s < T$,*

$$\lim_{t \rightarrow s^+} \|U(t, s) U_0 - U_0\|_0 = 0;$$

(IV) *if $U_0 \in D(A(s))$,*

$$\lim_{t \rightarrow s^+} \|U(t, s) U_0 - U_0\|_1 = 0;$$

(V) *the map $(t, s) \rightarrow U(t, s)$ is continuous from Δ_T to $\mathcal{L}(E_0, E_1)$; the map $(t, s, F) \rightarrow U(t, s) F$ is continuous from $\overline{\Delta_T} \times E_0$ to E_0 ;*

(VI) *let $U_0 \in E_0$; then, $t \rightarrow U(t, s) U_0 \in C^1([s, T]; E_0) \cap$*

$\cap C([s, T]; E_1)$ and, for $s < t \leq T$,

$$\partial_t U(t, s) U_0 - \mathcal{A}(t) U(t, s) U_0 = 0,$$

$$\gamma(\mathcal{B}_j(t) U(t, s) U_0) = 0, (1 \leq j \leq m).$$

(VII) let $\theta \in]0, \beta/2 - \nu[$, G_θ a space of type θ between E_0 and E_1 ; then, $(t, s) \rightarrow U(t, s) - T(t - s, s) \in C(\overline{\Delta_T}; \mathcal{L}(E_0, G_\theta))$.

PROOF. (I) follows immediately from 1.7(II) and 1.18(I).

Analogously, (II) is a consequence of 1.7(II) and 1.18(II).

(III) follows from 1.7(III) and 1.18(I).

(IV) follows from 1.15(V) and 1.15(III).

(VI) follows from 1.24(II).

It remains to prove (V) and (VII); first of all, $(t, s) \rightarrow T(t - s, s) \in C(\Delta_T; \mathcal{L}(E_0, E_1))$; this follows from 1.7(I) and 1.7(VIII). Analogously, one has from 1.7(IX) and 1.7(XIV) that, for $j = 1, \dots, m$, $K_j \in C(\Delta_T; \mathcal{L}(E_{\mu_j}, E_1))$. This implies that, for $0 \leq j, k \leq m$, $N_{jk} \in C(\Delta_T; \mathcal{L}(E_{\mu_k}, E_{\mu_j}))$. From the estimates of [8]II.4.2 we have also that the series (1.31) converges uniformly in Δ_T ; this implies that for every $(j, k) R_{jk} \in C(\Delta_T; \mathcal{L}(E_{\mu_k}, E_{\mu_j}))$. Now, let $j \in \{0, \dots, m\}$; then, if $0 \leq s < t \leq T$,

$$(1.36) \quad \int_s^t K_j(t - \sigma, \sigma) R_{j0}(\sigma, s) d\sigma = \int_s^t K_j(t - \sigma, \sigma) [R_{j0}(\sigma, s) - R_{j0}(t, s)] d\sigma \\ + \int_s^t [K_j(t - \sigma, \sigma) - K_j(t - \sigma, t)] R_{j0}(t, s) d\sigma + K_j^{(-1)}(t - s, t) R_{j0}(t, s).$$

The function $(t, s) \rightarrow K_j^{(-1)}(t - s, t) R_{j0}(t, s)$ belongs to $C(\Delta_T; \mathcal{L}(E_0, E_1))$. The continuity of the other summands with values in $\mathcal{L}(E_0; E_1)$ follows from 1.19. So $(t, s) \rightarrow U(t, s)$ is continuous from Δ_T to $\mathcal{L}(E_0, E_1)$. The second statement of (V) follows from the first, 1.7(III), 1.7(VIII) and 1.18(I).

Finally, from 1.18 one has that, if $\theta < \beta/2 - \nu$, $\|U(t, s) - T(t - s, s)\|_{\mathcal{L}(E_0, G_\theta)} \rightarrow 0$ as $t - s \rightarrow 0$ uniformly in s . This proves (VII).

1.21 PROPOSITION (THE VARIATION OF PARAMETER FORMULA). (I) Let U be the mild solution in $[s, T]$ ($0 \leq s < T$) with data $(0, F, (0)_{1 \leq j \leq m})$;

then, if $s \leq t \leq T$

$$U(t) = \int_s^t U(t, \sigma) F(\sigma) d\sigma.$$

PROOF. We have

$$U(t) = \int_s^t T(t - \sigma, \sigma) R(\sigma) d\sigma + \sum_{j=1}^m \int_s^t K_j(t - \sigma, \sigma) S_j(\sigma) d\sigma$$

with

$$R(t) = f(t) + \int_s^t N_{00}(t, \tau) R(\tau) d\tau + \sum_{k=1}^m \int_s^t N_{0k}(t, \tau) S_k(\tau) d\tau,$$

$$S_j(t) = \int_s^t N_{j0}(t, \tau) R(\tau) d\tau + \sum_{k=1}^m \int_s^t N_{jk}(t, \tau) S_k(\tau) d\tau$$

for $1 \leq j \leq m$. It follows from (1.30)

$$R(t) = F(t) + \int_s^t R_{00}(t, \tau) F(\tau) d\tau,$$

$$S_j(t) = \int_s^t R_{j0}(t, \tau) F(\tau) d\tau, (1 \leq j \leq m),$$

so that

$$\begin{aligned} U(t) &= \int_s^t T(t - \sigma, \sigma) \left[F(\sigma) + \int_s^\sigma R_{00}(\sigma, \tau) F(\tau) d\tau \right] d\sigma + \\ &+ \sum_{j=1}^m \int_s^t K_j(t - \sigma, \sigma) \left[\int_s^\sigma R_{j0}(\sigma, \tau) F(\tau) d\tau \right] d\sigma = \\ &= \int_s^t \left[T(t - \sigma, \sigma) + \int_\sigma^t T(t - \tau, \tau) R_{00}(\tau, \sigma) d\tau + \right. \\ &\left. + \sum_{j=1}^m \int_\sigma^t K_j(\sigma - \tau, \tau) R_{j0}(\tau, \sigma) d\tau \right] F(\sigma) d\sigma = \int_s^t U(t, \sigma) F(\sigma) d\sigma. \end{aligned}$$

1.22 COROLLARY. Assume that the problem (1.8) with $\gamma g_j \equiv 0$ for $j = 1, \dots, m$, has a classical solution U ; then, U coincides with the mild solution of the same problem.

PROOF. Owing to 1.15(V) and 1.21, if $0 < h < T - s$ and $s + h \leq t \leq T$,

$$U(t) = U(t, s + h) U(s + h) + \int_{s+h}^t U(t, \sigma) F(\sigma) d\sigma.$$

Letting $h \rightarrow 0^+$ and using 1.20(I) and 1.20(V), we obtain that, if $s < t \leq T$,

$$U(t) = U(t, s) U_0 + \int_s^t U(t, \sigma) F(\sigma) d\sigma.$$

1.23 DEFINITION. If $(t, s) \in \overline{\Delta_T}$, we put

$$(1.37) \quad V(t, s) = \begin{cases} \int_s^t U(\sigma, s) d\sigma & \text{if } s < t, \\ 0 & \text{if } s = t. \end{cases}$$

1.24 PROPOSITION. (I) For every $(t, s) \in \overline{\Delta_T}$ $V(t, s) \in \mathcal{L}(E_0, E_1)$ and there exists $C > 0$ such that $\|V(t, s)\|_{\mathcal{L}(E_0, E_1)} \leq C$ for every $(t, s) \in \overline{\Delta_T}$;

(II) the map $(t, s) \rightarrow V(t, s)$ is continuous from $\overline{\Delta_T}$ to $\mathcal{L}(E_0)$ and from Δ_T to $\mathcal{L}(E_0, E_1)$;

(III) the map $(t, s, F) \rightarrow V(t, s)F$ is continuous from $\overline{\Delta_T} \times E_0$ to E_1 .

PROOF. If $(t, s) \in \Delta_T$,

$$(1.38) \quad V(t, s) = T^{(-1)}(t - s, s) + \int_s^t [U(\tau, s) - T(\tau - s, s)] d\tau,$$

which implies (I) owing to 1.7(II) and 1.18(II).

(II) follows from 1.20(I), 1.20(V), 1.19, 1.17(I), 1.17(VIII) and (1.38).

