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Right ideals and derivations on multilinear polynomials

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Right Ideals and Derivations on Multilinear Polynomials.

VINCENZO DE FILIPPIS (*)

ABSTRACT - Let R be an associative prime ring with center $Z(R)$ and extended centroid C, $f(x_1, \ldots, x_n)$ a non-zero multilinear polynomial over C in n non-commuting variables, d a non-zero derivation of R, $m \geq 1$ a fixed integer and ϱ a non-zero right ideal of R. We prove that: (i) if $(d(f(x_1, \ldots, x_n)) - f(x_1, \ldots, x_n))^m$ is a differential identity for ϱ then $C\varrho = eRC$ for some idempotent element e in the socle of RC and $f(x_1, \ldots, x_n)$ is an identity for $eRCe$; (ii) if $(d(f(r_1, \ldots, r_n)) - f(r_1, \ldots, r_n))^m$ is central on R, for any $r_1, \ldots, r_n \in \varrho$, then $C\varrho = eRC$, for some idempotent element e in the socle of RC and either $f(x_1, \ldots, x_n)$ is central in $eRCe$ or $eRCe$ satisfies the standard identity $S_4(x_1, \ldots, x_4)$.

Let R be an associative prime ring with center $Z(R)$ and extended centroid C. Recall that an additive mapping d of R into itself is a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. In [5] J. Bergen proved that if g is an automorphism of R such that $(g(x) - x)^m = 0$, for all $x \in R$, where $m \geq 1$ is a fixed integer, then $g = 1$. Later Bell and Daif [3] proved some results which have the same flavour, when the automorphism is replaced by a non-zero deivation d. They showed that if R is a semiprime ring with a non-zero ideal I such that $d([x, y]) - [x, y] = 0$, or $d([x, y]) + [x, y] = 0$, for all $x, y \in I$, then I is central. More recently Hongan [13] proved that if R is a 2-torsion free semiprime ring and I a non-zero ideal of R, then I is central if and only if $d([x, y]) - [x, y] \in Z(R)$, or $d([x, y]) + [x, y] \in Z(R)$, for all $x, y \in I$.

In this paper we prove two results generalizing some of the previous ones. More precisely we consider the case when $f(x_1, \ldots, x_n)$ is a multili-

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near polynomial over C in n non-commuting variables, ϱ a non-zero right ideal of R and we show

THEOREM 1. *If* $(d(f(r_1, \ldots, r_n)) - f(r_1, \ldots, r_n))^m = 0$, *for any* $r_1, \ldots, r_n \in \varrho$, *then* $C\varrho = eRC$ *for some idempotent element* $e \in \mathrm{Soc}(RC)$ *and* $f(x_1, \ldots, x_n)$ *is a polynomial identity for* $eRCe$.

THEOREM 2. *If* $(d(f(r_1, \ldots, r_n)) - f(r_1, \ldots, r_n))^m \in Z(R)$, *for any* $r_1, \ldots, r_n \in \varrho$, *then* $C\varrho = eRC$ *for some idempotent element* $e \in \mathrm{Soc}(RC)$ *and either* $f(x_1, \ldots, x_n)$ *is central in* $eRCe$ *or* $eRCe$ *satisfies* $S_4(x_1, \ldots, x_4)$.

To prove these theorems we need some notations concerning quotient rings. Denote by Q the two-sided Martindale quotient ring of R and by C the center of Q, which is called the extended centroid of R. Note that Q is also a prime ring with C a field. We will make a frequent use of the following notation:

$$f(x_1, \ldots, x_n) = x_1 \cdot x_2 \cdots x_n + \sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}$$

for some $\alpha_\sigma \in C$ and we denote by $f^d(x_1, \ldots, x_n)$ the polynomial obtained from $f(x_1, \ldots, x_n)$ by replacing each coefficient α_σ with $d(\alpha_\sigma \cdot 1)$. Thus we write $d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n)$, for all $r_1, \ldots, r_n \in R$. We recall that any derivation of R can be uniquely extended to a derivation of Q, moreover by [19] the two-sided ideal I and Q satisfy the same differential identities. For this reason whenever R satisfies a differential identity, by replacing R by Q we will assume, without loss of generality, $R = Q$, $C = Z(R)$ and R will be a C-algebra centrally closed.

