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p-adic Completions and Automorphisms of Nilpotent Groups.

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ABSTRACT - Given a group G, a new construction of a torsion-free, nilpotent group H of class two is given such that $\mathrm{Aut}\, H/\mathrm{Stab}\, H \cong G$. When $G = \{e\}$, it is shown that $\mathrm{Aut}\, H = \mathrm{Inn}\, H$.

1. Introduction.

It was shown in [4] that any group G is the outer automorphism group $\mathrm{Aut}\, H/\mathrm{Inn}\, H$ of some torsion-free metabelian group H. If the given group G has infinite cardinality $< 2^{\aleph_0}$, then we may also assume that $|H| = |G|$ (see [5]). It is natural to ask whether this result can be strengthened to nilpotent groups of class two.

In [2] and [3], two different constructions of a group H are presented, wherein H is a torsion-free nilpotent group of class two and $\mathrm{Aut}\, H/\mathrm{Stab}\, H$ is a prescribed group. (If K is any group, the group $\mathrm{Stab}\, K$ is defined in this paper to be

$$\mathrm{Stab}\, K = \{\varphi \in \mathrm{Aut}\, K : \varphi \restriction_{Z(K)} = id_{Z(K)} \quad \text{and} \quad \varphi \restriction_{K/Z(K)} = id_{K/Z(K)}\}$$

where $Z(K)$ denotes the center of K. Clearly, $\mathrm{Inn}\, H \subseteq \mathrm{Stab}\, H$, if H is nilpotent of class two.) The first one made use of Zalesskii's construction of

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a torsion-free nilpotent group of class 2, with rank 3 but having no outer automorphisms. The second involved the creation of a 2-divisible, torsion-free abelian group X admitting an alternating bilinear map.

In this paper we consider the p-adic completion $\widehat{N}$ of a free nilpotent group N and construct a group H such that $N < H < \widehat{N}$ and Aut $H/$Stab H is a prescribed group. It is hoped that the given proof, based entirely on group theory, will give more insight into this replacement of Inn H by Stab H. The construction is very canonical and we can also show that in this setting *we can replace* Stab H *by* Inn H *if and only if the given group* $G = \{e\}$. Section 2 gives a description of the p-adic completion $\widehat{N}$ of a free nilpotent group N of class two and its elements. Section 3 contains a characterization of the elements of Inn $\widehat{N}$ and Stab $\widehat{N}$ and the main theorem.

2. p-adic completion of a free nilpotent group of class two.

If N is a nilpotent group and p is a prime, a topology on N, called the p-adic topology on N, can be defined by taking the set $\{N^{p^n}: n \in \omega\}$ as a base of open sets about $\{1\}$, with p-adic completion $\widehat{N}$ defined to be

$$\varprojlim_n N/N^{p^n} = \left\{(a_i)_{i<\omega} \in \prod_{i<\omega} N/N^{p^i}: \Pi^j_i(a_j) = a_i \text{ for all } i<j\right\},$$

where $\Pi^j_i: N/N^{p^j} \to N/N^{p^i}$ is defined to be $\Pi^j_i(aN^{p^j}) = aN^{p^i}$ $(i<j)$. If we take N to be free nilpotent of class two with free generating set $\{x_i: i \in I\}$, then the given base of open sets determines a Hausdorff topology on N and N embeds in $\widehat{N}$, where $x \in N$ is identified with $(xN^{p^i})_{i<\omega} \in \widehat{N}$. Moreover $\widehat{N}$ is also nilpotent of class two, $\widehat{N'} = \widehat{N}'$ and $\widehat{N}/\widehat{N}' \cong \widehat{N/N'}$ (see [7], p. 55). If N is free nilpotent, every element $g \in N$ can be represented as

$$\prod_{i\in I} x_i^{k_i} \prod_{i \neq j} [x_i, x_j]^{k_{ij}},$$

where only finitely many k_i, $k_{ij} \in \mathbb{Z}$ are non-zero. The set $\{x_i N': i \in I\}$ is a free set of generators for the free abelian group N/N'. If, in addition, I is a linearly ordered set, then $\{[x_i, x_j]: i<j\}$ is a free set of generators for the free abelian group N' and every element g can be uniquely represented as

$$g = x_{i_1}^{k_1} \dots x_{i_n}^{k_n} \prod_{i<j} [x_i, x_j]^{k_{ij}},$$

where $i_1 < \ldots < i_n$ and only finitely many k_{ij} are non-zero (see [6], p. 165).

If $x \in N$ and $\xi = (g_i + p^i \mathbb{Z})_{i<\omega} \in J_p$ is a p-adic integer, we let x^{ξ} denote the element $(a_i)_{i<\omega} \in \widehat{N}$, where $a_i = x^{g_i} N^{p^i}$. If $x, y \in N$, then

$$(xy)^{\xi} = x^{\xi} y^{\xi} [y, x]^{\frac{(\xi-1)\xi}{2}},$$

since $(ab)^n = a^n b^n [b, a]^{\frac{(n-1)n}{2}}$ for all $n \in \mathbb{Z}$. Let $\prod_{k<\omega} x_{i_k}^{\xi_k}$ denote the infinite product $x_{i_1}^{\xi_1} x_{i_2}^{\xi_2} x_{i_3}^{\xi_3} \ldots$. It is easy to see that every element of $\widehat{N}$ can be written (not necessarily uniquely) in the form

$$\prod_{k<\omega} x_{i_k}^{\xi_k} \prod_{j<k<\omega} [x_{i_j}, x_{i_k}]^{\xi_{jk}},$$

where $\xi_i, \xi_{jk} \in J_p$ with only countably many ξ_k and ξ_{jk} non-zero and for all n, p^n divides all but finitely many ξ_k and ξ_{jk}.

3. Prescribing automorphism groups.

In this section we show how a given group can be realized as the automorphism group of a torsion-free nilpotent group of class two modulo its stabilizer. We begin by defining some preliminary notions which have appeared in [1] and [4] within the framework of modules, but which are now formulated in the context of nilpotent groups.