(III) follows from (II), 1.7(VIII), 1.7(IV), 1.18(II) and 1.19.

2. Evolution operators for Volterra integrodifferential systems.

In this section we shall extend part of the results of the previous one to the case of Volterra integrodifferential systems. We continue to assume that the conditions (L1) – (L4) and (k5) are fulfilled and we keep the meaning of the notations (1.3), (1.4), (1.5), (1.6), (1.7), (1.10), (1.11), (1.12), (1.13), with the condition (1.9), so that (h1) – (h7) are still satisfied.

In the following the next result will be useful:

2.1 PROPOSITION. *There exists a sequence of operators $(P_r)_{r \in \mathbf{N}}$ in $\mathcal{L}(E_0, E_1)$ such that for every $F \in E_1$*

$$\lim_{r \rightarrow \infty} \|P_r F - F\|_1 = 0.$$

Moreover, there exists $C > 0$ such that for every $r \in \mathbf{N}$

$$\|P_r|_{E_1}\|_{\mathcal{L}(E_1)} \leq C.$$

PROOF. Let E be a common linear bounded extension operator from $W^{s,p}(\Omega)$ to $W^{s,p}(\mathbf{R}^n)$ for every $s \in [0, 2m]$ (see for this [TR] 3.3.4). Fix $x^0 \in \overline{\Omega}$ and consider the operator B of domain $\prod_{k=0}^{l-1} W^{(l-k)d,p}(\mathbf{R}^n)$ such that, for $V = (V_0, \dots, V_{l-1}) \in \prod_{k=0}^{l-1} W^{(l-k)d,p}(\mathbf{R}^n)$,

$$B(V_0, \dots, V_{l-1}) = \left(V_1, \dots, V_{l-1}, - \sum_{k=0}^{l-1} \alpha_{l-k}(0, x^0, \partial_x) V_k \right).$$

Then, by [6] 1.6, B is the infinitesimal generator of an analytic semigroup $\{e^{tB} | t \geq 0\}$ in the space $\prod_{k=0}^{l-1} W^{(l-k-1)d,p}(\mathbf{R}^n)$. So, if r_0 is sufficiently large, for every $r \in \mathbf{N}$ $r_0 + r \in \rho(B)$ and

$$\lim_{r \rightarrow \infty} \|(r + r_0)(r + r_0 - B)^{-1} F - F\|_{\prod_{k=0}^{l-1} W^{(l-k-1)d,p}(\mathbf{R}^n)} = 0$$

for every $F \in \prod_{k=0}^{l-1} W^{(l-k-1)d,p}(\mathbf{R}^n)$ and, as $D(B) = \prod_{k=0}^{l-1} W^{(l-k)d,p}(\mathbf{R}^n)$, for

every $F \in \prod_{k=0}^{l-1} W^{(l-k)d,p}(\mathbf{R}^n)$

$$\lim_{r \rightarrow \infty} \|(r + r_0)(r + r_0 - B)^{-1} F - F\|_{\prod_{k=0}^{l-1} W^{(l-k)d,p}(\mathbf{R}^n)} = 0.$$

Set now, for $r \in \mathbf{N}$ and $F = (F_0, \dots, F_{l-1}) \in E_0$,

$$P_r F := (r + r_0) R(r + r_0 - B)^{-1} (EF_0, \dots, EF_{l-1}),$$

where R is the restriction operator from \mathbf{R}^n to Ω . The sequence $(P_r)_{r \in \mathbf{N}}$ has the requested properties.

We introduce now an element $\mathcal{C} \in C(\Delta_T; \mathcal{L}(E_1, E_0))$ such that

(C) $\mathcal{C} \in C(\Delta_T; \mathcal{L}(E_1, E_0))$ is such that, for some $\alpha, \eta \in]0, 1[$, $C > 0$,

$$\|\mathcal{C}(t, s) - \mathcal{C}(\tau, s)\|_{\mathcal{L}(E_1, E_0)} \leq C(t - \tau)^\alpha (\tau - s)^{-\eta}$$

if $0 \leq s < \sigma < t \leq T$,

$$\|\mathcal{C}(t, s) - \mathcal{C}(t, \sigma)\|_{\mathcal{L}(E_1, E_0)} \leq C(t - \sigma)^{-\eta} (\sigma - s)^\alpha$$

if $0 \leq s \leq \sigma < t \leq T$. It is not restrictive to assume $\alpha < \eta$ and this is what we shall do in the following.

We introduce now the following integrodifferential system:

$$(2.1) \quad \begin{cases} U'(t) = A(t) U(t) + \int_s^t \mathcal{C}(t, \sigma) U(\sigma) d\sigma + F(t), & t \in]s, T], \\ U(s) = U_0. \end{cases}$$

Here $0 \leq s < T$, $F \in C([s, T]; E_0)$, $U_0 \in E_0$ and $A(t)$ was defined in (1.16).

2.2 LEMMA. *Let \mathcal{C} satisfy the assumption (C) with $\alpha < \eta$; then, for some $C > 0$, for every $(t, s) \in \Delta_T$,*

$$\|\mathcal{C}(t, s)\|_{\mathcal{L}(E_1, E_0)} \leq C(t - s)^{-\eta}.$$

PROOF. We have

$$\begin{aligned} \|\mathcal{C}(t, s)\|_{\mathcal{L}(E_1, E_0)} &\leq \|\mathcal{C}(t, s) - \mathcal{C}(t, 0)\|_{\mathcal{L}(E_1, E_0)} + \\ &+ \|\mathcal{C}(t, 0) - \mathcal{C}(T, 0)\|_{\mathcal{L}(E_1, E_0)} + \|\mathcal{C}(T, 0)\|_{\mathcal{L}(E_1, E_0)} \leq \\ &\leq C[(t - s)^{-\eta} s^\alpha + (T - t)^\alpha t^{-\eta} + 1] \leq C(t - s)^{-\eta}. \end{aligned}$$

Now we pass to define strict and classical solutions of (2.1):

2.3 DEFINITION. *A strict solution of (2.1) is an element U of $C^1([s, T]; E_0) \cap C([s, T]; E_1)$ such that $U(t) \in D(A(t))$ for every $t \in [s, T]$ and (2.1) is satisfied, again for every $t \in [s, T]$.*

A classical solution of (2.1) is an element U of $C^1(]s, T]; E_0) \cap C(]s, T]; E_1) \cap C([s, T]; E_0)$, such that $U(t) \in D(A(t))$ for every

$t \in]s, T]$, $U(s) = U_0$, $t \rightarrow \int_s^t U(\tau) d\tau \in C([s, T]; E_1)$, $\|U(t)\|_1 \leq C(t-s)^{-1}$ for every $t \in]s, T]$ and the first equation in (2.1) is satisfied for every $t \in]s, T]$.

In this case the integral in (2.1) exists (and is intended) in generalized sense,

$$(2.2) \quad \int_s^t \mathcal{C}(t, \sigma) U(\sigma) d\sigma := E_0 - \lim_{\varepsilon \rightarrow 0^+} \int_{s+\varepsilon}^t \mathcal{C}(t, \sigma) U(\sigma) d\sigma = \\ = \int_s^t [\mathcal{C}(t, \sigma) - \mathcal{C}(t, s)] U(\sigma) d\sigma + \mathcal{C}(t, s) \int_s^t U(\sigma) d\sigma.$$

Of course, if (2.1) has a strict solution, necessarily $U_0 \in D(A(s))$

2.4 LEMMA. Assume that the assumptions (L1)-(L7) and (k5) are satisfied; let $U_0 \in E_0$, $s \in [0, T]$; set, for $s \leq t \leq T$,

$$U(t) = U(t, s) U_0.$$

Then,

(I) $U \in C^1(]s, T]; E_0) \cap C(]s, T]; E_1) \cap C([s, T]; E_0)$ and, if $t \in]s, T]$, $U(t) \in D(A(t))$;

(II) $t \rightarrow \int_s^t U(\tau) d\tau \in C([s, T]; E_1)$ and there exists $C > 0$ such that for every $t \in]s, T]$ $\|U(t)\|_1 \leq C(t-s)^{-1}$;

(III) for $s < t \leq T$

$$U'(t) - A(t) U(t) - \int_s^t \mathcal{C}(t, \sigma) U(\sigma) d\sigma = M(t, s) U_0,$$

where, if $(t, s) \in \Delta_T$

$$(2.3) \quad M(t, s) := \mathcal{C}(t, s) V(t, s) + \int_s^t [\mathcal{C}(t, \sigma) - \mathcal{C}(t, s)] U(\sigma, s) d\sigma.$$

PROOF. The result follows easily from 1.20 and 1.24.

