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The Gross-Koblitz Formula Revisited.

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The formula in question gives an explicit value of Gauss sums using the p-adic gamma function of Morita. We give here an elementary proof of this formula (valid for all primes). Let me thank L. van Hamme who stimulated me to find such a proof, and A. Junod who helped me to understand [2], which has been my starting point.

1. Preliminary comments on numeration.

Let $q = p^f$ $(f \geqslant 1)$ be a power of a prime p. Each affine map

$$x \mapsto a + qx : \boldsymbol{Z}_p \to \boldsymbol{Z}_p \qquad (a \in \boldsymbol{Z}_p)$$

has a unique fixed point

$$a_* = \frac{a}{1-q} = a + aq + aq^2 + \ldots = a + q(\underbrace{a + aq + aq^2 + \ldots}_{a_*}).$$

When a is an integer in the interval $0 \leqslant a < q$, say with p-adic expansion

$$a = a_0 + a_1 p + \ldots + a_{f-1} p^{f-1} \qquad (0 \leqslant a_j < p),$$

the fixed point of the corresponding affine transformation has a periodic p-adic expansion given by $a + aq + aq^2 + \ldots$ (period of length f). Let us write

$$a_* = a_0 + p(a_1 + a_2 p + \ldots + a_{f-1} p^{f-2} + a_0 p^{f-1} + \ldots) = a_0 + p a'_*.$$

We recognize in a'_* the fixed point of the affine map corresponding to

$$a' = a_1 + a_2 p + \ldots + a_{f-1} p^{f-2} + a_0 p^{f-1},$$

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and we observe that a' is obtained from a by a cyclic permutation of its digits. Iterating the procedure, we can write

$$a'_* = a_1 + pa''_*, \qquad a''_* = \frac{a''}{1-q}, \dots$$

In this way, we obtain a cycle of integers in the interval $\{0, \dots, q-1\}$

$$a', a'', \dots, a^{(f-1)}, a^{(f)} = a$$

having p-adic expansions obtained by cyclic permutations from that of a.

2. p-adic extensions of quotients of factorials.

For any prime p and $0 \leqslant a < p$, the relation

$$(*) \quad \frac{(a+pn)!}{p^n n!} = \frac{(a+pn)!}{(p1)(p2)\dots(pn)} = (-1)^{a+pn+1}\Gamma_p(a+pn+1)$$

shows that

$$n \mapsto (-1)^{pn}\frac{(a+pn)!}{p^n n!}$$

has a continuous extension $\boldsymbol{Z}_p \to \boldsymbol{Z}_p^\times \subset \boldsymbol{Q}_p$ given by

$$x \mapsto (-1)^{a+1}\Gamma_p(a+px+1).$$

This simply follows from the definition of the Γ_p-function by Morita.

Let us generalize this observation to the case of quotients $m \mapsto (a+$ $+qm)!/m!$ when $0 \leqslant a < q = p^f$ $(f \geqslant 1)$.

We can introduce the p-adic expansion of a, say

$$a = a_0 + a_1 p + \dots + a_{f-1}p^{f-1},$$

and write

$$a + qm = a_0 + p\underbrace{(a_1 + \dots + a_{f-1}p^{f-2} + p^{f-1}m)}_{n_1}.$$

Put $n_0 = a + qm = a_0 + pn_1$ and successively

$$n_0 = a_0 + a_1 p + p^2 n_2, \qquad n_1 = a_1 + pn_2, \qquad \text{etc.}$$

hence with

$$n_1 = a_1 + \dots + a_{f-1}p^{f-2} + p^{f-1}m$$

$$= a_1 + p\underbrace{(a_2 + \dots + a_{f-1}p^{f-3} + p^{f-2}m)}_{n_2}, \qquad \text{etc.}$$