To obtain the conclusions required we will also make use of the following result:

CLAIM 1 [14]. Let R be a prime ring, d a non-zero derivation of R and I a non-zero two-sided ideal of R. Let $g(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n))$ a differential identity in I, that is

$$g(r_1, \ldots, r_n, d(r_1), \ldots, d(r_n)) = 0 \qquad \forall r_1, \ldots, r_n \in I .$$

Then one of the following holds:

1) either d is an inner derivation in Q, in the sense that there exists $q \in Q$ such that $d = ad(q)$ and $d(x) = ad(q)(x) = [q, x]$, for all $x \in$

$\in R$, and I satisfies the generalized polynomial identity

$$g(x_1, \ldots, x_n, [q, x_1], \ldots, [q, x_n]);$$

2) or I satisfies the generalized polynomial identity

$$g(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

We premit the following:

LEMMA 1. *Let ϱ be a non-zero right ideal of R and d a derivation of R. Then the following conditions are equivalent:* (i) *d is an inner derivation induced by some $b \in Q$ such that $b\varrho = 0$;* (ii) *$d(\varrho)\varrho = 0$ (For its proof we refer to* [6, *Lemma*]).

LEMMA 2. *If $(d(f(r_1, \ldots, r_n)) - f(r_1, \ldots, r_n))^m \in Z(R)$, for any $r_1, \ldots, r_n \in \varrho$, then R is a GPI-ring.*

PROOF. Assume R is not commutative, otherwise we conclude trivially that R is a GPI-ring. Suppose that d is a inner derivation, $d = ad(b)$, for some $b \in Q$, $d(x) = [b, x]$, for all $x \in Q$. Since $d \neq 0$, let $b \notin C$. Moreover, since R is not commutative, there exists $a \in \varrho - C$. Thus $[([b, f(ax_1, \ldots, ax_n)] - f(ax_1, \ldots, ax_n))^m, x_{n+1}]$ is a non-trivial GPI for R.

Let now d an outer derivation of R. If for all $r \in \varrho$, $d(r) \in rC$, then $[d(r), r] = 0$, that is R is commutative (see [4]). Therefore there exists $a \in \varrho$ such that $d(a) \notin aC$. Write

$$d(f(ax_1, \ldots, ax_n)) =$$

$$= f^d(ax_1, \ldots, ax_n) + \sum_i f(ax_1, \ldots, d(a)x_i + ad(x_i), \ldots, ax_n).$$

Thus

$$\Big[\Big(f^d(ax_1, \ldots, ax_n) +$$

$$+ \sum_i f(ax_1, \ldots, d(a)\, x_i + ad(x_i), \ldots, ax_n) - f(ax_1, \ldots, ax_n)\Big)^m, x_{n+1}\Big]$$

is a generalized differential identity for R. In particular, by Kharchenko's theorem in [14], since $d(a) \notin aC$, we have that

$$\Big[\Big(f^d(ax_1, \ldots, ax_n) + \sum_i f(ax_1, \ldots, d(a)\, x_i, \ldots, ax_n) - f(ax_1, \ldots, ax_n)\Big)^m, x_{n+1}\Big]$$

is a non-trivial GPI for R. ■

Before proceeding to he proof of main results, we need to resolve the simplest case, when $\varrho = R$.

LEMMA 3. *Let $R = M_k(F)$ be the ring of $k \times k$ matrices over the field F, with $k \geq 2$, d a non-zero inner derivation induced by a non-central element A of R. Theorems 1 and 2 hold if $\varrho = R$.*

PROOF. Suppose $k \geq 3$. Let e_{ij} the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. By the assumption

$$([A, f(r_1, \ldots, r_n)] - f(r_1, \ldots, r_n))^m \in Z(R) \qquad \forall r_1, r_2, \ldots, r_n \in R\,.$$