2.5 LEMMA. Let $s \in [0, T[$ and $R \in C_\mu^e(]s, T]; E_0$ for certain $\varrho, \mu \in]0, 1[$ with $\varrho < \mu$; set, for $s \leq t \leq T$,

$$U(t) := \int_s^t U(t, \sigma) R(\sigma) d\sigma;$$

then,

(I) $u \in C(]s, T]; E_1) \cap C^1(]s, T]; E_0) \cap C([s, T]; E_0)$ and, for some $C > 0$,

$$(2.4) \quad \|U(t)\|_1 \leq C(t-s)^{-\mu}$$

for every $t \in]s, T];$

(II) for every t in $]s, T]$ $U(t) \in D(A(t))$ and

$$U'(t) - A(t)U(t) = R(t);$$

(III) for every $t \in]s, T]$ the integral $\int_s^t \mathcal{C}(t, \sigma) U(\sigma) d\sigma$ exists in Bochner's sense and

$$\int_s^t \mathcal{C}(t, \sigma) U(\sigma) d\sigma = \int_s^t M(t, \sigma) R(\sigma) d\sigma.$$

PROOF. We have $U(t) = \int_s^t U(t, \sigma)[R(\sigma) - R(t)] d\sigma + \int_s^t U(t, \sigma) R(t) d\sigma$.
Now,

$$\left\| \int_s^t U(t, \sigma)[R(\sigma) - R(t)] d\sigma \right\|_1 \leq C \int_s^t (t-\sigma)^{\varrho-1} (\sigma-s)^{-\mu} d\sigma = C(t-s)^{\varrho-\mu}.$$

So, owing to 1.5(II) (applied in case $U_0 = 0, g_j = 0$ for $j = 1, \dots, m$ and F constant), we have that $U \in C(]s, T]; E_1)$. Moreover, by 1.7(II), 1.7(VIII), 1.18(II)

$$\begin{aligned} & \left\| \int_s^t U(t, \sigma) R(t) d\sigma \right\|_1 \leq \\ & \leq \|T^{(-1)}(t-s, t) R(t)\|_1 + \left\| \int_s^t [T(t-\sigma, \sigma) - T(t-\sigma, t)] R(t) d\sigma \right\|_1 + \\ & + \left\| \int_s^t [U(t, \sigma) - T(t-\sigma, \sigma)] R(t) d\sigma \right\|_1 \leq C \|R(t)\|_0 \leq C(t-s)^{-\mu}. \end{aligned}$$

Now we show that $U \in C^1(]s, T]; E_0)$; let $\varepsilon \in]0, T - s[$. If $s + \varepsilon < t \leq T$, we have from 1.20(VI), setting $U_\varepsilon(t) := \int_s^{t-\varepsilon} U(t, \sigma) R(\sigma) d\sigma$,

$$\begin{aligned} U'_\varepsilon(t) &= U(t, t - \varepsilon) R(t - \varepsilon) + \int_s^{t-\varepsilon} A(t) U(t, \sigma) R(\sigma) d\sigma = \\ &= R(t) + \int_s^t A(t) U(t, \tau) [R(\tau) - R(t)] d\tau + A(t) \int_s^t U(t, \tau) R(t) d\tau - \\ &- \int_{t-\varepsilon}^t A(t) U(t, \tau) [R(\tau) - R(t)] d\tau + U(t, t - \varepsilon) [R(t - \varepsilon) - R(t)] - \\ &- \mathcal{A}(t) \int_{t-\varepsilon}^t [U(t, \tau) - T(t - \tau, t)] R(t) d\tau + [U(t, t - \varepsilon) - T(\varepsilon, t - \varepsilon)] R(t) + \\ &+ [T(\varepsilon, t - \varepsilon) - T(\varepsilon, t)] R(t) + [T(\varepsilon, t) - 1 - A(t) T^{(-1)}(\varepsilon, t)] R(t). \end{aligned}$$

We have

$$\begin{aligned} \left\| \int_{t-\varepsilon}^t A(t) U(t, \tau) [R(\tau) - R(t)] d\tau \right\|_0 &\leq C\varepsilon^\varrho (t - \varepsilon - s)^{-\mu}, \\ \|U(t, t - \varepsilon) [R(t - \varepsilon) - R(t)]\|_0 &\leq C\varepsilon^\varrho (t - \varepsilon - s)^{-\mu}, \\ \left\| \mathcal{A}(t) \left(\int_{t-\varepsilon}^t [U(t, \tau) - T(t - \tau, t)] R(t) d\tau \right) \right\|_0 &\leq \\ \leq C \int_{t-\varepsilon}^t [\|U(t, \tau) - T(t - \tau, \tau)\|_{\mathcal{L}(E_1, E_0)} + \|T(t - \tau, \tau) - T(t - \tau, t)\|_{\mathcal{L}(E_1, E_0)}] \|R(t)\|_0 d\tau &\leq \\ &\leq C\varepsilon^\theta \|R(t)\|_0 \end{aligned}$$

for every θ less than $\beta + \nu - 1$,

$$\| [U(t, t - \varepsilon) - T(\varepsilon, t - \varepsilon)] R(t) \|_0 \leq C\varepsilon^\beta \|R(t)\|_0,$$

$$\| [T(\varepsilon, t - \varepsilon) - T(\varepsilon, t)] R(t) \|_0 \leq C\varepsilon^\beta \|R(t)\|_0.$$

Finally, owing to 1.7(VI),

$$[T(\varepsilon, t) - 1 - A(t) T^{(-1)}(\varepsilon, t)] R(t) = 0.$$

So we have that

$$\lim_{\varepsilon \rightarrow 0^+} \|U_\varepsilon(t) - U(t)\|_0 = 0,$$

$$\lim_{\varepsilon \rightarrow 0^+} \left\| U'_\varepsilon(t) - R(t) - \int_s^t A(t) U(t, \tau) [R(\tau) - R(t)] d\tau - A(t) \int_s^t U(t, \tau) R(t) d\tau \right\|_0 = 0$$

for every $t \in]s, T]$, uniformly in every interval $[s + \delta, T]$ for every $\delta \in]0, T - s[$. From the previous considerations we get (II).

To obtain (III), observe first that in our case, owing to (2.4), the integral (2.2) is defined even in Bochner's sense; moreover, for $t \in]s, T]$, owing to 2.1,

$$\begin{aligned} \int_s^t \mathcal{C}(t, \tau) U(\tau) d\tau &= \lim_{r \rightarrow \infty} \int_s^t \mathcal{C}(t, \tau) P_r U(\tau) d\tau = \\ &= \lim_{r \rightarrow \infty} \int_s^t \left(\int_s^\tau \mathcal{C}(t, \tau) P_r U(\tau, \sigma) R(\sigma) d\sigma \right) d\tau \end{aligned}$$

(owing to $\| \mathcal{C}(t, \tau) P_r U(\tau, \sigma) R(\sigma) \|_0 \leq C(r)(t - \tau)^{-\eta}(\sigma - s)^{-\mu}$)

$$= \lim_{r \rightarrow \infty} \left\{ \int_s^t \left(\int_\sigma^t [\mathcal{C}(t, \tau) - \mathcal{C}(t, \sigma)] P_r U(\tau, \sigma) d\tau \right) R(\sigma) d\sigma + \int_s^t \mathcal{C}(t, \sigma) P_r V(t, \sigma) R(\sigma) d\sigma \right\}$$

(owing to Fubini's theorem)

$$= \int_s^t M(t, \sigma) R(\sigma) d\sigma.$$

2.6 REMARK. Let $U \in C^1([s, T]; E_0) \cap C([s, T]; E_1)$ be such that $U(t) \in D(A(t))$ for every $t \in [s, T]$ (that is, $\gamma \beta_j(t) U(t) = 0$ for every $j = 1, \dots, m, t \in [s, T]$). Then, we have for every $t \in [s, T]$

$$\int_s^t \mathcal{C}(t, \sigma) U(\sigma) d\sigma = \int_s^t M(t, \sigma) R(\sigma) d\sigma + M(t, s) U(s)$$

if $R(t) = U'(t) - A(t) U(t)$.