Let us write a telescopic product ($n_0 = a + qm$, $n_f = m$)

$$\frac{(a+qm)!}{m!} = \frac{n_0!}{n_1!}\frac{n_1!}{n_2!}\cdots\frac{n_{f-1}!}{n_f!}$$

$$= \frac{(a_0+pn_1)!}{n_1!}\frac{(a_1+pn_2)!}{n_2!}\cdots\frac{(a_{f-1}+pm)!}{m!}$$

$$= \pm p^{n_1}\Gamma_p(a_0+pn_1+1)\cdot p^{n_2}\Gamma_p(n_1+1)\dots p^m\Gamma_p(n_{f-1}+1),$$

$$\frac{(a+qm)!}{p^{n_1+\dots+n_f}m!} = \pm\Gamma_p(\underbrace{a+qm}_{a_0+pn_1=n_0}+1)\,\Gamma_p(\underbrace{a_1+pn_2}_{n_1}+1)\dots\Gamma_p(\underbrace{a_{f-1}+\overbrace{pm}^{pn_f}}_{n_{f-1}}+1)$$

$$= \pm\prod_{0\leq i<f}\Gamma_p(\underbrace{a_i+pn_{i+1}}_{n_i}+1) = \pm\prod_{0\leq i<f}\Gamma_p(n_i+1).$$

Recalling $(*)$ in the form

$$\frac{(a+pn)!}{n!} = (-1)^{a+pn+1}p^n\Gamma_p(a+pn+1)$$

we see that the precise sign is $(-1)^{(n_0+1)+(n_1+1)+\dots+(n_{f-1}+1)}$. Moreover, the sum $\sigma = n_1+\dots+n_f$ may be computed as follows

$$\begin{array}{rcl} n_1 &=& \left\lfloor \frac{a}{p} \right\rfloor \quad + \quad p^{f-1}m \\ n_2 &=& \left\lfloor \frac{a}{p^2} \right\rfloor \quad + \quad p^{f-2}m \\ \dots &=& \dots \\ n_{f-1} &=& \left\lfloor \frac{a}{p^{f-1}} \right\rfloor \quad + \quad pm \\ n_f &=& \qquad\qquad m \\ \hline \sigma &=& \mathrm{ord}_p a! \quad + \quad \frac{q-1}{p-1}m \end{array}$$

so that $n_0 + \ldots + n_{f-1} = n_0 + \sigma - n_f = a + (q-1)m + \sigma$. Hence

$$\frac{(a+qm)!}{p^\sigma m!} = (-1)^{f+(q-1)m+a+\sigma} \prod_{0 \leqslant i < f} \Gamma_p(n_i+1),$$

$$\frac{(a+qm)!}{(-p)^\sigma m!} = (-1)^{f+(q-1)m+a} \prod_{0 \leqslant i < f} \Gamma_p(n_i+1).$$

THEOREM 1. *For a fixed power $q=p^f \, (f \geqslant 1)$ of p, the functions*

$$m \mapsto \frac{(a+qm)!}{(-p)^{\frac{q-1}{p-1}m} m!} \qquad (0 \leqslant a < q)$$

admit continuous extensions $\boldsymbol{Z}_p \to \boldsymbol{Q}_p$ given by

$$x \mapsto (-1)^{(q-1)m+f+a}(-p)^{\mathrm{ord}_p a!} \prod_{0 \leqslant i < f} \Gamma_p\Bigg(\underbrace{\left\lfloor \frac{a}{p^i} \right\rfloor + p^{f-i}x + 1}_{n_i(x)}\Bigg) \qquad \blacksquare$$

When the prime p is odd, $q-1$ is even and $(-1)^{(q-1)m} = +1$. Hence this sign is relevant only if $p=2$ in which case it is $\varepsilon(m) = (-1)^m$: let ε denote the character sign having kernel $2\boldsymbol{Z}_2$

$$\varepsilon(x) = \begin{cases} +1 & \text{if } x \in 2\boldsymbol{Z}_2 \\ -1 & \text{if } x \in 1 + 2\boldsymbol{Z}_2. \end{cases}$$