If assume $f(x_1, \ldots, x_n)$ not central in R, by [20, Lemma 2, proof of Lemma 3] there exist $r_1, \ldots, r_n \in R$ such that $f(r_1, \ldots, r_n) = ae_{ij}$, with $0 \neq a \in F$ and $i \neq j$. Since the subset $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in R\}$ is invariant under any F-automorphism, then for any $i \neq j$ there exist $t_1, \ldots, t_n \in R$ such that $f(t_1, \ldots, t_n) = ae_{ij}$. Thus, for any $i \neq j$

$$([A, ae_{ij}] - ae_{ij})^m \in Z(R)$$

moreover $([A, ae_{ij}] - ae_{ij})^m$ has rank ≤ 2, that is $([A, ae_{ij}] - ae_{ij})^m = 0$ in R. Right multiplying by e_{ij}

$$0 = (Aae_{ij} - ae_{ij}A - ae_{ij})^m e_{ij} = (ae_{ij}A)^m e_{ij}\,.$$

It follows that the (j,i)-entry of the matrix A is zero, for all $i \neq j$ and this means that the A is diagonal, that is $A = \sum_t \alpha_t e_{tt}$, with $\alpha_t \in F$. Now denote d the inner derivation induced by A. If χ is a F-automorphism of R, then the derivation $d_\chi = \chi^{-1} d \chi$ satisfies the same condition of d, that is

$$(d_\chi(f(r_1, \ldots, r_n)) - f(r_1, \ldots, r_n))^m \in Z(R) \qquad \text{for any } r_1, \ldots, r_n \in R\,.$$

Since the derivation d_χ is the one induced by the element $\chi(A) = \chi^{-1} A \chi$, then $\chi(A)$ is a diagonal matrix, according to the above argument. Fix now $i \neq j$ and $\chi(x) = (1 + e_{ij})\, x (1 - e_{ij})$, for all $x \in R$. Since $\chi(A) = (1 +$

$+ e_{ij})A(1 - e_{ij})$ must be diagonal then

$$\sum_t \alpha_t e_{tt} - \alpha_i e_{ij} + \alpha_j e_{ij} \quad \text{is diagonal}$$

that is $\alpha_i = \alpha_j$ and we get the contradiction that A is a central matrix. Therefore $f(x_1, \ldots, x_n)$ must be central in R.

Of course if $([A, f(r_1, \ldots, r_n)] - f(r_1, \ldots, r_n))^m = 0$, for all $r_1, \ldots, r_n \in R$, the above argument can be adapted to prove that $f(x_1, \ldots, x_n)$ is central, without any restriction on k. Moreover, since in this case $[A, f(r_1, \ldots, r_n)] = 0$, then $f^m(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \in R$ and so $f(x_1, \ldots, x_n)$ is an identity in R [20, Lemma 3, proof of Theorem 4]. ∎

LEMMA 4. *Theorem* 1 *holds if* $\varrho = R$.

PROOF. Let

$$g(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n)) = (d(f(x_1, \ldots, x_n)) - f(x_1, \ldots, x_n))^m =$$

$$= \Big(f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n) - f(x_1, \ldots, x_n)\Big)^m.$$

If d is not inner then, by Claim 1, R satisfies the differential identity

$$g(x_1, \ldots, x_n, y_1, \ldots, y_n) =$$

$$= \Big(f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n) - f(x_1, \ldots, x_n)\Big)^m.$$

In particular $f^m(x_1, \ldots, x_n)$ is an identity for R. In this case since R satisfies a polynomial identity, there exists a suitable field F such that R and $M_k(F)$ satisfy the same polynomial identities. It follows that $f(x_1, \ldots, x_n)$ must be an identity in $M_k(F)$ (see [20]) and so in R.

Now let d be an inner derivation induced by an element $A \in Q$.

Then, for any $r_1, r_2, \ldots, r_n \in R$, $([A, f(r_1, \ldots, r_n)] - f(r_1, \ldots, r_n))^m = = 0$. Since by [1] (see also [7]) R and Q satisfy the same generalized polynomial identities, we have $([A, f(r_1, \ldots, r_n)] - f(r_1, \ldots, r_n))^m = 0$, for any $r_1, r_2, \ldots, r_n \in Q$. Moreover, since Q remains prime by the primeness of R, replacing R by Q we may assume that $A \in R$ and $C = Z(Q)$ is just the center of R. In the present situation R is a centrally closed prime C-algebra [10], i.e. $RC = R$. By Martindale's theorem in [21], $RC = R$ is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D. Since R is primitive then there exist a vector space V and the division ring D such that R is dense of D-linear transformations over V.

Assume first that $dim_D V = \infty$. Recall that one can write $f(x_1, \ldots, x_n) = x_1 x_2 .. x_n + \sum_{\sigma \neq 1} \beta_\sigma x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$. We want to show that, for any $v \in V$, v and Av are linearly D-dependent.