In fact, by 1.15(V), for every $t \in [s, T]$ $U(t) = U(t, s) U(s) + \int_s^t U(t, \sigma) R(\sigma) d\sigma$ and we can repeat without change the proof of 2.5(III), observing that owing to 1.15(III), as $U(s) \in D(A(s))$,

$$\left\| \int_s^t U(t, \sigma) R(\sigma) d\sigma \right\|_1 \leq \|U(t)\|_1 + \|U(t, s) U(s)\|_1 \leq C.$$

Now we come back to the system (2.1) and look for a solution U in the form

$$(2.5) \quad U(t) = U(t, s) U_0 + \int_s^t U(t, \sigma) R(\sigma) d\sigma.$$

Owing to 2.4 and 2.5, if U is of the form (2.5), we have, at least formally,

$$(2.6) \quad \begin{aligned} U'(t) - A(t) U(t) - \int_s^t \mathcal{C}(t, \sigma) U(\sigma) d\sigma = \\ = R(t) - M(t, s) U_0 - \int_s^t M(t, \sigma) R(\sigma) d\sigma. \end{aligned}$$

Therefore, we are reduced to solve the Volterra integral equation

$$(2.7) \quad R(t) = \int_s^t M(t, \sigma) R(\sigma) d\sigma + F(t) + M(t, s) U_0.$$

2.7 LEMMA. (I) *There exists $C > 0$ such that for every $(t, s) \in \Delta_T$*

$$\|M(t, s)\|_{\mathcal{L}(E_0)} \leq C(t-s)^{-\eta};$$

(II) *for every $\theta < \min\{\alpha, 1 - \eta\}$ there exist $\delta(\theta) < 1$, $C(\theta) > 0$ such that, if $0 \leq s < \tau \leq t \leq T$*

$$\|M(t, s) - M(\tau, s)\|_{\mathcal{L}(E_0)} \leq C(\theta)(t-\tau)^\theta(\tau-s)^{-\delta(\theta)}.$$

PROOF. (I) We have

$$\begin{aligned} \|M(t, s)\|_{\mathcal{L}(E_0)} &\leq \|C(t, s)V(t, s)\|_{\mathcal{L}(E_0)} + \left\| \int_s^t [C(t, \sigma) - C(t, s)]U(\sigma, s) d\sigma \right\|_{\mathcal{L}(E_0)} \leq \\ &\leq C[(t-s)^{-\eta} + \int_s^t (t-\sigma)^{-\eta}(\sigma-s)^{\alpha-1} d\sigma] \leq C(t-s)^{-\eta}, \end{aligned}$$

using 2.2, 1.24(I) and 1.20(II).

(II) First of all,

$$\|V(t, s) - V(\tau, s)\|_{\mathcal{L}(E_0, E_1)} = \left\| \int_\tau^t U(\sigma, s) d\sigma \right\|_{\mathcal{L}(E_0, E_1)} \leq C(t-\tau)(\tau-s)^{-1}.$$

Recalling 1.24(I), we obtain that for some $C > 0$ and every $\xi \in [0, 1]$

$$\|V(t, s) - V(\tau, s)\|_{\mathcal{L}(E_0, E_1)} \leq C(t-\tau)^\xi(\tau-s)^{-\xi}.$$

It follows that

$$\begin{aligned} \|C(t, s)V(t, s) - C(\tau, s)V(\tau, s)\|_{\mathcal{L}(E_0)} &\leq \|C(t, s)\|_{\mathcal{L}(E_1, E_0)} \|V(t, s) - V(\tau, s)\|_{\mathcal{L}(E_0, E_1)} + \\ &+ \|C(t, s) - C(\tau, s)\|_{\mathcal{L}(E_1, E_0)} \|V(\tau, s)\|_{\mathcal{L}(E_0, E_1)} \leq \\ &\leq C[(t-\tau)^\xi(\tau-s)^{-\eta-\xi} + (t-\tau)^\alpha(\tau-s)^{-\eta}]. \end{aligned}$$

Observe that $\eta + \xi < 1$ if and only if $\xi < 1 - \eta$. Next,

$$\begin{aligned} \left\| \int_\tau^t [C(t, \sigma) - C(t, s)]U(\sigma, s) d\sigma \right\|_{\mathcal{L}(E_0)} &\leq C \int_\tau^t (t-\sigma)^{-\eta}(\sigma-s)^{\alpha-1} d\sigma \leq \\ &\leq C(t-\tau)^{1-\eta}(\tau-s)^{\alpha-1}. \end{aligned}$$

Finally, if $s < \sigma < \tau \leq t$,

$$\begin{aligned} \|C(t, \sigma) - C(t, s) - C(\tau, \sigma) + C(\tau, s)\|_{\mathcal{L}(E_1, E_0)} &\leq \\ &\leq C \min\{(\sigma-s)^\alpha, (t-\tau)^\alpha\}(\tau-\sigma)^{-\eta} \leq C(t-\tau)^{\theta\alpha}(\tau-s)^{(1-\theta)\alpha-\eta} \end{aligned}$$

for every $\theta \in]0, 1[$. So (II) is proved.

We come back to the integral equation (2.7); we introduce the notion of mild solution:

2.8 DEFINITION. Let $U_0 \in E_0$, $F \in C([s, T]; E_0)$. A mild solution of (2.1) is a function U of the form

$$U(t) = U(t, s) U_0 + \int_s^t U(t, \sigma) R(\sigma) d\sigma,$$

with $R \in L^1(]s, T[; E_0)$ solution of (2.7) (in the sense that it is satisfied for almost every t).

We have

2.9 PROPOSITION. For every $U_0 \in E_0$, $F \in C([s, T]; E_0)$ the problem (2.1) has a unique mild solution. In this case $R \in C(]s, T[; E_0)$ and there exists $C > 0$ such that, if $s < t \leq T$,

$$(2.8) \quad \|R(t)\|_0 \leq C(t-s)^{-\eta}.$$

PROOF. Owing to 2.7(I), for some $C > 0$, for every $t \in]s, T[$,

$$\|F(t) + M(t, s) U_0\|_0 \leq C(t-s)^{-\eta}.$$

Then we can apply 1.12 with $r = 0$, $X_0 = E_0$, $\gamma_{00} = \sigma_0 = \eta$.

2.10 PROPOSITION. Let U be the mild solution of (2.1); then, $U \in C([s, T]; E_0)$.

PROOF. It follows easily from 1.20(II), 1.20(V), (2.8) and 1.19.

2.11 PROPOSITION. Assume that, for some $\varepsilon > 0$, $F \in C^\varepsilon([s, T]; E_0)$. Then, the mild solution of (2.1) is a classical solution.

PROOF. If $s < \tau \leq t \leq T$,

$$\begin{aligned} & \left\| \int_s^t M(t, \sigma) R(\sigma) d\sigma - \int_s^\tau M(\tau, \sigma) R(\sigma) d\sigma \right\|_0 \leq \\ & \leq \left\| \int_\tau^t M(t, \sigma) R(\sigma) d\sigma \right\|_0 + \left\| \int_s^\tau [M(t, \sigma) - M(\tau, \sigma)] R(\sigma) d\sigma \right\|_0 \leq \\ & \leq C \left[\int_x^t (t-\sigma)^{-\eta} (\sigma-s)^{-\eta} d\sigma + \int_s^\tau (t-\tau)^\theta (\tau-\sigma)^{-\delta(\theta)} (\sigma-s)^{-\eta} d\sigma \right] \end{aligned}$$

for every $\theta < \min\{\alpha, 1 - \eta\}$ with $\delta(\theta) < 1$. This expression can be majorized with

$$C[(t - \tau)^{1-\eta}(\tau - s)^{-\eta} + (t - \tau)^\theta(\tau - s)^{1-\eta-\delta(\theta)}].$$

This, together with the assumptions on F , 2.7(II) and (2.7), shows that R satisfies the assumptions of 2.5. So the result follows from 2.4 and 2.5.

