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$\mathbb{Z}_{k+l} \times \mathbb{Z}_2$-Graded Polynomial Identities for $M_{k,l}(E) \otimes E$.

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ABSTRACT - Let $\mathbb{K}$ be a field of characteristic zero, and E be the Grassmann algebra over an infinite-dimensional $\mathbb{K}$-vector space. We endow $M_{k,l}(E) \otimes E$ with a $\mathbb{Z}_{k+l} \times \mathbb{Z}_2$-grading, and determine a generating set for the ideal of its graded polynomial identities. As a consequence, we prove that $M_{k,l}(E) \otimes E$ and $M_{k+l}(E)$ are PI-equivalent with respect to this grading. In particular, this leads to their ordinary PI-equivalence, a classical result obtained by Kemer.

1. Introduction.

Let $\mathbb{K}$ be a field of characteristic zero, and E be the Grassmann algebra over an infinite-dimensional $\mathbb{K}$-vector space. For fixed integers k, l $(k \geqslant l)$ we consider the $\mathbb{K}$-algebra $M_{k,l}(E)$, whose elements are the following block matrices with entries in the even and odd part of E, resp. E_0 and E_1:

$$\underbrace{\left(\begin{array}{c|c} E_0 & E_1 \\ \hline E_1 & E_0 \end{array}\right)}_{k \quad\; l} \begin{array}{l} \updownarrow k \\ \updownarrow l \end{array}$$

As follows by the results of Kemer [K], these algebras generate nontrivial prime varieties, and their study is essential in the theory of PI-algebras. Since $M_{k,l}(E)$ is a subalgebra of $M_{k+l}(E)$, the following inclusion

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for the ideals of polynomial identities follows: $T(M_{k,l}(E)) \supseteq T(M_{k+l}(E))$. It is somehow surprising that to get a PI-equivalence with $M_{k+l}(E)$ it suffices to consider the tensor product $M_{k,l}(E) \otimes E$, regardless to k, l, i.e. the T-ideals of the polynomial identities of these algebras are equal. Originally, this fact was proved by Kemer in [K1] as a consequence of his structure theory for varieties of algebras. Other proofs are in the papers of Regev [R] and Berele [B]. In this paper, we shall study $M_{k,l}(E) \otimes E$ as a graded algebra. Recall briefly that, for a given group G, a $\mathbb{K}$-algebra R is G-graded if, for each $g \in G$, there is a subspace R^g of R (the *g-homogeneous component* of R) such that

$$R = \sum_{g \in G} R^g \qquad \text{and} \qquad R^g R^h \subseteq R^{g+h} \ \text{ for all } g, h \in G.$$

We shall write $\partial_G(r) = g$ (or simply $\partial(r) = g$ if G is clear from the context) to denote the *G-homogeneous degree* of the homogeneous element $r \in R^g$.

The study of graded algebras is almost a standard approach in many problems of PI-theory, and many algebras have natural grading which enrich them with nice structure properties. The algebras $M_n(\mathbb{K})$, $M_{k,l}(E)$, $M_n(E)$, for instance, are $\mathbb{Z}_2$-graded algebras in a natural way. Before getting into details in the next section, we briefly recall some terminology:

Let G be a group; for each $g \in G$ let X^g be a countable set of non-commuting variables, and let X^G be their disjoint union. Then the algebra $\mathbb{K}\langle X^G \rangle$ is a free object in the class of G-graded algebras. A polynomial $f = f(x_1^{g_1}, \ldots, x_r^{g_r})$ with variables $x_i^{g_i} \in X^{g_i}$ is a graded polynomial identity for R if for all substitutions $x_i^{g_i} \to a_i \in R^{g_i}$ $(i = 1, \ldots, r)$ it results $f(a_1, \ldots, a_r) = 0$. The set of all graded polynomial identities for R is an ideal of $\mathbb{K}\langle X^G \rangle$ invariant under all endomorphisms of $\mathbb{K}\langle X^G \rangle$ preserving the homogeneous components; we call it the T_G-ideal of R, and denote it by $T_G(R)$. Now call:

$$V_r^G := \mathrm{span}_{\mathbb{K}} \langle x_{\sigma(1)}^{g_1} \ldots x_{\sigma(r)}^{g_r} \mid \sigma \in S_r, g_1, \ldots, g_r \in G \rangle.$$