If $Av = 0$ then $\{v, Av\}$ is D-dependent. Thus we may suppose that $Av \neq 0$. If v and Av are D-independent, since $dim_D V = \infty$, then there exist $w_3, \ldots, w_n \in V$ such that $v = w_1, Av = w_2, w_3, \ldots, w_n$ are also linearly independent. By the density of I, there exist $r_1, \ldots, r_n \in I$ such that

$$r_n w_2 = w_{n-1}$$

$$r_i w_i = w_{i-1} \quad \text{for} \quad 4 \leq i \leq n-1$$

$$r_3 w_3 = w_n$$

$$r_2 w_n = w_1$$

$$r_1 w_1 = w_1$$

$$r_i w_j = 0 \quad \text{for all other possible choices of } i, j\,.$$

Therefore

$$([A, f(r_1, \ldots, r_n)] - f(r_1, \ldots, r_n))\, v = -v$$

and we obtain the contradiction

$$0 = ([A, f(r_1, \ldots, r_n)] - f(r_1, \ldots, r_n))^m v = (-1)^m v \neq 0\,.$$

Hence A, Av must be D-dependent, for any $v \in V$.

Now we show that there exists $b \in D$ such that $Av = vb$, for any $v \in V$. Choose $v, w \in V$ linearly independent. Since $dim_D V = \infty$, there exists $u \in V$ such that v, w, u are linearly independent. By above argument, there exist $a_v, a_w, a_u \in D$ such that

$$Av = va_v,\; Aw = wa_w,\; Au = ua_u \;\text{ that is }\; A(v + w + u) = va_v + wa_w + ua_u\,.$$

Moreover $A(v + w + u) = (v + w + u)\, a_{v+w+u}$, for a suitable $a_{v+w+u} \in D$. Then $0 = v(a_{v+w+u} - a_v) + w(a_{v+w+u} - a_w) + u(a_{v+w+u} - a_u)$ and, because v, w, u are linearly independent, $a_u = a_w = a_v = a_{v+w+u}$, as required.

Let now $r \in R$ and $v \in V$. As we have just seen, there exists $b \in D$ such that $Av = vb$, $r(Av) = r(vb)$, and also $A(rv) = (rv)\, b$. Thus $0 = [A, r]\, v$, for any $v \in V$, that is $[A, r] V = 0$. Since V is a left faithful irreducible Rmodule, $[A, r] = 0$, for all $r \in R$, i.e. $A \in Z(R)$ and $d = 0$, which contradicts our hypothesis.

Therefore $\dim_D V$ must be a finite positive integer. In this case R is a simple GPI ring with 1, and so it is a central simple algebra finite dimensional over its center. From Lemma 2 in [16] it follows that there exists a suitable field F such that $R \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, and moreover $M_k(F)$ satisfies the generalized polynomial identity $([A, f(x_1, \ldots, x_n)] - f(x_1, \ldots, x_n))^m$.

As in Lemma 3 we conclude that $f(x_1, \ldots, x_n)$ is an identity in R. ■

LEMMA 5. *Theorem 2 holds if $\varrho = R$.*

PROOF. If, for every $r_1, r_2, \ldots, r_n \in I$, $(d(f(r_1, \ldots, r_n)) - f(r_1, \ldots, r_n))^m = 0$, by Lemma 4, $f(r_1, \ldots, r_n)$ is an identity in R. Otherwise, by our assumptions, $I \cap Z(R) \neq 0$. Let now K be a non-zero two-sided ideal of R_Z, the ring of the central quotients of R. Since $K \cap R$ is an ideal of R then $K \cap R \cap Z(R) \neq 0$, that is K contains an invertible element in R_Z, and so R_Z is simple with 1.