Let λ be a regular cardinal such that $\lambda \geq 2^{\aleph_0}$, and define the tree $T = {}^{\omega>}\lambda$ to be the set of all functions $\tau : n \to \lambda$ $(n < \omega)$. If σ and τ are two functions in T, define $\sigma \leq \tau$ if $\sigma \subseteq \tau$. Let G be any group such that $|G| \leq \lambda$ and e be the identity of G. Define N_G to be the free nilpotent group of class 2 with free generating set $\{g_\tau : g \in G, \tau \in T\}$. Note that G acts on N_G via

$$(g_\tau)^h = (gh)_\tau \tag{3.1}$$

and G embeds in Aut N_G. Moreover the same action makes N_G/N_G' a $\mathbb{Z}[G]$-module. Let p be an odd prime and $\widehat{N}_G$ be the p-adic completion of N_G. If $y \in \widehat{N}_G$, then

$$y = \prod_{(g,\tau) \in G \times T} g_\tau^{\xi_{g_\tau}} \prod_{g_\tau \neq h_\mu} [g_\tau, h_\mu]^{\xi_{g_\tau h_\mu}},$$

where countably many $\xi_{g_\tau}, \xi_{g_\tau h_\mu}$ are non-zero p-adic integers, and, for all n, p^n divides all but finitely many ξ_{g_τ} and $\xi_{g_\tau h_\mu} - \xi_{h_\mu g_\tau}$ $(g_\tau \neq h_\mu)$. Define the *T-support of y* to be

$$[y] = \{\tau, \mu \in T : \xi_{g_\tau} \neq 0 \text{ or } \xi_{g_\tau h_\mu} - \xi_{h_\mu g_\tau} \neq 0\}.$$

Hence $[y]$ is the smallest subset S of T such that $y \in \langle g_\tau : g \in G, \tau \in S \rangle$. Define the *norm of y* to be

$$\|y\| = \min\{\nu \subseteq \lambda : [y] \subseteq {}^{\omega>}\nu\}.$$

A *branch v of T* is defined to be a linearly ordered sequence $v = (v_n)_{n<\omega}$, where $v_n : n \to \lambda$ and $v_n \leqslant v_{n+1}$. Note that $v \in {}^{\omega}\lambda \setminus T$ and $v \restriction_n = v_n$. If $X \subseteq T$, the set of all branches of T contained in X will be denoted by $Br(X)$. If v is a branch of T and ν is an ordinal such that $\nu < \lambda$, define *the part of v to the right of ν* to be

$${}_\nu[v] = \{v_n : \|v_n\| > \nu\}.$$

If $v = (v_n)_{n<\omega}$ is a branch of T, define

$$v^1 = \prod_{i<\omega} e_{v_i}^{p^i} \in \widehat{N}_G.$$

If H is a group such that $N < H < \widehat{N}$, define the *purification H_* of H in $\widehat{N}$* to be

$$H_* = \{x \in \widehat{N} : x^{p^k} \in H \text{ for some } k \in \mathbb{Z}\}.$$

If p is an odd prime and $x, y \in \widehat{N}$ such that $x^{p^k}, y^{p^k} \in H$, then we obtain the equation $(xy)^{p^{2k}} = x^{p^{2k}} y^{p^{2k}} [y^{p^k}, x^{p^k}]^{\frac{p^{2k}-1}{2}}$. Thus it is clear that H_* is a subgroup of $\widehat{N}$, if p is an odd prime. Moreover, if p is an odd prime and N is any nilpotent group of class two, then if x and y are p-th powers in N, then so is xy. Define a *canonical subgroup P of $\widehat{N}_G$* to be a subgroup of the form

$$N_{T_0} = \langle g_\tau : g \in G, \tau \in T_0 \rangle,$$

for some countable subset T_0 of T. We identify each $\tau \in T$ with the element e_τ of N.

DEFINITION 3.1. *A trap (f, P, φ) is a triple, where $f : {}^{\omega>}\omega \to T$ is a tree embedding, P is a canonical subgroup of $\widehat{N}_G$, and $\varphi \in \operatorname{Aut} \widehat{P}$ such that*

(i) $\operatorname{Im} f \subseteq P$,

(ii) $[P] \subseteq P$, *where* $[P]$ *is a subtree of* T,

(iii) $cf(\|P\|) = \omega$, *and*

(iv) $\|v\| = \|P\|$ *for all* $v \in \mathrm{Br}(\mathrm{Im}\, f)$.

We state without proof the following theorem (see [1]), which holds in ordinary set theory ZFC.

THEOREM 3.2 (The Black Box). *For some ordinal* λ^* *of cardinality* λ, *there exists a transfinite sequence of traps* $(f_\alpha, P_\alpha, \varphi_\alpha)$ $(\alpha < \lambda^*)$ *such that for* $\alpha, \beta < \lambda^*$,

(i) *if* $\beta < \alpha$, *then* $\|P_\beta\| \leq \|P_\alpha\|$;

(ii) *if* $\beta \neq \alpha$, *then* $\mathrm{Br}(\mathrm{Im} f_\alpha) \cap \mathrm{Br}(\mathrm{Im} f_\beta) = \emptyset$;

(iii) *if* $\beta + 2^{\aleph_0} \leq \alpha$, *then* $\mathrm{Br}(\mathrm{Im}\ \mathrm{f}_\alpha) \cap \mathrm{Br}([\mathrm{P}_\beta]) = \emptyset$;

(iv) *for all* $X \subset \widehat{N}$ *with* $|X| \leq \aleph_0$ *and for all* $\varphi \in \mathrm{Aut}\, \widehat{N}$, *there exists* $\alpha < \lambda^*$ *such that*

$$X \subset \widehat{P}_\alpha, \ \|X\| < \|P_\alpha\|, \qquad \varphi \restriction_{P_\alpha} = \varphi_\alpha.$$

We now describe the construction of the torsion-free nilpotent group, which will possess the desired automorphism group. Let $\langle x^G \rangle$ denote the subgroup generated by the set $\{x^g : g \in G\}$.