2.12 PROPOSITION. *Let $s \in [0, T[$, $U_0 \in D(A(s))$, $F \in C^\varepsilon([s, T]; E_0)$ for some $\varepsilon > 0$. Then the mild solution of (2.1) is a strict solution.*

PROOF. Let $U_0 \in D(A(s))$; then, if $s < t \leq T$,

$$\|V(t, s) U_0\|_1 = \left\| \int_s^t U(\tau, s) U_0 d\tau \right\|_1 \leq C(t - s),$$

owing to 1.15(III) and 1.15(V). Moreover, if $s < \tau \leq t \leq T$,

$$\|[V(t, s) - V(\tau, s)] U_0\|_1 = \left\| \int_\tau^t U(\sigma, s) U_0 d\sigma \right\|_1 \leq C(t - \tau).$$

It follows that, if $s < \tau \leq t \leq T$,

$$\begin{aligned} & \| \mathcal{C}(t, s) V(t, s) U_0 - \mathcal{C}(\tau, s) V(\tau, s) U_0 \|_0 \leq \\ & \leq \| \mathcal{C}(t, s) [V(t, s) - V(\tau, s)] U_0 \|_0 + \| [\mathcal{C}(t, s) - \mathcal{C}(\tau, s)] V(\tau, s) U_0 \|_0 \leq \\ & \leq C[(t - s)^{-\eta}(t - \tau) + (t - \tau)^\alpha(\tau - s)^{1-\eta}] \leq C[(t - \tau)^{1-\eta} + (t - \tau)^\alpha]. \end{aligned}$$

Moreover,

$$\left\| \int_\tau^t [\mathcal{C}(t, \sigma) - \mathcal{C}(t, s)] U(\sigma, s) U_0 d\sigma \right\|_0 \leq C \int_\tau^t (t - \sigma)^{-\eta} (\sigma - s)^\alpha d\sigma \leq C(t - \tau)^{1-\eta},$$

and, if $s \leq \sigma < \tau \leq t \leq T$,

$$\| \mathcal{C}(t, \sigma) - \mathcal{C}(t, s) - \mathcal{C}(\tau, \sigma) + \mathcal{C}(\tau, s) \|_{\mathcal{L}(E_1, E_0)} \leq C(t - \tau)^\alpha (\tau - \sigma)^{-\eta},$$

so that

$$\left\| \int_s^\tau [\mathcal{C}(t, \sigma) - \mathcal{C}(t, s) - \mathcal{C}(\tau, \sigma) + \mathcal{C}(\tau, s)] U(\sigma, s) U_0 d\sigma \right\|_0 \leq C(t - \tau)^\alpha.$$

The previous estimates show that $t \rightarrow F(t) + M(t, s) U_0 \in C^\varepsilon([s, T]; E_0)$

for some positive ε . Usual arguments (using (2.7)) give that R is bounded with values in E_0 and consequently hölder continuous in $[s, T]$. So the result follows from 1.15(III) and 1.15(V).

2.13 PROPOSITION. *The system (2.1) has at most one strict solution; such solution, if existing, coincides with the mild solution of the same problem.*

PROOF. Obviously, the first statement is a consequence of the second. So, let U be a strict solution of (2.1). Set $g(t) := \int_s^t \mathcal{C}(t, \sigma) U(\sigma) d\sigma$. Then U is the strict solution of

$$(2.9) \quad \begin{cases} U'(t) = A(t) U(t) + F(t) + g(t), & t \in [s, T], \\ U(s) = U_0. \end{cases}$$

We indicate now with M the mild solution of (2.1); then, if $s \leq t \leq T$,

$$M(t) = U(t, s) U_0 + \int_s^t U(t, \sigma) r(\sigma) d\sigma,$$

where

$$r(t) = F(t) + M(t, s) U_0 + \int_s^t M(t, \sigma) r(\sigma) d\sigma.$$

Now we set $\phi(t) := r(t) - F(t)$; then, for $t \in]s, T]$,

$$\begin{aligned} \phi(t) &= M(t, s) U_0 + \int_s^t M(t, \sigma) r(\sigma) d\sigma = \\ &= M(t, s) U_0 + \int_s^t M(t, \sigma) F(\sigma) d\sigma + \int_s^t M(t, \sigma) \phi(\sigma) d\sigma. \end{aligned}$$

On the other hand, by 2.6 and (2.9),

$$g(t) = M(t, s) U_0 + \int_s^t M(t, \sigma) F(\sigma) d\sigma + \int_s^t M(t, \sigma) g(\sigma) d\sigma.$$

So ϕ and g are solutions of the same Volterra integral equation; it follows from 1.12 that $\phi \equiv g$, so that $r \equiv F + g$ and, by 1.15(V), $M \equiv U$.

2.14 Concerning the mild solution U of (2.1), we have that, if $s < t \leq T$,

$$U(t) = U(t, s) U_0 + \int_s^t U(t, \sigma) R(\sigma) d\sigma,$$

where R solves (2.7). With the considerations following 1.15, we get

$$(2.10) \quad R(t) = M(t, s) U_0 + F(t) + \int_s^t A(t, \sigma) [M(\sigma, s) U_0 + F(\sigma)] d\sigma,$$

where, if $(t, s) \in \Delta_T$,

$$(2.11) \quad A(t, s) = \sum_{k=1}^{\infty} M_k(t, s),$$

with

$$(2.12) \quad M_1(t, s) = M(t, s), M_{k+1}(t, s) = \\ = \int_s^t M(t, \sigma) M_k(\sigma, s) d\sigma = \int_s^t M_k(t, \sigma) M(\sigma, s) d\sigma.$$

We have also

$$(2.13) \quad A(t, s) = M(t, s) + \int_s^t M(t, \sigma) A(\sigma, s) d\sigma = \\ = M(t, s) + \int_s^t A(t, \sigma) M(\sigma, s) d\sigma.$$

2.15 PROPOSITION. (I) *there exists $C > 0$ such that, if $(t, s) \in \Delta_T$,*

$$\|A(t, s)\|_{\mathcal{L}(E_0)} \leq C(t-s)^{-\eta};$$

(II) let $\theta < \min \{1 - \eta, \alpha\}$; then there exist $C > 0$ and $\delta(\theta) < 1$ such that, if $0 \leq s < \tau \leq t \leq T$,

$$\|A(t, s) - A(\tau, s)\|_{\mathcal{L}(E_0)} \leq C(t - \tau)^\theta (\tau - s)^{-\delta(\theta)}.$$

PROOF. It follows with usual arguments from 1.12 and 2.7.

2.16 We consider now the mild solution U with data $(U_0, 0)$, with $U_0 \in E_0$; it is easily seen that, if $s < t \leq T$,

$$U(t) = S(t, s) U_0,$$

where, for $(t, s) \in \overline{\Delta_T}$, we set

$$(2.14) \quad S(t, s) := U(t, s) + \int_s^t U(t, \sigma) A(\sigma, s) d\sigma.$$

2.17 LEMMA. (I) there exists $C > 0$ such that for every $(t, s) \in \overline{\Delta_T}$

$$\|S(t, s) - T(t - s, s)\|_{\mathcal{L}(E_0)} \leq C(t - s)^{\min\{\beta, 1 - \eta\}};$$

(II) for every $(t, s) \in \Delta_T$ $S(t, s) \in \mathcal{L}(E_0, E_1)$ and for every $\theta < \min\{\beta + \nu - 2, \alpha - 1, -\eta\}$ there exists $C(\theta) > 0$ such that

$$\|S(t, s) - T(t - s, s)\|_{\mathcal{L}(E_0, E_1)} \leq C(\theta)(t - s)^\theta.$$

PROOF. (I) follows from 1.18(I) and 2.15(I).

Concerning (II), we have

$$(2.15) \quad \int_s^t U(t, \sigma) A(\sigma, s) d\sigma = \\ = \int_s^t U(t, \sigma)[A(\sigma, s) - A(t, s)] d\sigma + \int_s^t [U(t, \sigma) - T(t - \sigma, \sigma)] d\sigma A(t, s) + \\ + \int_s^t [T(t - \sigma, \sigma) - T(t - \sigma, t)] d\sigma A(t, s) + T^{(-1)}(t - s, t) A(t, s)$$

and the result follows from 1.20(II), 2.15(II), 1.18(II), 2.15(I), 1.7(IV).