We know that for any $r_1, r_2, \ldots, r_n \in R$, $(d(f(r_1, \ldots, r_n)) - f(r_1, \ldots, r_n))^m \in Z(R)$, i.e.

$$[(d(f(r_1, \ldots, r_n)) - f(r_1, \ldots, r_n))^m, s] = 0 \quad \text{for any } s \in R\,.$$

Thus R satisfies the differential identity

$$g(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n)) =$$

$$= \left[\left(f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, d(x_i), \ldots, x_n) - f(x_1, \ldots, x_n)\right)^m, x_{n+1}\right].$$

If the derivation is not inner, by Claim 1, R satisfies the polynomial identity

$$g(x_1, \ldots, x_n, y_1, \ldots, y_n) =$$

$$= \left[\left(f^d(x_1, \ldots, x_n) + \sum_i f(x_1, \ldots, y_i, \ldots, x_n) - f(x_1, \ldots, x_n)\right)^m, x_{n+1}\right]$$

and in particular R satisfies

$$\left[\left(\sum_i f(x_1, \ldots, y_i, \ldots, x_n) - f(x_1, \ldots, x_n)\right)^m, x_{n+1}\right]$$

and so $[f^m(x_1, \ldots, x_n), x_{n+1}]$. Therefore R is a prime PI-ring. For $a \in$

$\in R - Z(R)$, we have that R satisfies

$$\left[\left(\sum_i f(x_1, \ldots, [a, x_i], \ldots, x_n) - f(x_1, \ldots, x_n)\right)^m, x_{n+1}\right] =$$

$$= [([a, f(x_1, \ldots, x_n)] - f(x_1, \ldots, x_n))^m, x_{n+1}]$$

and in this situation we get the required conclusion by lemma 3.

Now let d be an inner derivation induced by an element $A \in Q$. Also in this case we will prove that either $f(x_1, \ldots, x_n)$ is central in R or R satisfies $S_4(x_1, \ldots, x_4)$.

By localizing R at $Z(R)$ it follows that $([A, f(r_1, \ldots, r_n)] - f(r_1, \ldots, r_n))^m \in Z(R_Z)$, for all $r_1, r_2, \ldots, r_n \in R_Z$.

Since R and R_Z satisfy the same polynomial identities, in order to prove that R satisfies $[f(x_1, \ldots, x_n), x_{n+1}]$, we may assume that R is simple with 1.

In this case, $([A, f(r_1, \ldots, r_n)] - f(r_1, \ldots, r_n))^m \in Z(R)$, for all $r_1, r_2, \ldots, r_n \in R$. Therefore R satisfies a generalized polynomial identity and it is simple with 1, which implies that $Q = RC = R$ and R has a minimal right ideal. Thus $A \in R = Q$ and R is simple artinian that is $R = D_k$, where D is a division ring finite dimensional over $Z(R)$ [21]. From Lemma 2 in [16] it follows that there exists a suitable field F such that $R \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, and moreover $M_k(F)$ satisfies the generalized polynomial identity $[([A, f(x_1, \ldots, x_n)] - f(x_1, \ldots, x_n))^m, x_{n+1}]$. We end up again by lemma 3. ■

REMARK. *In all that follows we prefer to write the polynomial* $f(x_1, \ldots, x_n)$ *by using the following notation:*

$$f(x_1, \ldots, x_n) = \sum_i g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\, x_i$$

where any g_i *is a multilinear polynomial of degree* $n-1$ *and* x_i *never appears in any monomial of* g_i. *Note that if there exists an idempotent* $e \in H = \mathrm{Soc}(Q)$ *such that any* g_i *is a polynomial identity for* eHe, *then we get the conclusion that* $f(x_1, \ldots, x_n)$ *is a polynomial identity for* eHe. *Thus we suppose that there exists an index* i *and* $r_1, \ldots, r_{n-1} \in eHe$ *such that* $g_i(r_1, \ldots, r_{n-1}) \neq 0$. *Now let* $f(x_1, \ldots, x_n) = g_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\, x_i + h(x_1, \ldots, x_n)$ *where* g_i *and* h *are multilinear polynomials,* x_i *never appears in any monomials of* g_i *and* x_i *never appears as last variable in any monomials of* h. *Without loss of generality we assume* $i = n$, *say* $g_n(x_1, \ldots, x_{n-1}) = t(x_1, \ldots, x_{n-1})$ *and* *so* $f(x_1, \ldots, x_n) = t(x_1, \ldots, x_{n-1})\, x_n + h(x_1, \ldots, x_n)$ *where* $t(eHe) \neq 0$.