Choose a transfinite sequence $(f_\alpha, P_\alpha, \varphi_\alpha)_{\alpha < \lambda^*}$ satisfying the conclusion of the Black Box. Let $H_0 = N$. Let $\mu \leq \lambda^*$ and assume we have found an ascending continuous chain of G-invariant subgroups

$$H_\alpha = \langle N, g_\beta^G : \beta < \alpha \rangle_* \qquad (\alpha < \mu)$$

of $\widehat{N}_G$ such that the following hold inductively:

If $\mu = \alpha + 1$, choose a branch $v_\alpha \in \mathrm{Br}(\mathrm{Im} f_\alpha)$. Let $g_\alpha = v_\alpha^1$, if $(v_\alpha^1)^{\varphi_\alpha} \notin \langle H_\alpha, (v_\alpha^1)^G \rangle_*$. Otherwise, let $g_\alpha = x_\alpha \cdot v_\alpha^1$, where $x_\alpha^\varphi \notin \langle H_\alpha, x_\alpha^G \rangle_*$ and $\|x_\alpha\|$, $\|x_\alpha\|^\varphi < \|v_\alpha\|$. If $H_{\alpha+1} = \langle H_\alpha, g_\alpha^G \rangle_*$, we also require that

$$(\dagger) \qquad g_\alpha^{\varphi_\alpha} \notin H_{\alpha+1}$$

$$(*) \qquad g_\beta^{\varphi_\beta} \notin H_{\alpha+1}, \qquad \text{if } g_\beta^{\varphi_\beta} \notin H_\alpha \qquad (\beta < \alpha)$$

If $(\dagger)$ does not occur, take $H_{\alpha+1} = H_\alpha$. If μ is a limit ordinal, take $H_\mu = \bigcup_{\alpha < \mu} H_\alpha$. Finally, let $H_{\lambda^*} = \langle N_G, g_\alpha^G : \alpha < \lambda^* \rangle_*$.

The next, by now standard, argument shows that $(*)$ can be arranged, while $(\dagger)$ depends on the choice of φ_α. Hence we will always choose

g_α as above with ($*$) and, whenever possible, with (†). The latter case is called the *strong case* in [1]. The following theorem shows that condition ($*$) holds for every ordinal α.

THEOREM 3.3. *Suppose H_α is defined as above. Then there exists a branch $v \in \mathrm{Br}(\mathrm{Im} f_\alpha)$ such that $H_{\alpha+1} = \langle H_\alpha, g_\alpha^G \rangle_*$ satisfies ($*$), where $g_\alpha = v^1$.*

PROOF. Suppose that the above conclusion does not hold, i.e., if $v \in \mathrm{Br}(\mathrm{Im} f_\alpha)$ and $H_v = \langle H_\alpha, (v^1)^G \rangle_*$, there exists $\beta = \beta(v) < \alpha$ such that $g_\beta^{\varphi_\beta} \in H_v \setminus H_\alpha$. Then for some integer s, the element $g_\beta^{\varphi_\beta p^s}$ is a product of elements $g_v^{h_i}$, $g_{\beta_i}^{h_j}$ and elements from N_G. Using commutators, there exist integers s_v, n_i (not all zero), $n_{\beta_{ij}}$; $h_i \in G$; n, $u_{ik} \in N_G$; and $h_{vi} \in H_\alpha$ such that $g_\beta^{\varphi_\beta p^s}$ is equal to

$$g_v^{n_1 h_1} \dots g_v^{n_k h_k} n \prod_i g_{\beta_i}^{n_{\beta i1} h_1} \dots g_{\beta_i}^{n_{\beta ik} h_k} \prod_i [u_{i1}, g_{\beta_i}^{h_1}] \dots [u_{ik}, g_{\beta_i}^{h_k}] \cdot$$

$$\cdot \prod_v [h_{v1}, g_v^{h_1}] \dots [h_{vk}, g_v^{h_k}].$$

Let $\nu < \|v\|$. Since ${}_\nu[v]$ is an infinite subset of v and not all n_i's are zero, an infinite subset of v is contained in $[g_\beta^{\varphi_\beta}] \subseteq [P_\beta]$. This means that $[v] \subseteq [P_\beta]$ and so $v \in \mathrm{Br}(\mathrm{Im} f_\alpha) \cap \mathrm{Br}([P_\beta])$. By condition (iii) of the Black Box, $\alpha < \beta + 2^{\aleph_0}$. Hence if $v \in \mathrm{Br}(\mathrm{Im} f_\alpha)$, there exist $\beta(v) < \alpha$; n, $u_{ik} \in N_G$; $h_{vi} \in H_\alpha$; $h_i \in G$ and integers s_v, n_i (not all zero) such that $\beta(v) < \alpha < \beta(v) + 2^{\aleph_0}$ and

$$g_v^{-n_k h_k} \dots g_v^{-n_1 h_1} g_{\beta(v)}^{\varphi_{\beta(v)} p^{s_v}} \prod_v [g_v^{h_1}, h_{v1}] \dots [g_v^{h_k}, h_{vk}] \in H_\alpha.$$

Hence there exist distinct branches v, $w \in \mathrm{Br}(\mathrm{Im} f_\alpha)$ such that $\beta(v) = \beta(w) = \beta$ (see [1], p. 457). Then

$$g_v^{-n_k h_k} \dots g_v^{-n_1 h_1} g_\beta^{\varphi_\beta p^{s_v}} \prod_v [g_v^{h_1}, h_{v1}] \dots [g_v^{h_k}, h_{vk}] \tag{3.2}$$

$$g_w^{-m_k h_k} \dots g_w^{-m_1 h_1} g_\beta^{\varphi_\beta p^{s_w}} \prod_w [g_w^{h_1}, h_{w1}] \dots [g_w^{h_k}, h_{wk}] \tag{3.3}$$

are both in H_α. Taking the p^{s_w}-th power of (3.2) and the p^{s_v}-th power of (3.3), we obtain

$$(g_v^{-n_k h_k} \dots g_v^{-n_1 h_1})^{p^{s_w}} g_\beta^{\varphi_\beta p^{s_v + s_w}} \Big(\prod_v [g_v^{h_1}, h_{v1}] \dots [g_v^{h_k}, h_{vk}] \Big)^{p^{s_w}} \cdot \tag{3.4}$$

$$\cdot [g_\beta^{\varphi_\beta p^{s_v}}, g_v^{-n_k h_k} \dots g_v^{-n_1 h_1}]^{\frac{(p^{s_w} - 1) p^{s_w}}{2}}$$

$$(3.5)\qquad (g_w^{-m_k h_k}\cdots g_w^{-m_1 h_1})^{p^{s_v}} g_\beta^{\varphi\beta p^{s_w+s_v}} \Big(\prod_w [g_w^{h_1}, h_{w1}]\dots[g_w^{h_k}, h_{wk}]\Big)^{p^{s_v}}$$

$$[g_\beta^{\varphi\beta p^{s_w}}, g_w^{-m_k h_k}\cdots g_w^{-m_1 h_1}]^{\frac{(p^{s_v}-1)p^{s_v}}{2}}.$$