We shall be interested in the *inverse* of the preceding functions. Thus we define continuous functions $G_a \colon \boldsymbol{Z}_p \to \boldsymbol{Q}_p$ $(0 \leqslant a < q)$ with

$$G_a(x) = \varepsilon(x)(-1)^{f+a} \Big/ (-p)^{\mathrm{ord}_p a!} \prod_{0 \leqslant i < f} \Gamma_p(n_i(x)+1),$$

$$G_a(m) = (-p)^{\frac{q-1}{p-1}m} \frac{m!}{(a+qm)!} \qquad (m \geqslant 0)$$

$(\varepsilon = 1$ if $p \neq 2)$. Let us use the Legendre formula to simplify the preceding expressions. When $p \geqslant 3$ is odd, $\varepsilon = 1$ and

$$\frac{1}{\Gamma_p(n_i(x)+1)} = (-1)^{1+a_i} \Gamma_p(-n_i(x)).$$

Moreover $\sum a_i \equiv \sum a_i p^i = a \bmod 2$, so that

$$G_a(x) = \frac{1}{(-p)^{\mathrm{ord}_p a!}} \prod_{0 \leqslant i < f} \Gamma_p(-n_i(x)).$$

This formula is also true when $p = 2$, because the Legendre formula is now

$$\frac{1}{\Gamma_2(n_i(x)+1)} = (-1)^{1+a_i+a_{i+1}} \Gamma_2(-n_i(x)),$$

and the product leads to an exponent of -1 equal to

$$f + (a_0 + a_1) + (a_1 + a_2) + \ldots + (a_{f-1} + x_0) \equiv f + a_0 + x_0 \bmod 2 .$$

Since $\varepsilon(x) = (-1)^{x_0}$ we have $\varepsilon(x)(-1)^{f+a}(-1)^{f+a_0+x_0} = 1$ and there only remains

$$G_a(x) = \frac{1}{(-2)^{\mathrm{ord}_2 a!}} \prod_{0 \leqslant i < f} \Gamma_2(-n_i(x)).$$

3. Mahler coefficients of the functions G_a.

Let us choose a nonzero root $\pi \in \boldsymbol{C}_p$ of $X + \frac{1}{p} X^p = 0$. We have

$$\pi^{p-1} = -p \quad \text{and} \quad (-p)^{\mathrm{ord}_p a!} = \pi^{a - S_p(a)},$$

so that

$$\pi^a G_a(x) = \pi^{S_p(a)} \prod_{0 \leqslant i < f} \Gamma_p(-n_i(x))$$

for all primes p. This expression is especially simple at the fixed point $x = a_*$ of the map $x \mapsto a + qx$, since in this case

$$n_i(x) = n_i(a_*) = a_i + a_{i+1} p + \ldots = \frac{a^{(i)}}{1-q}$$

are obtained by a cyclic permutation from a_*

$$\pi^a G_a(a_*) = \pi^{S_p(a)} \prod_{0 \leqslant i < f} \Gamma_p\left(\frac{a^{(i)}}{q-1}\right).$$

It turns out that the Mahler coefficients of the functions G_a are linked to the coefficients of the Dwork exponential

$$\Theta_q(T) = e^{\pi(T - T^q)} = \sum_{n \geq 0} A_n T^n = 1 + \pi T + T^2(\dots).$$

THEOREM 2. *For $0 \leq a < q$, the Mahler expansion of $G_a: \boldsymbol{N} \to \boldsymbol{Q} \subset \subset \boldsymbol{Q}_p$ is*

$$G_a(x) = \sum_{k \geq 0} \frac{A_{a+kq}}{\pi^{a+k}} k! \binom{x}{k},$$

$$\tilde{G}_a(x) = \pi^a G_a(x) = \sum_{k \geq 0} \frac{A_{a+kq}}{\pi^k} (x)_k = A_a + \frac{A_{a+q}}{\pi} x + \dots$$

The proof of this result obviously involves some formal manipulations of power series. These are made easier if we use the *Atkin operators* (*).