The following result generalizes 1.20:

2.18 PROPOSITION. (I) *There exists $C > 0$ such that for every $(t, s) \in \overline{\Delta_T}$*

$$\|S(t, s)\|_{\mathcal{L}(E_0)} \leq C;$$

S is differentiable with respect to t with values in $\mathcal{L}(E_0)$ in Δ_T and

$$\|\partial_t S(t, s)\|_{\mathcal{L}(E_0)} \leq C(t-s)^{-1};$$

(II) *for every $(t, s) \in \Delta_T$ $S(t, s) \in \mathcal{L}(E_0, E_1)$ and*

$$\|S(t, s)\|_{\mathcal{L}(E_0, E_1)} \leq C(t-s)^{-1};$$

(III) *for every $U_0 \in E_0$, $0 \leq s < T$*

$$\lim_{t \rightarrow s^+} \|S(t, s) U_0 - U_0\|_0 = 0;$$

(IV) *if $U_0 \in D(A(s))$,*

$$\lim_{t \rightarrow s^+} \|S(t, s) U_0 - U_0\|_1 = 0;$$

(V) *$(t, s) \rightarrow S(t, s) \in C(\Delta_T; \mathcal{L}(E_0, E_1))$ and $(t, s, F) \rightarrow S(t, s) F \in C(\overline{\Delta_T} \times E_0; E_0)$;*

(VI) *if $U_0 \in E_0$ and $s \in [0, T[$, then $t \rightarrow S(t, s) U_0 \in C^1(]s, T[; E_0) \cap C(]s, T[; E_1)$, $S(t, s) U_0 \in D(A(t))$ for every $t \in]s, T[$, $\int_s^t \mathcal{C}(t, \sigma) S(\sigma, s) U_0 d\sigma$ is defined in generalized sense and*

$$\partial_t S(t, s) U_0 = A(t) S(t, s) U_0 + \int_s^t \mathcal{C}(t, \sigma) S(\sigma, s) U_0 d\sigma;$$

(VII) *if $\theta \in]0, ((\min\{\beta, 1-\eta\})/(\min\{\beta, 1-\eta\} + \max\{2-\beta-\nu, \eta, 1-\alpha\}))]$ and G_θ is a space of type θ between E_0 and E_1 , $(t, s) \rightarrow S(t, s) - T(t-s, s) \in C(\overline{\Delta_T}; \mathcal{L}(E_0; G_\theta))$.*

PROOF. The first statement in (I) follows from 2.17(I) and 1.7(II); (II) follows from 2.17(II) and 1.7(II); (III) follows from 2.10; (IV) follows from 2.12; (V) follows from (2.14), (2.15), 1.20(V), 1.18; (VI) follows from 2.11; a consequence of (VI), together with (II), 2.4 and 2.17(II) is the second statement in (I); (VII) follows from 2.17 by interpolation.

2.19 PROPOSITION (THE VARIATION OF PARAMETER FORMULA). *Let U be the mild solution of (2.1) in $[s, T]$ ($0 \leq s < T$) with data (U_0, F) ; then, if $s < t \leq T$,*

$$U(t) = S(t, s) U_0 + \int_s^t S(t, \sigma) F(\sigma) d\sigma.$$

PROOF. Analogous to the proof of 1.21.

We conclude this section considering classical solutions:

2.20 PROPOSITION. *Assume that (2.1) has a classical solution U ; then U coincides with the mild solution of the same problem.*

PROOF. Let $\varepsilon \in]0, T - s[$; we set, for $s + \varepsilon \leq t \leq T$,

$$g_\varepsilon(t) := \int_s^{s+\varepsilon} \mathcal{C}(t, \sigma) U(\sigma) d\sigma.$$

As $U \in C^1(]s, T[; E_0) \cap C(]s, T[; E_1)$, $g_\varepsilon \in C(]s + \varepsilon, T[; E_0)$. From 2.13 and 2.19, we have, for $s + \varepsilon \leq t \leq T$,

$$U(t) = S(t, s + \varepsilon) U(s + \varepsilon) + \int_{s+\varepsilon}^t S(t, \sigma) F(\sigma) d\sigma + \int_{s+\varepsilon}^t S(t, \sigma) g_\varepsilon(\sigma) d\sigma.$$

From 2.18(I) and (V) we have that

$$S(t, s + \varepsilon) U(s + \varepsilon) = S(t, s + \varepsilon)[U(s + \varepsilon) - U_0] + S(t, s + \varepsilon) U_0$$

tends to $S(t, s) U_0$ in E_0 as $\varepsilon \rightarrow 0^+$. Clearly,

$$\lim_{\varepsilon \rightarrow 0^+} \left\| \int_{s+\varepsilon}^t S(t, \sigma) F(\sigma) d\sigma - \int_s^t S(t, \sigma) F(\sigma) d\sigma \right\|_0 = 0$$

and from

$$\begin{aligned} \|g_\varepsilon(\tau)\|_0 &\leq \left\| \int_s^{s+\varepsilon} [\mathcal{C}(\tau, \sigma) - \mathcal{C}(\tau, s)] U(\sigma) d\sigma \right\|_0 + \left\| \mathcal{C}(\tau, s) \int_s^{s+\varepsilon} U(\sigma) d\sigma \right\|_0 \leq \\ &\leq C \left[(\tau - s - \varepsilon)^{-\eta} \varepsilon^\alpha + (\tau - s)^{-\eta} \left\| \int_s^{s+\varepsilon} U(\sigma) d\sigma \right\|_1 \right] \end{aligned}$$

we get also

$$\lim_{\varepsilon \rightarrow 0^+} \left\| \int_{s+\varepsilon}^t S(t, \sigma) g_\varepsilon(\sigma) d\sigma \right\|_0 = 0$$

$$\text{and } U(t) = S(t, s) U(s) + \int_s^t S(t, \sigma) F(\sigma) d\sigma.$$

3. Volterra integrodifferential boundary value problems of higher order in time.

Now we apply the results of the second section to the problem

$$(3.1) \quad \begin{cases} \mathcal{A}(t, x, \partial_t, \partial_x) u(t, x) = \int_s^t \mathcal{C}(t, \sigma, x, \partial_\sigma, \partial_x) u(\sigma, x) d\sigma + f(t, x), \\ s < t \leq T, x \in \Omega, \\ \mathcal{B}_j(t, x', \partial_t, \partial_x) u(t, x') = 0, 1 \leq j \leq m, t \in]s, T], x' \in \partial\Omega, \\ \partial_t^j u(s, x) = u_j(x), x \in \Omega, j = 0, \dots, l-1. \end{cases}$$

under the following assumptions:

\alpha) the conditions (L1)-(L4) are satisfied;

$$\beta) \mathcal{C}(t, s, x, \partial_s, \partial_x) = \sum_{k=0}^{l-1} \mathcal{C}_{l-k}(t, s, x, \partial_x) \partial_s^k = \sum_{k=0}^{l-1} \sum_{|\beta| \leq (l-k)d} c_{k\beta}(t, s, x) \partial_x^\beta \partial_s^k$$

is a linear partial differential operator with coefficients defined and continuous in $\Delta_T \times \bar{\Omega}$ such that, for every k and β , for certain $C > 0$, $\alpha, \eta \in]0, 1[$, if $0 \leq s \leq \sigma < t \leq T$,

$$|c_{k\beta}(t, \sigma, x) - c_{k\beta}(t, s, x)| \leq C(t - \sigma)^{-\eta} (\sigma - s)^\alpha$$

and, if $0 \leq s < \tau \leq t \leq T$,

$$|c_{k\beta}(t, s, x) - c_{k\beta}(\tau, s, x)| \leq C(t - \tau)^\alpha (\tau - s)^{-\eta}.$$

We take $f \in C([s, T]; L^p(\Omega))$, for some $p \in]1, +\infty[$, and $u_0 \in W^{2m-d, p}(\Omega), \dots, u_j \in W^{2m-(j+1)d, p}(\Omega), \dots, u_{l-1} \in L^p(\Omega)$.

In this context we introduce the notions of strict and classical solution for (3.1):

3.1 DEFINITION. A strict solution u of (3.1) is an element of $\prod_{k=0}^l C^k([s, T]; W^{2m-kd, p}(\Omega))$, satisfying all the conditions in (3.1), the two first even for $t = s$.