Multiplying (3.5) on the right by the inverse of (3.4), we have

$$(g_w^{-m_k h_k}\cdots g_w^{-m_1 h_1})^{p^{s_v}} (g_v^{n_1 h_1}\cdots g_v^{n_k h_k})^{p^{s_w}}\cdot$$

$$\cdot\Big(\prod_w [g_w^{h_1}, h_{w1}]\dots[g_w^{h_k}, h_{wk}]\Big)^{p^{s_v}} \Big(\prod_v [h_{v1}, g_v^{h_1}]\dots[h_{vk}, g_v^{h_k}]\Big)^{p^{s_w}}\cdot$$

$$\cdot[g_\beta^{\varphi\beta p^{s_w}}, g_w^{-m_k h_k}\cdots g_w^{-m_1 h_1}]^{\frac{(p^{s_v}-1)p^{s_v}}{2}} [g_\beta^{\varphi\beta p^{s_v}}, g_v^{n_k h_k}\cdots g_v^{n_1 h_1}]^{\frac{(p^{s_w}-1)p^{s_w}}{2}},$$

which is an element of H_α. By the definition of the supports of the elements of H_α, an infinite subset of v is contained in w or an infinite subset of w is contained in v. This gives a contradiction, since v and w have finite intersection. ■

Recall from equation (3.1) that G embeds in Aut N_G. By continuity, G also embeds in Aut $\widehat{N}_G$. Since the intersection $G \cap$ Stab $\widehat{N}_G$ contains only the identity map and Stab $\widehat{N}_G$ is a normal subgroup of Aut $\widehat{N}_G$, then the semi-direct product Stab $\widehat{N}_G \rtimes G$ also embeds in Aut $\widehat{N}_G$. The following theorem describes the automorphisms which do not extend to the purification of every G-invariant extension of H_α.

THEOREM 3.4. *If* $\varphi \in \operatorname{Aut}\widehat{N}_G \backslash (\operatorname{Stab}(\widehat{N}_G) \rtimes G)$, *then there exists* $x \in \widehat{N}_G$ *such that* $x^\varphi \notin \langle N_G, g_\beta^G, x^G \colon \beta < \alpha\rangle_*$, *where the* g_β*'s are defined as in Theorem* 3.3.

PROOF. Let $H_\alpha = \langle N_G, g_\beta^G \colon \beta < \alpha\rangle_*$ and suppose that $x^\varphi \in \langle H_\alpha, x^G\rangle_*$ for all $x \in \widehat{N}_G$. Let τ and δ be distinct elements of T and $1, \xi, \varrho \in J_p$ be algebraically independent over $\mathbb{Z}$. Then there exists $k \in \mathbb{Z}$ such that

$$x_\tau^{\varphi p^k} \equiv \prod_i g_{\beta_i}^{a_i} \prod_\mu e_\mu^{b_\mu} \bmod \widehat{N}_G'$$

$$y_\delta^{\varphi p^k} \equiv \prod_i g_{\beta_i}^{c_i} \prod_\mu e_\mu^{d_\mu} \bmod \widehat{N}_G'$$

$$(x_\tau^\xi y_\delta^\varrho)^{\varphi p^k} \equiv \prod_i g_{\beta_i}^{e_i} \prod_\mu e_\mu^{f_\mu} (x_\tau^\xi y_\delta^\varrho)^n \bmod \widehat{N}_G'$$

for some a_i, b_μ, c_i, d_μ, n in $\mathbb{Z}[G]$. Since $(x_\tau^\xi y_\delta^\varrho)^\varphi = x_\tau^{\varphi\xi} y_\delta^{\varphi\varrho}$, we have

$$\prod g_{\beta_i}^{a_i\xi} \prod e_\mu^{b_\mu\xi} \prod g_{\beta_i}^{c_i\varrho} \prod e_\mu^{d_\mu\varrho} \equiv \prod g_{\beta_i}^{e_i} \prod e_\mu^{f_\mu} (x_\tau^\xi y_\delta^\varrho)^n \bmod \widehat{N}_G'$$

Thus $\prod g_{\beta_i}^{a_i \xi - e_i + c_i \varrho} \prod e_{\mu}^{b_\mu \xi - f_\mu + d_\mu \varrho} x_\tau^{\xi n} y_\delta^{\varrho n} \equiv 1 \bmod \widehat{N}'_G$. Since the elements $g_\beta \widehat{N}'_G$ and $h_\tau \widehat{N}'_G$ $(\beta < \alpha,\ h \in G,\ \tau \in T)$ are independent in $\widehat{N}_G / \widehat{N}'_G$ and $1, \xi, \varrho$ are algebraically independent over $\mathbb{Z}$, we have that

$$a_i = e_i = c_i = 0 \quad \text{for all } i \quad \text{and} \quad b_\mu = f_\mu = d_\mu = 0 \quad \text{for all } \mu \neq \tau, \delta .$$

Hence

$$x_\tau^{\varphi p^k} \equiv e_\tau^{b_\tau} e_\delta^{b_\delta}, \quad y_\delta^{\varphi p^k} \equiv e_\tau^{d_\tau} e_\delta^{d_\delta}, \quad e_\tau^{b_\tau \xi} e_\delta^{b_\delta \xi} e_\tau^{d_\tau \varrho} e_\delta^{d_\delta \varrho} \equiv e_\tau^{f_\tau} e_\delta^{f_\delta} (x_\tau^\xi y_\delta^\varrho)^n .$$