Let us recall their definition and formal properties. The operator U_q is defined on formal Laurent series by

$$f = \sum a_n T^n \mapsto U_q(f) = \sum a_{qn} T^n.$$

Obviously

$$T^j U_q(f) = U_q(T^{qj} f), \qquad g(T)\, U_q(f) = U_q(g(T^q)\, f).$$

For example, remplacing f by $e^{\pi T} f$ and letting $g = e^{-\pi T}$, we find

$$e^{-\pi T} U_q(e^{\pi T} f) = U_q(e^{-\pi T^q} e^{\pi T} f) = U_q(\Theta_q(T)\, f).$$

This is the reason for the appearance of the Dwork exponential in this context. Observe that the action of the Atkin operator forgets all coefficients a_i having an index i not multiple of q e.g. $U_q\left(\sum_{n > -q} a_n T^n\right) = U_q\left(\sum_{n \geq 0} a_n T^n\right)$, and also

$$U_q\left(\sum_{n \geq -a} a_n T^n\right) = U_q\left(\sum_{n \geq 0} a_n T^n\right) \qquad (0 \leq a < q).$$

We shall use twice this observation in the next computation (and indicate it by a «!» on the concerned equality).

(*) Also called «Dwork ψ-operators» or «Hecke» operators.

PROOF OF THEOREM 2. Let us recall the *Boole relation* linking the values of a function f to its Mahler coefficients $c_k(f)$

$$e^{-T}\sum_{m\geq 0} f(m)\frac{T^m}{m!} = \sum_{k\geq 0} c_k(f)\frac{T^k}{k!}.$$

Take $f = G_a$ and remplace the indeterminate T by πT

$$e^{-\pi T}\sum_{m\geq 0} G_a(m)\frac{\pi^m T^m}{m!} = \sum_{k\geq 0} c_k(G_a)\frac{(\pi T)^k}{k!}.$$

Let us now compute the left-hand side, recalling that $(-p)^{\frac{1}{p-1}} = \pi$

$$e^{-\pi T}\sum_{m\geq 0} G_a(m)\frac{\pi^m T^m}{m!}$$

$$= e^{-\pi T}\sum_{m\geq 0}\frac{\pi^{(q-1)m}m!}{(a+qm)!}\frac{\pi^m T^m}{m!} = e^{-\pi T}\sum_{m\geq 0}\frac{\pi^{qm}\not{m}!}{(a+qm)!}\frac{T^m}{\not{m}!}$$

$$= e^{-\pi T}U_q\left(\sum_{m\geq 0 \text{ or } -a}\frac{\pi^{a+m}}{(a+m)!}\frac{T^m}{\pi^a}\right) \overset{!}{=} e^{-\pi T}U_q\left(\sum_{n\geq 0}\frac{\pi^n}{n!}\frac{T^{n-a}}{\pi^a}\right)$$

$$= e^{-\pi T}U_q\left(\sum_{n\geq 0}\frac{\pi^n T^n}{n!}\frac{T^{-a}}{\pi^a}\right) = e^{-\pi T}U_q\left(e^{\pi T}\frac{T^{-a}}{\pi^a}\right)$$

$$= U_q\left(\Theta_q(T)\frac{T^{-a}}{\pi^a}\right) = U_q\left(\sum_{n\geq 0}\frac{A_n}{\pi^a}T^{n-a}\right)$$

$$= U_q\left(\sum_{n\geq -a \text{ or } 0}\frac{A_{n+a}}{\pi^a}T^n\right) \overset{!}{=} U_q\left(\sum_{n\geq 0}\frac{A_{a+n}}{\pi^a}T^n\right)$$

$$= \sum_{k\geq 0}\frac{A_{a+kq}}{\pi^a}T^k = \sum_{k\geq 0}\underbrace{\frac{A_{a+kq}}{\pi^{a+k}}k!}_{c_k(G_a)}\frac{(\pi T)^k}{k!}.$$