We call V_r^G the space of graded multilinear polynomials, and it is easily seen that the usual left action of S_r endows V_r^G with the structure of left S_r-module as in the ordinary case. Moreover, since the field $\mathbb{K}$ is of characteristic zero, standard arguments yield that $T_G(R)$ is generated by its multilinear parts, i.e. by the S_r-submodules $V_r^G \cap T_G(R)$ for all $r \in \mathbb{N}$. There are many more examples of these and other concepts related to graded algebras; for shortness, we introduce those who are related to this paper. The first is the natural $\mathbb{Z}_n$-grading for the algebra $M_n(\mathbb{K})$:

$$(M_n(\mathbb{K}))^t := \mathrm{span}_{\mathbb{K}}\langle \boldsymbol{e}_{ij} \mid \overline{j-i} = t \in \mathbb{Z}_n \rangle.$$

Vasilovsky in [V] proved that its $T_{\mathbb{Z}_n}$-ideal is generated by the following multilinear polynomials:

$$[x_1^0, x_2^0] \quad x_1^t x^{-t} x_2^t - x_2^t x^{-t} x_1^t \quad (t \in \mathbb{Z}_n).$$

The second instance is about the algebra $M_n(E) \cong M_n(\mathbb{K}) \otimes E$, which has the natural $\mathbb{Z}_n \times \mathbb{Z}_2$-grading

$$(M_n(E))^{(t,\lambda)} := M_n(\mathbb{K})^t \otimes E_\lambda$$

where the first component is the t-homogeneous component of $M_n(\mathbb{K})$ in the previous grading for $M_n(\mathbb{K})$. The authors in [DVN] found a system of generators for the $T_{\mathbb{Z}_n \times \mathbb{Z}_2}$-ideal of its graded polynomial identities.

In this paper, for $n = k + l$, we define a $\mathbb{Z}_n \times \mathbb{Z}_2$-grading for $M_{k,l}(E) \otimes E$ and describe a set of generators for $T_{\mathbb{Z}_n \times \mathbb{Z}_2}(M_{k,l}(E) \otimes E)$. In particular it turns out that this set generates $T_{\mathbb{Z}_n \times \mathbb{Z}_2}(M_n(E))$ as well. Hence $M_{k,l}(E) \otimes E$ and $M_n(E)$ are equivalent as graded PI-algebras. General arguments lead to their ordinary PI-equivalence, and we obtain a new proof for the mentioned result of Kemer, using only elementary tools.

2. Preliminaries.

Consider the $\mathbb{K}$-algebra $M_{k,l}(E)$, and let $n := k + l$ in the following. We may start from the natural $\mathbb{Z}_n$-grading on $M_n(\mathbb{K})$ in order to endow $M_{k,l}(E)$ with the following $\mathbb{Z}_n \times \mathbb{Z}_2$-grading:

$$(M_{k,l}(E))^{(t,\lambda)} := \mathrm{span}_{\mathbb{K}}\langle E_\lambda \boldsymbol{e}_{ij} \mid \overline{j-i} = t \in \mathbb{Z}_n \rangle \cap M_{k,l}(E).$$

Of course some of the graded components may be trivial (for instance, $(M_{k,l}(E))^{(0,1)} = 0$). It is easy to verify, however, that this is actually a grading for $M_{k,l}(E)$. Next, consider $M_{k,l}(E) \otimes E$ and define

$$(M_{k,l}(E) \otimes E)^{(t,\lambda)} := (M_{k,l}(E))^{(t,\lambda)} \otimes E_0 \oplus (M_{k,l}(E))^{(t,\lambda+1)} \otimes E_1.$$

Then $M_{k,l}(E) \otimes E$ is $\mathbb{Z}_n \times \mathbb{Z}_2$-graded, and we shall prove that it is PI-equivalent to the algebra $M_n(E)$ with the $\mathbb{Z}_n \times \mathbb{Z}_2$-grading

$$(M_n(E))^{(t,\lambda)} = \mathrm{span}_{\mathbb{K}}\langle \boldsymbol{e}_{ij} \otimes E_\lambda \mid \overline{j-i} = t \in \mathbb{Z}_n \rangle.$$

In order to have a clearer view of the problem, the following considerations are useful:

DEFINITION 2.1. Let $\varepsilon : \{1, \ldots, n\} \times \{1, \ldots, n\} \to \mathbb{Z}_2$ be the map defined via

$$\varepsilon(i,j) := \begin{cases} 0 & \text{if } i,j \leqslant k \text{ or } i,j > k \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, let $\mathcal{E}_0$ be the natural $\mathbb{K}$-basis for E_0, and $\mathcal{E}_1$ be the corresponding basis for E_1.