Applying once more the algebraic independence of $1, \xi, \varrho$ over $\mathbb{Z}$ and the independence of g_τ modulo $\widehat{N}_G / \widehat{N}'_G$ $(g \in G,\ \tau \in T)$ to the preceding congruence, we obtain $b_\delta = d_\tau = 0$, $e_\tau^{b_\tau} = x_\tau^n$, $e_\delta^{d_\delta} = y_\delta^n$,

$$x_\tau^{\varphi p^k} \equiv x_\tau^n \quad \text{and} \quad y_\delta^{\varphi p^k} \equiv y_\delta^n .$$

Suppose now that $n = \sum_{i=1}^{k} w_i h_i \in \mathbb{Z}[G]$. Then $x_\tau^{\sum w_i h_i} \equiv \prod (x h_i)_\tau^{w_i}$, by equation (3.1). Similarly, we obtain

$$x_\tau^{\varphi p^k} = (x h_1)_\tau^{w_1} \ldots (x h_k)_\tau^{w_k} \cdot m_1 \tag{3.6}$$

$$y_\delta^{\varphi p^k} = (y h_1)_\delta^{w_1} \ldots (y h_k)_\delta^{w_k} \cdot m_2 \tag{3.7}$$

$$(x_\tau^\xi y_\delta^\varrho)^{\varphi p^k} = (x_\tau^\xi y_\delta^\varrho)^{w_1 h_1} \ldots (x_\tau^\xi y_\delta^\varrho)^{w_k h_k} \cdot q \tag{3.8}$$

for some $m_i \in H'_\alpha$ and for some $q \in \langle H_\alpha, (x_\tau^\xi y_\delta^\varrho)^G \rangle'$. Let

$$q = m_3 \prod_{h_i} [a_i, x_\tau^\xi y_\delta^\varrho]^{h_i} \prod_{i<j} [(x_\tau^\xi y_\delta^\varrho)^{h_i}, (x_\tau^\xi y_\delta^\varrho)^{h_j}]^{b_{ij}},$$

where $m_3 \in H'_\alpha$, $a_i \in H_\alpha$ and $b_{ij} \in \mathbb{Z}$. But

$$(x_\tau^\xi y_\delta^\varrho)^{\varphi p^k} = x_\tau^{\xi \varphi p^k} y_\delta^{\varrho \varphi p^k} [y_\delta^{\varrho \varphi}, x_\tau^{\xi \varphi}]^{\frac{(p^k - 1) p^k}{2}} = \tag{3.9}$$

$$= ((x h_1)_\tau^{w_1} \ldots (x h_n)_\tau^{w_n})^\xi ((y h_1)_\delta^{w_1} \ldots (y h_n)_\delta^{w_n})^\varrho m_1^\xi m_2^\varrho [y_\delta^{\varrho \varphi}, x_\tau^{\xi \varphi}]^{\frac{(p^k - 1) p^k}{2}} =$$

$$= (x h_1)_\tau^{w_1 \xi} \ldots (x h_n)_\tau^{w_n \xi} (y h_1)_\delta^{w_1 \varrho} \ldots (y h_n)_\delta^{w_n \varrho} m_1^\xi m_2^\varrho [y_\delta^{\varrho \varphi}, x_\tau^{\xi \varphi}]^{\frac{(p^k - 1) p^k}{2}}$$

$$\cdot \prod_{i<j} [(x h_j)_\tau^{w_j}, (x h_i)_\tau^{w_i}]^{\frac{(\xi - 1) \xi}{2}} \prod_{i<j} [(y h_j)_\delta^{w_j}, (y h_i)_\delta^{w_i}]^{\frac{(\varrho - 1) \varrho}{2}} .$$

Now equation (3.8) can be rewritten as

$$(3.10)\qquad (xh_1)_\tau^{w_1\xi}\dots(xh_n)_\tau^{w_n\xi}(yh_1)_\delta^{w_1\varrho}\dots(yh_n)_\delta^{w_n\varrho}m_3\prod_i[a_i,x_\tau^\xi y_\delta^\varrho]^{h_i}$$

$$\cdot\prod_{i<j}[(x_\tau^\xi y_\delta^\varrho)^{h_i},(x_\tau^\xi y_\delta^\varrho)^{h_j}]^{b_{ij}}\prod_i[(yh_i)_\delta^\varrho,(xh_i)_\tau^\xi]^{\frac{(w_i-1)w_i}{2}}\prod_{i<j}[(yh_i)_\delta^\varrho,(xh_j)_\tau^\xi]^{w_iw_j}.$$

Equations (3.9) and (3.10) yield

$$(3.11)\qquad m_1^\xi m_2^\varrho[y_\delta^{\varrho\varphi},x_\tau^{\xi\varphi}]^{\frac{(p^k-1)p^k}{2}}\prod_{i<j}[(xh_j)_\tau^{w_j},(xh_i)_\tau^{w_i}]^{\frac{(\xi-1)\xi}{2}}\prod_{i<j}\cdot$$

$$\cdot[(yh_j)_\delta^{w_j},(yh_i)_\delta^{w_i}]^{\frac{(\varrho-1)\varrho}{2}}=m_3\prod_i[a_i,x_\tau^\xi y_\delta^\varrho]^{h_i}\prod_{i<j}[(x_\tau^\xi y_\delta^\varrho)^{h_i},(x_\tau^\xi y_\delta^\varrho)^{h_j}]^{b_{ij}}\cdot$$

$$\cdot\prod_i[(yh_i)_\delta^\varrho,(xh_i)_\tau^\xi]^{\frac{(w_i-1)w_i}{2}}\prod_{i<j}[(yh_i)_\delta^\varrho,(xh_j)_\tau^\xi]^{w_iw_j}.$$

Taking the p^k-th power of equation (3.11) and collecting the commutators with $\xi\varrho$, we use equations (3.6) and (3.7) to get

$$(3.12)\qquad [(yh_1)_\delta^{w_1}\dots(yh_n)_\delta^{w_n},(xh_1)_\tau^{w_1}\dots(xh_n)_\tau^{w_n}]^{\frac{p^k-1}{2}}$$

$$=\prod_{i<j}[(xh_i)_\tau,(yh_j)_\delta]^{b_{ij}}[(yh_i)_\delta,(xh_j)_\tau]^{b_{ij}}$$

$$\cdot\prod_i[(yh_i)_\delta,(xh_i)_\tau]^{\frac{(w_i-1)w_i}{2}}\prod_{i<j}[(yh_i)_\delta,(xh_j)_\tau]^{w_iw_j}.$$