This proves the announced formula. ■

COMMENT. Note that the coefficients A_n of the expansion of the Dwork exponential Θ_q depend on the power $q = p^f$ *and* the choice of root π such that $\pi^{p-1} = -p$. If we replace π by another choice $\zeta\pi$ where $\zeta^{p-1} = 1$, the coefficient A_n is replaced by $\zeta^n A_n$. Since $\zeta = \zeta^p = \ldots = \zeta^q$

implies $\zeta^k = \zeta^{qk}$, we see that the coefficients $\frac{A_{a+kq}}{\pi^{a+k}} k!$ are unchanged. On the other hand, these coefficients belong to $\boldsymbol{Q}_p$ simply since they are Mahler coefficients of a $\boldsymbol{Q}_p$-valued continuous function.

4. Gauss sums.

The Gross-Koblitz formula concerns the Gauss sums

$$-\sum_{\varepsilon^q = \varepsilon \neq 0} \varepsilon^{-a} \Theta_q(\varepsilon) = -\sum_{\varepsilon^{q-1} = 1} \varepsilon^{-a} \sum_{n \geqslant 0} A_n \varepsilon^n = -\sum_{n \geqslant 0} A_n \sum_{\varepsilon^{q-1} = 1} \varepsilon^{n-a}$$

(the sign « $-$ » is chosen in order to give it the value $+1$ when $a = 0$). The sum on roots of unity is $q - 1$ if $n - a$ is a multiple of $q' = q - 1$ and is 0 otherwise. If $a = q - 1 = q'$, we have to take into account the value $k = -1$. Let us assume that $0 \leqslant a < q'$, so that only the values $k \geqslant 0$ occur

$$-\sum_{\varepsilon^q = \varepsilon \neq 0} \varepsilon^{-a} \Theta_q(\varepsilon) = (1 - q) \sum_{k \geqslant 0} A_{a + kq'} .$$

The above Mahler series involve the coefficients of the Dwork exponential having indices in arithmetic progressions of ratio q, whereas we are looking for a summation formula for these coefficients with indices in an arithmetic progression of ratio $q' = q - 1$. Here is a link between the two.

LEMMA. *We have* $nA_n = \pi A_{n-1}$ $(1 \leqslant n < q)$, $nA_n = \pi(A_{n-1} - qA_{n-q})$ $(n \geqslant q)$.

PROOF. We differentiate the defining identity

$$\sum_{n \geqslant 0} A_n T^n = \Theta_q(T) = e^{\pi(T - T^q)}$$

$$\sum_{n \geqslant 0} nA_n T^{n-1} = \Theta_q(T)' = e^{\pi(T - T^q)}(\pi - q\pi T^{q-1})$$

$$\sum_{n \geqslant 0 \text{ or } 1} nA_n T^{n-1} = \Theta_q(T)(\pi - q\pi T^{q-1}) = \sum_{n \geqslant 0} A_n T^n (\pi - q\pi T^{q-1}) .$$

The identification of the coefficients of T^{n-1} leads to the result. ∎

Let us define functions $\widetilde{G}_\alpha$ for all integers $\alpha \geqslant 0$ by

$$\widetilde{G}_\alpha(x) = \sum_{k \geqslant 0} \frac{A_{\alpha+kq}}{\pi^k}(x)_k = A_\alpha + \frac{A_{\alpha+q}}{\pi}x + \frac{A_{\alpha+2q}}{\pi^2}x(x-1) + \dots$$

This definition extends the preceding one (given only for $\alpha = a < q$), but let us emphasize that when $\alpha \geqslant q$, these functions are not simply given by products of Γ_p as in the previous case.