A classical solution u of (3.1) is an element of $\prod_{k=0}^l C^k([s, T]; W^{2m-kd, p}(\Omega)) \cap \prod_{k=0}^{l-1} C^k([s, T]; W^{2m-(k+1)d, p}(\Omega))$, such that

(a) there exists $C > 0$ so that $\sum_{k=0}^l \|\partial_t^k u(t)\|_{2m-kd, p, \Omega} \leq C(t-s)^{-1}$ for $s < t \leq T$;

(b) $t \rightarrow \int_s^t u(\sigma) d\sigma \in C([s, T]; W^{2m, p}(\Omega))$;

(c) (3.1) is satisfied.

3.2 REMARK. If (3.1) has a strict solution, necessarily $u_0 \in W^{2m, p}(\Omega), \dots, u_j \in W^{2m-jd, p}(\Omega), \dots, u_{l-1} \in W^{2m-(l-1)d, p}(\Omega) = W^{d, p}(\Omega)$.

Concerning classical solutions, the integral in the first equation of (3.1) is understood in generalized sense, as

$$\lim_{\varepsilon \rightarrow 0^+} \int_{s+\varepsilon}^t \mathcal{C}(t, \sigma, \dots, \partial_\sigma, \partial_x) u(\sigma) d\sigma \text{ in } L^p(\Omega) \text{ and equals}$$

$$\int_s^t [\mathcal{C}(t, \sigma, \dots, \partial_\sigma, \partial_x) - \mathcal{C}(t, s, \dots, \partial_\sigma, \partial_x)] u(\sigma) d\sigma + \mathcal{C}_l(t, s, \dots, \partial_x) \int_s^t u(\sigma) d\sigma +$$

$$+ \sum_{k=1}^{l-1} \mathcal{C}_{l-k}(t, s, \dots, \partial_x) [\partial_t^{k-1} u(t) - u_{k-1}].$$

We have:

3.3 THEOREM. Assume that the conditions (L1)-(L4) and (k5) are satisfied. Let $s \in [0, T[$, $\varepsilon > 0$, $f \in C^\varepsilon([s, T]; L^p(\Omega))$, $u_0 \in W^{2m-d, p}(\Omega), \dots, u_j \in W^{2m-(j+1)d, p}(\Omega), \dots, u_{l-1} \in L^p(\Omega)$. Then, the problem (3.1) has a unique classical solution. If, moreover, $u_0 \in W^{2m, p}(\Omega), \dots, u_j \in W^{2m-jd, p}(\Omega), \dots, u_{l-1} \in W^{d, p}(\Omega)$ and, for $j = 1, \dots, m$

$$(3.2) \quad \gamma \left(\sum_{k=0}^{l-1} \mathcal{B}_{jk}(s, \dots, \partial_x) u_k \right) = 0,$$

the classical solution is strict.

PROOF. Given a classical solution u of (3.1), set

$$U(t) := (u(t), \dots, \partial_t^{l-1} u(t)).$$

Then, clearly, U is a classical solution of

$$(3.3) \quad \begin{cases} \partial_t U(t) = \mathcal{A}(t) U(t) + \int_s^t \mathcal{C}(t, \sigma) U(\sigma) d\sigma + F(t), \\ \gamma(\mathcal{B}_j(t) U(t)) = 0, \quad 1 \leq j \leq m, \\ U(s) = (u_0, \dots, u_{l-1}) \end{cases}$$

in the sense of 2.3 and viceversa if we define, for $(t, s) \in \Delta_T$, $U = (U_0, \dots, U_{l-1}) \in E_1$,

$$(3.4) \quad \mathcal{C}(t, s) U = \left(0, \dots, 0, \sum_{k=0}^{l-1} \mathcal{C}_{l-k}(t, s, \dots, \partial_x) U_k \right)$$

and take $F(t) := (0, \dots, f(t))$. The same happens for strict solutions. Then, we can apply 2.11, 2.12, 2.13, 2.20 and get the result.

3.4 Now we drop assumption (k5). Assume that, for some $j \in \{1, \dots, m\}$, $2m - (r + 1)d \leq \sigma_j < 2m - rd$, with $0 \leq r \leq l - 1$. In this case $\mathcal{B}_{jk} \equiv 0$ if $k \geq l - r$. We assume that, for $0 \leq k \leq l - r - 1$, the coefficients of \mathcal{B}_{jk} are in $C([0, T]; C^{2m - \sigma_j}(\overline{\Omega})) \cap C^1([0, T]; C^{2m - \sigma_j - d}(\overline{\Omega})) \cap \dots \cap C^r([0, T]; C^{2m - \sigma_j - rd}(\overline{\Omega}))$. With the same argument of [6] 4.3, one can verify that, if a strict solution of (3.1) exists, necessarily,

$$(3.5) \quad \gamma \left(\sum_{k=0}^{l-r-1} \sum_{\varrho=0}^j \binom{j}{\varrho} \mathcal{B}_{jk}^{(j-\varrho)}(0, \dots, \partial_x) u_{k+\varrho} \right) = 0 \quad \text{for } 0 \leq j \leq r,$$

where we indicate with $\mathcal{B}_{jk}^{(\varrho)}(t, x, \partial_x)$ the operator obtained differentiating the coefficients of $\mathcal{B}_{jk}(t, x, \partial_x)$ ϱ times with respect to t . In the same way, if $2m - (r + 1)d \leq \sigma_j < 2m - rd$, a necessary condition for the existence of a classical solution is

$$(3.6) \quad \gamma \left(\sum_{k=0}^{l-r-1} \sum_{\varrho=0}^j \binom{j}{\varrho} \mathcal{B}_{jk}^{(j-\varrho)}(0, \dots, \partial_x) u_{k+\varrho} \right) = 0 \quad \text{for } 0 \leq j \leq r - 1,$$

(no conditions if $r = 0$).

To see that the conditions (3.5) and (3.6) together with the hölder continuity in time of f , guarantee the existence and uniqueness of a strict and a classical solution respectively, we introduce by local charts a

strongly elliptic operator H of order d and coefficients in $C^{2m}(\partial\Omega)$ in $\partial\Omega$ and an operator $\mathcal{K}(x, \partial_x)$ of order d and coefficients in $C^{2m}(\overline{\Omega})$, such that for every $u \in W^{d+1,p}(\Omega)$

$$\gamma(\mathcal{K}(\cdot, \partial_x) u) = H\gamma u.$$

We need the following

3.5 LEMMA. *Assume that the conditions (L1)-(L4) are satisfied; then, there exist operators H and K satisfying the previous conditions. Define*

$$(3.7) \quad \mathcal{B}_j^*(t, x, \partial_x, \partial_t) := (\partial_t - K(x, \partial_x))^r \mathcal{B}_j(t, x, \partial_x, \partial_t).$$

Then, replacing \mathcal{B}_j^ with \mathcal{B}_j , we obtain a system satisfying (L1)-(L4) and (k5).*

PROOF. See [6] 4.4.

As a consequence, we deduce the following result:

3.6 THEOREM. *Consider the problem (3.1) with a fixed $p \in]1, +\infty[$ under the assumptions α) and β).*

Then, for every $f \in C^\varepsilon([s, T]; L^p(\Omega))$ ($\varepsilon > 0$), $(u_0, \dots, u_{l-1}) \in \prod_{k=0}^{l-1} W^{2m-(k+1)d,p}(\Omega)$, such that (3.6) is satisfied (3.1) has a unique classical solution, which is strict if $(u_0, \dots, u_{l-1}) \in \prod_{k=0}^{l-1} W^{2m-kd,p}(\Omega)$ and (3.5) holds.

PROOF. If a classical (strict) solution u exists, u is also a classical (strict) solution of

$$(3.8) \quad \left\{ \begin{array}{l} \mathcal{A}(t, x, \partial_t, \partial_x) u(t, x) = \int_s^t \mathcal{C}(t, \sigma, x, \partial_\sigma, \partial_x) u(\sigma, x) d\sigma + f(t, x), \\ (t, x) \in]0, T] \times \Omega, \\ \gamma(\mathcal{B}_j(t, \dots, \partial_t, \partial_x) u(t, \cdot)) = 0, j \in \{1, \dots, m\}, \quad \text{if } \sigma_j \geq 2m - d, t \in]0, T], \\ \gamma(\mathcal{B}_j^*(t, \dots, \partial_t, \partial_x) u(t, \cdot)) = 0, j \in \{1, \dots, m\}, \quad \text{if } \sigma_j < 2m - d, t \in]0, T], \\ u(s, \cdot) = u_0, \\ \dots \\ \partial_t^{l-1} u(s, \cdot) = u_{l-1}. \end{array} \right.$$

Owing to 3.3, (3.8) has a unique classical solution, which is strict if the conditions (3.5) are satisfied; the same arguments of [6] show that u is a classical (strict) solution of (3.1).