Since the commutators $\{[(yh_i)_\delta,(xh_j)_\tau]: i,j=1,\dots,n\}$ form a linearly independent set in $\widehat{N}'_G$, combine like commutators to get the equations

$$(3.13)\qquad w_i^2\cdot\frac{p^k-1}{2}=\frac{(w_i-1)w_i}{2}$$

$$(3.14)\qquad w_iw_jp^k+b_{ij}p^k=w_iw_j\cdot\frac{p^k-1}{2}$$

$$(3.15)\qquad -b_{ij}p^k=w_iw_j\cdot\frac{p^k-1}{2}.$$

Equation (3.13) implies that either $w_i=0$ or $w_i=p^k$. Equations (3.14) and (3.15) imply that $w_i w_j = 0$ for all i, j. Since φ is assumed to be an automorphism, there is an i such that $w_i = p^k$. Thus it follows that $n = p^k h$ for some $h \in G$ and

$$x_\tau^\varphi \equiv (xh)_\tau \quad \text{and} \quad y_\delta^\varphi \equiv (yh)_\delta \bmod \widehat{N}'_G .$$

Hence there exists $h \in G$ such that for all $\tau \in T$ and for all $x \in G$, $(x_\tau)^\varphi \equiv \equiv (xh)_\tau \bmod \widehat{N}'_G$, i.e., φ induces h on $\widehat{N}_G / \widehat{N}'_G$. Since $\widehat{N}_G$ is nilpotent of class two, $[x_\tau, y_\delta]^\varphi = [(xh)_\tau, (yh)_\delta] = [x_\tau, y_\delta]^h$. Thus φ also induces h on $\widehat{N}'_G$, and so $\varphi \in \operatorname{Stab}(\widehat{N}_G) \rtimes G$. ■

Let $\varphi = \varphi_{gh}$ be the automorphism defined for all $x \in \widehat{N}_G$ to be $x^\varphi = x[h, x^g]$, for some fixed $g \in G$ and $h \in \widehat{N}_G$. Clearly $\varphi_{eh^{-1}}$ is conjugation by h.

COROLLARY 3.5. *If $h \in H_{\lambda^*}$ and g is any element of G, then $\varphi_{gh} \restriction_{H_{\lambda^*}} \in \operatorname{Stab} H_{\lambda^*}$ and extends to any extension of H_{λ^*} which is G-invariant.*

The corollary shows that if G is non-trivial, then there exist elements of $\operatorname{Stab} H_{\lambda^*}$ which are not in $\operatorname{Inn} H_{\lambda^*}$.

COROLLARY 3.6. *Suppose $G = \{e\}$ and $H = \langle N_G, g_\beta : \beta < \alpha \rangle$ is defined as in Theorem 3.3. If $\varphi \in \operatorname{Aut} \widehat{N}_G \setminus \operatorname{Inn} \widehat{N}_G$, then there exists $x \in \widehat{N}_G$ such that $x^\varphi \notin \langle H, x \rangle_*$.*

PROOF. If $G = \{e\}$, then $e_\tau^\varphi \equiv e_\tau \bmod \widehat{N}'_G$ for all $\tau \in T$, by Theorem 3.4. Let τ and δ be distinct elements of T. Then there exists an integer k such that $e_\tau^{\varphi p^k} = e_\tau^{p^k} h_1$ and $e_\tau^{\xi \varphi p^k} = e_\tau^{\xi p^k} h_2 [e_\tau^\xi, h]$, for some $h_i \in H'$ and $h \in H$. By continuity of φ, we also have $e_\tau^{\xi \varphi p^k} = e_\tau^{\xi p^k} h_1^\xi$. Thus $h_2 = 1$ and $h_1 = = [e_\tau, h]$. As in Theorem 3.4, we take and compare the images e_τ^φ, e_δ^φ and $(e_\tau^\xi e_\delta^\xi)^\varphi$ to show that for any pair τ, δ of distinct elements of T, there exist an integer k and $h \in H$ such that

$$e_\tau^{\varphi p^k} = e_\tau^{p^k} [e_\tau, h] \quad \text{and} \quad e_\delta^{\varphi p^k} = e_\delta^{p^k} [e_\delta, h] .$$

By taking three distinct elements $\tau, \delta, \mu \in T$ and applying the preceding observation to the three distinct pairs of elements of T, it is easy to see that there exist $g \in H$ and an integer k such that

$$e_\tau^{\varphi p^k} = e_\tau^{p^k} [e_\tau, g] \quad \text{for all } \tau \in T .$$

Since $e_\tau^{\varphi p^k}, e_\tau^{p^k} \in \widehat{N}_G^{p^k}$ for all τ, it follows that $[e_\tau, g] \in \widehat{N}_G^{p^k}$ for all τ. Thus there exists $g_* \in \widehat{N}_G$ such that $g_*{}^{p^k} \equiv g \bmod \widehat{N}'_G$ and $e_\tau^\varphi = e_\tau[e_\tau, g_*] = e_\tau^{g_*}$ for all τ, i.e., $\varphi \in \operatorname{Inn} \widehat{N}_G$. ■

THEOREM 3.7. *Let* $H_{\lambda^*} = \langle N_G, g_\alpha^G \colon \alpha < \lambda^* \rangle_*$. *Then* $\operatorname{Aut} H_{\lambda^*} \cong \cong \operatorname{Stab}(H_{\lambda^*}) \rtimes G$.