THEOREM 3. *For $\alpha \geqslant 0$, $\alpha_* = \dfrac{\alpha}{1-q}$, and $q' = q-1$ we have*

$$(1-q) \sum_{0 \leqslant k < N} A_{\alpha+kq'} = \widetilde{G}_\alpha(\alpha_*) - \widetilde{G}_{\alpha+Nq'}(\alpha_* - N) \qquad (N \geqslant 1).$$

PROOF. The crucial case is $N = 1$:

$$\widetilde{G}_\alpha(\alpha_*) - \widetilde{G}_{\alpha+q'}(\alpha_* - 1) = (1-q)\,A_\alpha.$$

To compute $\widetilde{G}_\alpha(x) - \widetilde{G}_{\alpha+q'}(x-1)$, we first transform its second term

$$\widetilde{G}_{\alpha+q'}(x-1) = \sum_{k \geqslant 0} A_{\alpha+q'+kq} \frac{(x-1)_k}{\pi^k}.$$

Since $\alpha + q' + kq = \alpha + (k+1)\,q - 1 = n - 1$, we can use the relation (lemma)

$$A_{n-1} = \frac{n}{\pi}A_n + qA_{n-q} \qquad (n \geqslant q),$$

to bring back the sequence of indices into arithmetic progressions of ratio q

$$\widetilde{G}_{\alpha+q'}(x-1) = \sum_{k \geqslant 0}\left[\frac{\alpha+(k+1)\,q}{\pi}A_{\alpha+(k+1)q} + qA_{\alpha+kq}\right]\frac{(x-1)_k}{\pi^k}$$

$$= \sum_{k \geqslant 1} \frac{\alpha+kq}{\pi}A_{\alpha+kq}\frac{(x-1)_{k-1}}{\pi^{k-1}} + \sum_{k \geqslant 0} qA_{\alpha+kq}\frac{(x-1)_k}{\pi^k}.$$

Hence $\widetilde{G}_\alpha(x) - \widetilde{G}_{\alpha+q'}(x-1)$ is equal to

$$A_\alpha + \sum_{k \geqslant 1} A_{\alpha+kq}\frac{(x-1)_{k-1}}{\pi^k}(x-\alpha-kq) - \sum_{k \geqslant 0} qA_{\alpha+kq}\frac{(x-1)_{k-1}}{\pi^k}(x-k).$$

A miracle happens when x is equal to the fixed point α_*:

$$\alpha_* - \alpha - kq = q(\alpha_* - k),$$

so that all terms compensate except $k = 0$, whence the first formula in the theorem. Summing up consecutive expressions and noting that $(\alpha + q')_* = \alpha_* - 1$, we obtain a telescopic sum

$$\widetilde{G}_\alpha(\alpha_*) - \widetilde{G}_{\alpha + Nq'}(\alpha_* - N) = (1 - q) \sum_{0 \leqslant k < N} A_{\alpha + kq'}. \qquad \blacksquare$$

More generally,

$$\frac{x - \alpha - kq}{x - k} - q = \frac{x - \alpha - kq - qx + qk}{x - k} = \frac{x - (\alpha + qx)}{x - k}$$

and remembering $\alpha = (1 - q)\alpha_*$

$$\frac{x - \alpha - kq}{x - k} - q = \frac{x - \alpha_* + q\alpha_* - qx}{x - k} = (x - \alpha_*)\frac{1 - q}{x - k},$$

hence the more general formula

$$\widetilde{G}_\alpha(x) - \widetilde{G}_{\alpha + q'}(x - 1) = (1 - q)A_\alpha + (x - \alpha_*)\frac{1 - q}{\pi}\sum_{k \geqslant 0}\frac{A_{\alpha + (k+1)q}}{\pi^k}(x - 1)_k.$$

It is well known that the Dwork exponential converges in a ball of radius > 1, hence $A_n \to 0$ $(n \to \infty)$ so that we may go to the limit

$$(1 - q)\sum_{k \geqslant 0} A_{\alpha + kq'} \qquad \text{(Gauss-Dwork sum)}$$

$$= \widetilde{G}_\alpha(\alpha_*) - \lim_{N \to \infty} \widetilde{G}_{\alpha + Nq'}(\alpha_* - N).$$

The limit vanishes in view of the following lemma since $\alpha_* - N \in \boldsymbol{Z}_p$.