In the following we set

$$(3.9) \quad \mathcal{B}_j^*(t, x, \partial_x) := \mathcal{B}_j(t, x, \partial_x)$$

in case $\sigma_j \geq 2m - d$.

We conclude the paper giving a formula of representation of a solution:

3.7 THEOREM. *Under the assumptions of 3.6 with the notation (3.9), the classical (strict) solution of (3.1) can be represented in the form*

$$(3.10) \quad u(t) = \sum_{r=0}^{l-1} S_r(t, s) u_r + \int_s^t S_{l-1}(t, \sigma) f(\sigma) d\sigma$$

with

a) $\partial_i^i S_r(t, s) \in C(\Delta_T; \mathcal{L}(W^{(l-r-1)d, p}(\Omega), W^{(l-i)d, p}(\Omega)))$ for every $r \in \{0, \dots, l-1\}$, $i \in \{0, \dots, l\}$;

b) there exists $C > 0$ such that, under the conditions on r and i in a), for every $(t, s) \in \Delta_T$,

$$\|\partial_i^i S_r(t, s)\|_{\mathcal{L}(W^{(l-r-1)d, p}(\Omega), W^{(l-i)d, p}(\Omega))} \leq C(t-s)^{-1};$$

c) if $0 \leq r \leq l-1$, $0 \leq i \leq l-1$, $(t, s) \in \Delta_T$,

$$\|\partial_i^i S_r(t, s)\|_{\mathcal{L}(W^{(l-r-1)d, p}(\Omega), W^{(l-i-1)d, p}(\Omega))} \leq C;$$

d) if $0 \leq r \leq l-1$, $u_r \in W^{(l-r-1)d, p}(\Omega)$, $0 \leq s < t \leq T$,

$$\sum_{k=0}^l \mathcal{A}_{l-k}(t, \dots, \partial_x) \partial_t^k S_r(t, s) u_r = \lim_{\varepsilon \rightarrow 0^+} \int_{s+\varepsilon}^t \sum_{k=0}^{l-1} \mathcal{C}_{l-k}(t, \sigma, \dots, \partial_x) \partial_\sigma^k S_r(\sigma, s) u_r d\sigma,$$

with the limit in $L^p(\Omega)$ and, if $j = 1, \dots, m$, $0 \leq s < t \leq T$,

$$\gamma \left(\sum_{k=0}^{l-1} \mathcal{B}_{jk}^*(t, \dots, \partial_x) \partial_t^k S_r(t, s) u_r \right) = 0.$$

PROOF. Replacing (if necessary) $\mathcal{B}_j(t, x, \partial_t, \partial_x)$ with $\mathcal{B}_j^*(t, x, \partial_t, \partial_x)$, we can assume that (k5) is satisfied.

Let now $S(t, s)$ ($(t, s) \in \overline{\Delta_T}$) be the operator defined in 2.16 under the conditions α) and β). Then, if $(u_0, \dots, u_{l-1}) \in E_0$,

$$(3.11) \quad S(t, s)(u_0, \dots, u_{l-1}) = \left(\sum_{r=0}^{l-1} S_{0,r}(t, s)u_r, \dots, \sum_{r=0}^{l-1} S_{l-1,r}(t, s)u_r \right).$$

Owing to 2.18(II), for every $i, r \in \{0, \dots, l-1\}, (t, s) \in \Delta_T, S_{i,r}(t, s) \in \mathcal{L}(W^{(l-r-1)d,p}(\Omega), W^{(l-i)d,p}(\Omega))$ and there exists $C > 0$ such that

$$\|S_{i,r}\|_{\mathcal{L}(W^{(l-r-1)d,p}(\Omega), W^{(l-i)d,p}(\Omega))} \leq C(t-s)^{-1},$$

$$\|S_{i,r}\|_{\mathcal{L}(W^{(l-r-1)d,p}(\Omega), W^{(l-i-1)d,p}(\Omega))} \leq C.$$

Moreover, as, for $r, i = 0, \dots, l-1, \partial_t^i S_{0,r}(t, s) = S_{i,r}(t, s)$, for every $s \in [0, T[$ $t \rightarrow S_{0,r}(t, s)$ belongs to $\bigcap_{i=0} C^i([s, T]); \mathcal{L}(W^{(l-r-1)d,p}(\Omega), W^{(l-i)d,p}(\Omega))$ and

$$\|\partial_t^i S_{0,r}(t, s)\|_{\mathcal{L}(W^{(l-r-1)d,p}(\Omega), L^p(\Omega))} = \|\partial_t S_{l-1,r}(t, s)\|_{\mathcal{L}(L^p(\Omega))} \leq C(t-s)^{-1}$$

by 2.18(I). Coming back to the problem (3.1), we obtain (3.10) from 2.19, setting $S_r(t, s) := S_{0,r}(t, s)$. So we have proved a) - c). d) follows from the fact that, if $r \in \{0, \dots, l-1\}, u_j = 0$ for $j \neq r$ and $f \equiv 0$, then $t \rightarrow S_r(t, s)u_r$ is a classical solution of (3.8).

REFERENCES

- [1] P. ACQUISTAPACE, *Existence and maximal time regularity for linear parabolic integrodifferential equations*, Journ. Int. Eq., **10** (1985), pp. 5-43.
- [2] P. ACQUISTAPACE - B. TERRENI, *Linear parabolic equations in Banach spaces with variable domain but constant interpolation spaces*, Ann. Sc. Norm. Sup. Pisa, ser. IV, **13** (1986), pp. 75-107.
- [3] P. ACQUISTAPACE - B. TERRENI, *Existence and sharp regularity results for linear parabolic non-autonomous integro-differential equations*, Israel Journ. Math. vol. **53**, n. 3 (1986), pp. 257-303.
- [4] G. DI BLASIO, *Nonautonomous integrodifferential equations in L^p spaces*, Journ. Int. Eq., **10** (1985), pp. 111-121.
- [5] D. GUIDETTI, *Abstract linear parabolic problems with nonhomogeneous boundary conditions*, in *Semigroup theory and evolution equations*, (ed. P. Clement, E. Mitidieri, B. de Pagter), Lec. Notes in Pure and Appl. Math. vol. 135, Marcel Dekker Inc. (1991).
- [6] D. GUIDETTI, *On boundary value problems for parabolic equations of higher order in time*, Journ. Diff. Eq., vol. **124**, n. 1 (1996), pp. 1-26.

- [7] T. KATO - H. TANABE, *On the abstract evolution equation*, Osaka Math. Journ. **14** (1962), pp. 107-133.
- [8] S. G. KREIN, *Linear differential equations in Banach spaces*, Translations of Mathematical Monographs, vol. **29** (1971).
- [9] A. LUNARDI, *Regular solutions for time dependent abstract integrodifferential equations with singular kernel*, Journ. Math. Anal. Appl., **130** (1988), 1-21.
- [10] A. LUNARDI - E. SINISTRARI, *C^{α} -regularity for nonautonomous linear integrodifferential equations*, J. Diff. Eq., **63** (1986), pp. 88-116.
- [11] J. PRÜSS, *On resolvent operators for linear integrodifferential equations of Volterra type*, Journ. Integral Eq., **5** (1983), 211-236.
- [12] H. TANABE, *Volterra integro-differential equations of parabolic type of higher order in t* , Journ. Fac. Science, Tokyo Univ., Sec. IA, vol. **34**, n. 1 (1987), pp. 115-125.
- [13] H. TANABE, *On fundamental solutions of linear parabolic equations of higher order in time and associated Volterra equations*, Journ. Diff. Eq., **73** (1988), 288-308.
- [TR] H. TRIEBEL, *Theory of function spaces*, Monographs in Mathematics, Birkhäuser (1983).

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