PROOF. Let $\varphi \in \operatorname{Aut} H_{\lambda^*}$. Then φ extends to $\varphi \in \operatorname{Aut} \widehat{N}_G$. Note that

$$H_{\lambda^*} \cap \widehat{N}'_G = Z(H_{\lambda^*}),\ (H'_{\lambda^*})_* = Z(H_{\lambda^*}), \quad \text{and} \quad H_{\lambda^*}/Z(H_{\lambda^*}) \cong H_{\lambda^*}\widehat{N}'_G/\widehat{N}'_G.$$

If $\varphi \in \operatorname{Stab}(\widehat{N}_G) \rtimes G$, then $\varphi \restriction_{\widehat{N}_G/\widehat{N}'_G} = g \restriction_{\widehat{N}_G/\widehat{N}_G'}$ and $\varphi \restriction_{\widehat{N}'_G} = g$, for some $g \in G$. Hence $\varphi \restriction_{H_{\lambda^*}/Z(H_{\lambda^*})} = g$ and $\varphi \restriction_{Z(H_{\lambda^*})} = g$, i.e., $\varphi \in \operatorname{Stab}(H_{\lambda^*}) \rtimes G$. Suppose $\varphi \in \operatorname{Aut} \widehat{N}_G \backslash (\operatorname{Stab}(\widehat{N}_G) \rtimes G)$. By Theorem 3.4, there exists $x \in \widehat{N}_G$ such that $x^\varphi \notin \langle H_{\lambda^*}, x^G \rangle_*$. Theorem 3.2 implies that there exists $\alpha < \lambda^*$ such that $x, x^\varphi \in \widehat{P}_\alpha$; $\|x\|, \|x^\varphi\| < \|P_\alpha\|$ and $\varphi_\alpha = \varphi \restriction_{\widehat{P}_\alpha}$. We show that there exists $g_\alpha \in \widehat{P_\alpha}$ and $v_\alpha \in \operatorname{Br}(\operatorname{Im} f_\alpha)$ such that $\|x\| < \|g_\alpha\|$ and if $H_{\alpha+1} = \langle H_\alpha, g_\alpha^G \colon \beta < \alpha \rangle_*$ with $g_\beta^{\varphi_\beta} \notin H_\alpha$ for all $\beta < \alpha$, then $g_\alpha^{\varphi_\alpha}, g_\beta^{\varphi_\beta} \notin H_{\alpha+1}$ for all $\beta < \alpha$. Let $v \in \operatorname{Br}(\operatorname{Im} f_\alpha)$, then $v \neq v_\beta$ for all $\beta < \alpha$. We show that either $(v^1)^\varphi \notin \langle H_\alpha, (v^1)^G \rangle_*$ or $(v^1 x)^\varphi \notin \langle H_\alpha, (v^1 x)^G \rangle_*$, i.e., the sought after g_α is either v^1 or $v^1 x$. Suppose that the preceding is false. Then there exists an integer k such that

$$(v^1 x)^{\varphi p^k} = (v^1 x)^{n_1 g_1} \dots (v^1 x)^{n_l g_l} h_1 \prod_g [h_{2g}, (v^1 x)^g] \tag{3.16}$$

$$(v^1)^{\varphi p^k} = (v^1)^{m_1 g_1} \dots (v^1)^{m_l g_l} h_3 \prod_g [h_{4g}, (v^1)^g] \tag{3.17}$$

for some integers m_i, n_i, $g_i \in G$ and h_i, $h_{ig} \in H_\alpha$. Then we have the following congruences mod $\widehat{N}'_G$:

$$(v^1 x)^{\varphi p^k} \equiv (v^1)^{\varphi p^k} x^{\varphi p^k} \equiv (v^1)^{m_1 g_1} \dots (v^1)^{m_l g_l} h_3 x^{\varphi p^k}$$

$$(v^1 x)^{\varphi p^k} \equiv (v^1 x)^{n_1 g_1} \dots (v^1 x)^{n_l g_l} h_1 \equiv (v^1)^{n_1 g_1} \dots (v^1)^{n_l g_l} x^{n_1 g_1} \dots x^{n_l g_l} h_1.$$

By choice of support and, hence, norm of v, it follows that $n_i = m_i$ ($i = = 1, \dots, l$) and

$$x^{\varphi p^k} \equiv h_3^{-1} x^{n_1 g_1} \dots x^{n_l g_l} h_1 \bmod \widehat{N}'_G. \tag{3.18}$$

So

$$(3.19)\qquad (v^1x)^{\varphi p^{2k}} = (v^1)^{\varphi p^{2k}} x^{\varphi p^{2k}} [x^\varphi, (v^1)^\varphi]^{\frac{(p^{2k}-1)p^{2k}}{2}} =$$

$$= ((v^1)^{m_1 g_1} \dots (v^1)^{m_l g_l})^{p^k} h_3^{p^k} x^{\varphi p^{2k}} \prod_g [h_{4g}, (v^1)^g]^{p^k} \cdot$$

$$\cdot [h_3, (v^1)^{m_1 g_1} \dots (v^1)^{m_l g_l}]^{\frac{(p^k-1)p^k}{2}} [x^{\varphi p^k}, (v^1)^{\varphi p^k}]^{\frac{p^{2k}-1}{2}}.$$

But $[x^{\varphi p^k}, (v^1)^{\varphi p^k}] = [h_3^{-1} x^{m_1 g_1} \dots x^{m_l g_l} h_1, (v^1)^{m_1 g_1} \dots (v^1)^{m_l g_l} h_3]$. We also have $(v^1 x)^{\varphi p^{2k}}$ equal to

$$(3.20)\qquad ((v^1x)^{m_1 g_1} \dots (v^1 x)^{m_l g_l})^{p^k} h_1^{p^k} [h_1, (v^1x)^{m_1 g_1} \dots (v^1 x)^{m_l g_l}]^{\frac{(p^k-1)p^k}{2}} \cdot$$

$$\cdot \prod_g [h_{2g}, (v^1 x)^g]^{p^k}.$$

Since $\|x\|$, $\|x^\varphi\| < \|v\|$ and $v \neq v_\beta$ for all $\beta < \alpha$, it follows from equations (3.17), (3.18), (3.19) and (3.20) that

$$x^{\varphi p^{2k}} = h_3^{-p^k} (x^{m_1 g_1} \dots x^{m_l g_l})^{p^k} h_1^{p^k} [h_1, x^{m_1 g_1} \dots x^{m_l g_l}]^{\frac{(p^k-1)p^k}{2}} \cdot$$

$$\cdot \prod_g [h_{2g}, x^g]^{p^k} [h_3, x^{m_1 g_1} \dots x^{m_l g_l} h_1]^{\frac{p^{2k}-1}{2}},$$

which is an element of $\langle H_\alpha, x^G \rangle \subset \langle H_{\lambda^*}, x^G \rangle$. This contradicts the assumption that $x^\varphi \notin \langle H_{\lambda^*}, x^G \rangle_*$. Therefore $\varphi \in \mathrm{Stab}(\widehat{N}_G) \rtimes G$ and $\mathrm{Aut}\, H_{\lambda^*} = \mathrm{Stab}(H_{\lambda^*}) \rtimes G$. ■

COROLLARY 3.8. *If $G = \{e\}$, there exists a torsion-free nilpotent group H_* of class two such that* $\mathrm{Aut}\, H_* = \mathrm{Inn}\, H_*$.