PROOF OF THEOREM 1. Suppose first that $f(x_1, \ldots, x_n)\, x_{n+1}$ is not an identity for ϱ. We proceed to derive a contradiction. Since by lemma 2 R is a GPI ring, so is also Q (see [1] and [7]). By [21] Q is a primitive ring with $H = \mathrm{Soc}\,(Q) \neq 0$, moreover we may assume that $f(x_1, \ldots, x_n)\, x_{n+1}$ is not an identity for ϱH, otherwise by [1] and [7] it should be an identity also for ϱQ, which is a contradiction. Let $a_1, \ldots, a_{n+1} \in \varrho H$ such that $f(a_1, \ldots, a_n)\, a_{n+1} \neq 0$. Since H is a regular ring, then for all $a \in H$ there exists $e^2 = e \in H$ such that $eH = a_1 H + a_2 H + \ldots + a_{n+1} H$, $e \in eH$, $a = ea$ and $a_i = ea_i$ for all $i = 1, \ldots, n+1$. Therefore we have $f(eHe) = f(eH)\, e \neq 0$. By our assumption and by [19] we also assume that $(d(f(x_1, \ldots, x_n)) - f(x_1, \ldots, x_n))^m$ is an identity for ϱQ. In particular $(d(f(x_1, \ldots, x_n)) - f(x_1, \ldots, x_n))^m$ is an identity for eH. It follows that, for all $r_1, \ldots, r_n \in H$,

$$0 = (d(ef(er_1, \ldots, er_n)) - f(er_1, \ldots, er_n))^m =$$
$$= (d(e)\, f(er_1, \ldots, er_n) + ed(f(er_1, \ldots, er_n)) - f(er_1, \ldots, er_n))^m.$$

As we said above, write $f(x_1, \ldots, x_n) = t(x_1, \ldots, x_{n-1}) x_n + h(x_1, \ldots, x_n)$, where x_n never appears as last variable in any monomials of h. Let $r \in H$ and pick $r_n = r(1-e)$. Hence we have:

$$\begin{aligned}
0 = (&d(e)\, t(er_1, \ldots, er_{n-1})\, er(1-e) + ed(t(er_1, \ldots, er_{n-1}))\, er(1-e) + \\
&+ et(er_1, \ldots, er_{n-1})\, d(e)\, r(1-e) + et(er_1, \ldots, er_{n-1})\, ed(r)(1-e) + \\
&+ et(er_1, \ldots, er_{n-1})\, erd(1-e) - t(er_1, \ldots, er_{n-1})\, er(1-e))^m = \\
&= (d(e)\, t(er_1, \ldots, er_{n-1})\, er(1-e) + ed(t(er_1, \ldots, er_{n-1}))\, er(1-e) + \\
&+ et(er_1, \ldots, er_{n-1})\, d(e) r(1-e) + et(er_1, \ldots, er_{n-1})\, ed(r)(1-e) + \\
&+ et(er_1, \ldots, er_{n-1})\, erd(1-e) - t(er_1, \ldots, er_{n-1})\, er(1-e)) \cdot \\
&\cdot (d(e)\, t(er_1, \ldots, er_{n-1})\, er(1-e))^{m-1}.
\end{aligned}$$

Left multiplying by $(1-e)$ we obtain

$$0 = (1-e)(d(e)\, t(er_1, \ldots, er_{n-1})\, er(1-e))^m$$

and so $((1-e)\, d(e)\, t(er_1, \ldots, er_{n-1})\, er)^{m+1} = 0$ that is

$$((1-e)\, d(e)\, t(er_1, \ldots, er_{n-1})\, eH)^{m+1} = 0$$

and, by [11], $(1-e)\,d(e)\,t(er_1, \dots, er_{n-1})\,eH = 0$ which implies

$$((1-e)\,d(e)\,t(er_1 e, \dots, er_{n-1} e) = 0\,.$$

Since eHe is a simple artinian ring and $t(eHe) \neq 0$ is invariant under the action of all inner automorphisms of eHe, by [8, lemma 2], $(1-e)\,d(e) = 0$ and so $d(e) = ed(e) \in eH$. Thus $d(eH) \subseteq d(e)\,H + ed(H) \subseteq eH \subseteq \varrho H$ and $d(a) = d(ea) \in d(eH) \subseteq eH$. This means that $d(\varrho H) \subseteq \varrho H$. Therefore the derivation d induced another one δ, which is defined in the prime ring $\overline{\varrho H} = \dfrac{\varrho H}{\varrho H \cap l_H(\varrho H)}$, where $l_h(\varrho H)$ is the left annihilator in H of ϱH, and $\delta(\overline{x}) = \overline{d(x)}$, for all $x \in \varrho H$. Moreover we obviously have that $(d(f(x_1, \dots, x_n)) - f(x_1, \dots, x_n))^m$ is a differential identity for $\overline{\varrho H}$. So, by lemma 4, one of the following holds: either $\delta = \overline{0}$, or $f(x_1, \dots, x_n)$ is an identity for $\overline{\varrho H}$.