LEMMA. *We have* $\|\widetilde{G}_\alpha\| \to 0$ $(\alpha \to \infty)$. *More precisely*

$$\|\widetilde{G}_\alpha\| \leqslant \begin{cases} r_p^{\alpha/q} & \text{if } p \geqslant 3 \\ r_p^{(\alpha - q)/2q} & \text{if } p = 2\,. \end{cases}$$

PROOF. The norm used here is the sup norm on the unit ball, so that

$$\|\tilde{G}_a\| \leqslant \sup_{k \geqslant 0} \left| \frac{A_{a+kq}}{\pi^k/k!} \right|$$

(the Mahler theorem states that this is in fact an equality, provided that the sup norm is taken on the unit ball of $\boldsymbol{C}_p$). But

$$\left| \frac{\pi^k}{k!} \right| = \frac{r_p^k}{|p|^{\mathrm{ord}_p k!}} = r_p^{k-(k-S_p(k))} = r_p^{S_p(k)}.$$

On the other hand, the Dwork series $\Theta_q(T) = e^{\pi(T-T^q)} = \sum_{n \geqslant 0} A_n T^n$ is bounded by 1 on the ball of radius $|p|^{\frac{1-p}{pq}} > 1$

$$|A_n|\,|p|^{n\frac{1-p}{pq}} \leqslant 1\,, \qquad |A_n| \leqslant |p|^{n\frac{p-1}{pq}} = r_p^{n\frac{(p-1)^2}{pq}}.$$

This leads to

$$\left| \frac{A_{a+kq}}{\pi^k/k!} \right| \leqslant \frac{r_p^{(a+kq)\frac{(p-1)^2}{pq}}}{r_p^{S_p(k)}}.$$

(1) *Case $p \geqslant 3$ is odd.* In this case, we use the minoration $r_p^{S_p(k)} \geqslant r_p^k$ of the denominator. The exponent of r_p is easily estimated

$$(a+kq)\frac{(p-1)^2}{pq} - k = a\frac{(p-1)^2}{pq} + k\left(\frac{(p-1)^2}{p} - 1\right).$$

As $p \geqslant 3$, $\frac{p-1}{p} > \frac{1}{2}$ and

$$(a+kq)\frac{(p-1)^2}{pq} - k \geqslant a\frac{p-1}{2q} + k\frac{p-3}{2} \geqslant a\frac{1}{q}.$$

Hence this exponent of r_p is greater or equal to a/q whence the first assertion.

(2) *Case* $p=2$. The preceding minoration of the denominator is not precise enough to lead to the result. This is why we keep

scrupulously the exponent $S_2(k)$ and have now to estimate

$$(a+kq)\frac{(p-1)^2}{pq} - S_p(k) = (a+kq)\frac{1}{2q} - S_2(k) = \frac{a}{2q} + \frac{k}{2} - S_2(k).$$

But the following table

k	0	1	2	3	$\geqslant 4$
$\frac{k}{2} - S_2(k)$	0	$-1/2$	0	$-1/2$	>0

shows $\frac{k}{2} - S_2(k) \geqslant -1/2$ (it is a simple exercise to prove it formally) which finishes the proof. ∎

Summing up, we have obtained the main result.

THEOREM 4 (GROSS-KOBLITZ). *For $0 \leqslant a < q-1$ ($q = p^f$, $f \geqslant 1$), we have*

$$-\sum_{\varepsilon^q = \varepsilon \neq 0} \varepsilon^{-a}\Theta_q(\varepsilon) = \pi^{S_p(a)} \prod_{0 \leqslant i < f} \Gamma_p\left(\frac{a^{(i)}}{q-1}\right)$$

where the integers $0 \leqslant a^{(i)} < q-1$ have p-adic expansions obtained by cyclic permutation from that of a, and $S_p(a)$ is the sum of digits of a in base p.

Since the values of Γ_p are p-adic units, we deduce the following result.