PROOF. Suppose $G = \{e\}$. Let $H = \langle N_G, g_\beta \colon \beta < \lambda^* \rangle$ be as in Theorem 3.7 and H_* its purification in $\widehat{N}_G$. Suppose $\varphi \in \mathrm{Aut}\, H_*$. Then φ extends to an automorphism of $\widehat{N}_G$. Using Corollary 3.6 and the same argument as in Theorem 3.7, it follows that $\varphi \in \mathrm{Inn}\, \widehat{N}_G$. Thus φ is conjugation by some element x in the normalizer $N_{\widehat{N}_G}(H_*)$ of H_* in $\widehat{N}_G$. So $[g, x] \in H_*$ for all $g \in H_*$. If τ and μ are distinct, there exists an integer k such that $[e_\tau, x]^{p^k}$ and $[e_\mu, x]^{p^k}$ are both in H, i.e.,

$$[e_\tau, x]^{p^k} = [e_\tau, h_\tau] \quad \text{and} \quad [e_\mu, x]^{p^k} = [e_\mu, h_\mu],$$

for some h_τ, $h_\mu \in H$. It follows from these equations that $x^{p^k} \equiv e_\tau^{\xi_\tau} h_\tau$ and $x^{p^k} \equiv e_\mu^{\xi_\mu} h_\mu \bmod \widehat{N}'_G$, for some ξ_μ, $\xi_\tau \in J_p$. Thus $e_\mu^{-\xi_\mu} e_\tau^{\xi_\tau} \in H\widehat{N}'_G$. By the choice of the supports of the elements of H, elements in H with finite support must be in N_G. Hence it must be that ξ_μ and ξ_τ are integers, $x_\tau^{\xi_\tau} \in H$ and $x^{p^k} \equiv x_\tau^{\xi_\tau} h_\tau \in H$. It follows that $x \in (H\widehat{N}'_G)_*$ and so $N_{\widehat{N}_G}(H_*) = (H\widehat{N}'_G)_*$.

Finally we show that $(H\widehat{N}'_G)_* = H_*\widehat{N}'_G$, for then $\operatorname{Aut} H_* = N_{\widehat{N}_G}(H_*)/\widehat{N}'_G = \operatorname{Inn} H_*$. Clearly $H_*\widehat{N}'_G \subseteq (H\widehat{N}'_G)_*$. Since $\widehat{N}'_G \subset H_*\widehat{N}'_G$, it suffices to show that if $\prod_{i<\omega} e_{\tau_i}^{\xi_i} \in (H\widehat{N}'_G)_*$, then there exists $g \in H_*$ and $\eta \in \widehat{N}'_G$ such that $\prod_{i<\omega} e_{\tau_i}^{\xi_i} = g\eta$. By the definition of $\widehat{N}_G$, for all n, p^n divides ξ_i for all but finitely many ξ_i in J_p. Now there exists an integer k such that

$$\prod_{i<\omega} e_{\tau_i}^{\xi_i p^k} \equiv \prod_{i=1}^{m} e_{\tau_i}^{a_i} \prod_{i=1}^{m} g_{\beta_i}^{b_i} \in H\,,$$

for some integers a_i, b_i. However

$$\prod_{i=1}^{m} e_{\tau_i}^{a_i} \prod_{i=1}^{m} g_{\beta_i}^{b_i} \equiv \prod_{i<\omega} e_{\tau_i}^{\xi_i p^k} \prod_{i<j<\omega} [e_{\tau_i}, e_{\tau_j}]^{\xi_{ij}}$$

and p^k divides ξ_{ij} for almost all ξ_{ij}. If p^k does not divide some ξ_{ij}, then there exists an integer n_{ij} such that $\xi_{ij} + n_{ij} \in p^k J_p$. Let $\mu_{ij} = \xi_{ij}$, if p^k divides ξ_{ij}, and $\mu_{ij} = \xi_{ij} + n_{ij}$, otherwise. Then

$$\prod e_{\tau_i}^{\xi_i p^k} \prod_{i<j} [e_{\tau_i}, e_{\tau_j}]^{\mu_{ij}} \in H \cap \widehat{N}_G^{p^k}.$$

Hence it must equal $(\prod e_{\tau_i}^{\xi_i} \prod_{i<j} [e_{\tau_i}, e_{\tau_j}]^{\delta_{ij}})^{p^k}$, for some $\delta_{ij} \in J_p$. Thus $\prod e_{\tau_i}^{\xi_i} \prod_{i<j} [e_{\tau_i}, e_{\tau_j}]^{\delta_{ij}}$ is an element of H_*, and so $\prod_{i<\omega} e_{\tau_i}^{\xi_i} \in H_*\widehat{N}'_G$. ■

COROLLARY 3.9. *Let $N = \langle X \rangle$ be a free nilpotent group with basis X and G a non-trivial group such that G acts faithfully on X. If $N \subset H \subset \widehat{N}$, $H_* = H$ and H is G-invariant, then* $\operatorname{Inn} H \neq \operatorname{Stab} H$ *and* $G \cap \operatorname{Stab} H = 1$.

PROOF. Let $g \in G \backslash \{e\}$ and $h \in H \backslash Z(H)$. Define the map φ by $x^\varphi = x[h, x^g]$, for all $x \in H$. Clearly $\varphi \in \operatorname{Stab} H$ and $\varphi \notin \operatorname{Inn} H$. Since G acts faithfully on X, the intersection $G \cap \operatorname{Stab} H$ must contain only the identity map. ■

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