If $\delta = \overline{0}$ then $d(\varrho H) \subseteq l_H(\varrho H)$ that is $d(\varrho H)\,\varrho H = 0$. By lemma 1, d is an inner derivation induced by an element $b \in Q$ such that $b\varrho = 0$. Thus, for all $r_1, \dots, r \in \varrho H$,

$$0 = (d(f(r_1, \dots, r_n)) - f(r_1, \dots, r_n))^m = (f(r_1, \dots, r_n)\,b - f(r_1, \dots, r_n))^m =$$
$$= (-1)^{m-1} f(r_1, \dots, r_n)^m b + (-1)^m f(r_1, \dots, r_n)^m.$$

Right multiplying by $f(r_1, \dots, r_n)$ we have $f(r_1, \dots, r_n)^{m+1} = 0$ and, as a consequence of main theorem in [8] we get the contradiction $f(r_1, \dots, r_n)\,\varrho H = 0$. Also in the case $f(x_1, \dots, x_n)$ is an identity for $\overline{\varrho H}$ we obtain the contradiction that $f(x_1, \dots, x_n)\,x_{n+1}$ is an identity for ϱH.

Finally we are in the case when $f(r_1, \dots, r_n)\,r_{n+1} = 0$ for all $r_1, \dots, r_{n+1} \in \varrho$. In this case, the proof of theorem 6 of [18, page 17, rows 3-8] shows that there exists an idempotent element $e \in Soc(RC)$ such that $C\varrho = eRC$ and $f(x_1, \dots, x_n)$ is an identity for $eRCe$. ∎

Proof of Theorem 2. Consider first the case when $[f(x_1, \dots, x_n), x_{n+1}]\,x_{n+2}$ is an identity for ϱ. By [18, proposition] $C\varrho = eRC$ for some idempotent element $e \in \mathrm{Soc}(RC)$. Moreover, by [7], theorem 2, $[f(x_1, \dots, x_n), x_{n+1}]x_{n+2}$ is also an identity in ϱR and so in ϱQ. In particular it is an identity for $\varrho C = eRC$, that is $[f(er_1, \dots, er_n), er_{n+1}]\,er_{n+2} = 0$, for all $r_1, \dots, r_{n+2} \in RC$ and so, for all $r_1, \dots, r_{n+1} \in RC$, $[f(er_1 e, \dots, er_n e), er_{n+1} e] = 0$. This means that $f(x_1, \dots, x_n)$ is central-valued in $eRCe$ and we are done.

Suppose now that $[f(x_1, \dots, x_n), x_{n+1}]\,x_{n+2}$ is not an identity for ϱ.

As in proof of theorem 1, since by lemma 2 R is a GPI ring and so is also Q ([1], [6]), Q is a primitive ring with socle $H = \mathrm{Soc}(Q) \neq 0$ [21] and $[f(x_1, \ldots, x_n), x_{n+1}] x_{n+2}$ is not an identity for ϱH, otherwise we have the contradiction that $[f(x_1, \ldots, x_n), x_{n+1}] x_{n+2}$ should be an identity for ϱQ. Let $a_1, \ldots, a_{n+2} \in \varrho H$ such that $[f(a_1, \ldots, a_n), a_{n+1}] a_{n+2} \neq 0$. By the regularity of H, for all $a \in \varrho H$, there exists an idempotent element $g \in \varrho H$ such that $a = ga$, $a_i = ga_i$, for all $i = 1, \ldots, n+2$. Moreover, by [19], $[(d(f(x_1, \ldots, x_n)) - f(x_1, \ldots, x_n))^m, x_{n+1}]$ is an identity in ϱQ, in ϱH and also in gH. As above we write $f(x_1, \ldots, x_n) = t(x_1, \ldots, x_{n-1}) x_n + h(x_1, \ldots, x_n)$, where t and h are multilinear polynomials, x_n never appears in any monomials of t, x_n never appears as last variable in any monomials of h and let $r_1, \ldots, r_n \in H$, with $r_n = r(1-g)$. Thus $f(gr_1, \ldots, gr_n) = t(gr_1, \ldots, gr_{n-1}) gr(1-g)$ and again

$$(1) \quad (d(f(gr_1, \ldots, gr_n)) - f(gr_1, \ldots, gr_n))^m =$$
$$= (d(t(gr_1, \ldots, gr_{n-1}) gr(1-g)) - t(gr_1, \ldots, gr_{n-1}) gr(1-g)) \cdot$$
$$\cdot (d(g)\, t(gr_1, \ldots, gr_{n-1}) gr(1-g))^{m-1} \in C.$$