COROLLARY 1 (STICKELBERGER). *For $0 \leqslant a < q$, the p-adic absolute value of the Gauss sum $\sum_{\varepsilon^q = \varepsilon \neq 0} \varepsilon^{-a}\Theta_q(\varepsilon)$ is*

$$|\pi^{S_p(a)}| = r_p^{S_p(a)} = |p|^{\frac{S_p(a)}{p-1}}. \quad ∎$$

COROLLARY 2. *When $p \equiv 1 \bmod n$, the values of Γ_p at the rational numbers $\frac{m}{n}$ are algebraic numbers. More precisely*

$$\Gamma_p\left(\frac{m}{n}\right) \in \boldsymbol{Q}(\mu_{np}, \sqrt[n]{-p}).$$

PROOF. By the functional equation of Γ_p, it is enough to establish this when $0 \leqslant m < n$. If we write $p-1 = ln$ and $\frac{m}{n} = \frac{lm}{p-1}$, we can use the Gross-Koblitz formula for $q = p$ and $a = lm$. ■

APPENDIX 1. For an odd prime $p \geqslant 3$, the Legendre relation for Γ_p is

$$\Gamma_p(x)\,\Gamma_p(1-x) = (-1)^{R(x)}$$

where $R(x) \in \{1, \ldots, p\}$ is in the class of $x \bmod p$. Let us write it in the equivalent form

$$\Gamma_p(-x)\,\Gamma_p(x+1) = (-1)^{R(-x)} = (-1)^{p-x_0} = (-1)^{1+x_0} \qquad (x = x_0 + x_1 p + \ldots)$$

For $p = 2$ and $x = x_0 + x_1 2 + x_2 2^2 + \ldots$, we have

$$\Gamma_2(x)\,\Gamma_2(1-x) = (-1)^{1+x_1},$$

$$\Gamma_2(-x)\,\Gamma_2(x+1) = (-1)^{1+x_0+x_1}.$$

One way of unifying the two cases consists in writing

$$\Gamma_p(-x)\,\Gamma_p(x+1) = (-1)^{1+x_0+(p-1)x_1}.$$

APPENDIX 2. It is well known that the $\Theta_q(\varepsilon) \in \boldsymbol{C}_p$ are pth roots of unity (Dwork's theorem). We can observe

$$\Theta_q(T) = \Theta_p(T)\,\Theta_p(T^p) \ldots \Theta_p(T^{q/p})$$

$$= 1 + \pi(T + T^p + \ldots + T^{q/p}) + \ldots$$

so that

$$\Theta_q(\varepsilon) \equiv 1 + \pi(\varepsilon + \varepsilon^p + \ldots + \varepsilon^{q/p}) \bmod \pi^2$$

$$\Theta_q(t(x)) = \xi_\pi^{t(Trx)}, \qquad \xi_\pi = \Theta_p(1) \quad (t\text{: Teichmüller})$$

and the Gauss sums considered here are precisely Gauss sums for the field $\boldsymbol{F}_q$.

APPENDIX 3. The Atkin operators still satisfy

$$U_q(f)(T^q) = \sum a_{qn} T^{qn} = q^{-1} \sum_{\zeta \in \mu_q} f(\zeta T)$$

(often used for $q = p$). On the other hand, the operator $\delta = T(d/dT)$ is the *degree operator*: it sends T^n onto nT^n hence

$$\delta = T\frac{d}{dT} : \sum_{n \geq 0} a_n T^n \mapsto \sum_{n \geq 0} n a_n T^n .$$

From this, the relation $U_q \circ \delta = q(\delta \circ U_q)$ immediately follows

$$U_q \circ \delta \Big(\sum_{n \geq 0} a_n T^n \Big) = \sum_{n \geq 0} q n a_{qn} T^n = q \sum_{n \geq 0} n a_{qn} T^n = q\delta \circ U_q \Big(\sum_{n \geq 0} a_n T^n \Big) .$$

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