Therefore, by commuting (1) with $gr(1-g)$, we have

$$0 = gr(1-g)(d(g)t(gr_1, \ldots, gr_{n-1}) gr(1-g))^{m-1}$$

that is

$$((1-g)\, d(g)t(gr_1, \ldots, gr_{n-1})\, gH)^{m+1} = 0$$

and by [12] $(1-g)\, d(g)\, t(gr_1, \ldots, gr_{n-1})\, gH$. Since gHg is a simple artinian ring and $t(gHg) \neq 0$ is invariant under the action of all the inner automorphisms of gHg, by [8, lemma 2], $(1-g)\, d(g) = 0$, that is $d(g) = gd(g) \in gH$. Therefore $d(gH) \subseteq d(g) H + gd(H) \subseteq gH \subseteq \varrho H$ and so $d(\varrho H) \subseteq \varrho H$. Therefore the derivation d induced another one δ, which is defined in the prime ring $\overline{\varrho H} = \dfrac{\varrho H}{\varrho H \cap l_H(\varrho H)}$, where $l_H(\varrho H)$ is the left annihilator in H of ϱH, and $\delta(\overline{x}) = \overline{d(x)}$, for all $x \in \varrho H$. Moreover we obviously have that $[(d(f(x_1, \ldots, x_n)) - f(x_1, \ldots, x_n))^m, x_{n+1}]$ is a differential identity for $\overline{\varrho H}$. By lemma 5, one of the following holds: either $\delta(\overline{\varrho H}) = 0$ or $f(x_1, \ldots, x_n)$ is central-valued in $\overline{\varrho H}$ or $\overline{\varrho H}$ satisfies the standard identity $S_4(x_1, \ldots, x_4)$.

If $f(x_1, \ldots, x_n)$ is central-valued in $\overline{\varrho H}$ we get the contradiction that

$$[f(x_1, \ldots, x_n), x_{n+1}] x_{n+2}$$

is an identity for ϱ. On the other hand, if $\delta(\overline{\varrho H}) = 0$, as in the proof of theorem 1, we have that d is an inner derivation induced by an element $b \in Q$ such that $b\varrho = 0$ and for all $r_1, \ldots, r_n \in \varrho H$

$$(2) \quad (d(f(r_1, \ldots, r_n)) - f(r_1, \ldots, r_n))^m = (f(r_1, \ldots, r_n) b - f(r_1, \ldots, r_n))^m =$$

$$= (-1)^{m-1} f(r_1, \ldots, r_n)^m b + (-1)^m f(r_1, \ldots, r_n)^m \in C .$$

By commuting (2) with $f(r_1, \ldots, r_n)$ we get $(-1)^{m-1} f(r_1, \ldots, r_n)^{m+1} b = 0$.

In this case, the main theorem in [8] says that $f(r_1, \ldots, r_n)\, \varrho H b = 0$, for all $r_1, \ldots, r_n \in \varrho H$. Since H is prime and $b \neq 0$, it follows that $f(r_1, \ldots, r_n)\, \varrho H = 0$, and a fortiori $[f(r_1, \ldots, r_n), r_{n+1}] r_{n+2} = 0$, for all $r_1, \ldots, r_n \in \varrho H$, a contradiction.

Finally we consider the last case when $S_4(x_1, \ldots, x_4)$ is an identity for $\overline{\varrho H}$. In this condition $S_4(x_1, \ldots, x_4) x_5$ is an identity for ϱH and so also for ϱC. By [18, proposition], there exists an idempotent element $e \in \mathrm{Soc}(RC)$ such that $\varrho C = eRC$ and so $S_4(eRC)\, eRC = 0$, which means $S_4(eRCe) = 0$, as required. ■

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