It is immediate to see that

$$\mathcal{A} := \{ a\boldsymbol{e}_{ij} \mid i,j \leqslant n, \ a \in \mathcal{E}_{\varepsilon(i,j)} \}$$

is a $\mathbb{K}$-basis for $M_{k,l}(E)$, and

$$\mathcal{B} := \{ a\boldsymbol{e}_{ij} \otimes b \mid i,j \leqslant n, \ a \in \mathcal{E}_{\varepsilon(i,j)}, \ b \in \mathcal{E}_\lambda \}$$

is a $\mathbb{K}$-basis for $M_{k,l}(E) \otimes E$. Moreover, writing a^λ as a shorthand for $a \in \mathcal{E}_\lambda$, it holds:

$$(M_{k,l}(E))^{(t,\lambda)} = \mathrm{span}_{\mathbb{K}}\langle a^{\varepsilon(i,j)} \boldsymbol{e}_{ij} \mid \overline{j-i} = t \in \mathbb{Z}_n, \quad \varepsilon(i,j) = \lambda \rangle$$

and

$$(M_{k,l}(E) \otimes E)^{(t,\lambda)} = \mathrm{span}_{\mathbb{K}}\langle a^{\varepsilon(i,j)} \boldsymbol{e}_{ij} \otimes b^{\lambda+\varepsilon(i,j)} \mid \overline{j-i} = t \in \mathbb{Z}_n \rangle.$$

By use of these definitions, the fact that $M_{k,l}(E) \otimes E$ is a $\mathbb{Z}_n \times \mathbb{Z}_2$-graded algebra follows easily. By the way, we find useful to remark a couple of lemmas which will be of help in the following of this part.

LEMMA 2.2. *Let*

$$\boldsymbol{A}_s := a_s^{\varepsilon(i_s,j_s)} \boldsymbol{e}_{i_s j_s} \otimes b_s^{\lambda_s + \varepsilon(i_s,j_s)} \in \mathcal{B} \ \textit{for} \ s = 1, 2.$$

If $\boldsymbol{A}_1 \boldsymbol{A}_2$ *is not zero, then there exists* $c \in \{1, -1\}$ *such that*

$$c\boldsymbol{A}_1 \boldsymbol{A}_2 \in \mathcal{B}.$$

In particular it holds:

$$j_1 = i_2; \qquad \varepsilon(i_1, j_1) + \varepsilon(i_2, j_2) = \varepsilon(i_1, j_2); \qquad \partial(\boldsymbol{A}_1 \boldsymbol{A}_2) = (\overline{j_2 - i_1}, \lambda_1 + \lambda_2).$$

PROOF. Suppose $\boldsymbol{A}_1 \boldsymbol{A}_2 \neq 0$ and look at $\varepsilon(i_1, j_1) := \varepsilon_1$ and $\varepsilon(i_2, j_2) := \varepsilon_2$. If $\varepsilon_1 = \varepsilon_2 = 1$, we know that i_1 and $j_1 = i_2$ are by opposite side with respect to k and this forces that j_2 and i_1 are by the same side with respect to k, so $\varepsilon(i_1, j_2) = 0 = \varepsilon_1 + \varepsilon_2$. Apply the same argument for the other cases to get $\varepsilon_1 + \varepsilon_2 = \varepsilon(i_1, j_2)$. Then, say V the infinite-dimensional vector space which generates the Grassmann algebra E, and say

$$a_1 = v_{l_1} \dots v_{l_r} \qquad \text{and} \qquad a_2 = v_{m_1} \dots v_{m_t}$$

where $v_{l_1}, \dots, v_{l_r}, v_{m_1}, \dots, v_{m_t}$ are pairwise-distinct vectors in an ordered basis for V since $\boldsymbol{A}_1 \boldsymbol{A}_2 \neq 0$, with $v_{l_1} < v_{l_2} < \dots < v_{l_h}$ and $v_{m_1} < v_{m_2} < \dots < v_{m_t}$. Then we may rearrange the entries in the word $a_1 a_2$ and obtain an element of $\mathcal{E}_{\varepsilon_1 + \varepsilon_2}$ which is equal to $a_1 a_2$ up to its sign. The same arguments apply to the b's, and using the first part of this Lemma we get the result. ■

DEFINITION 2.3. Let m be a monomial in $V_r^{\mathbb{Z}_n \times \mathbb{Z}_2}$, and let $\mathcal{S}:(\boldsymbol{A}_1, \boldsymbol{A}_2, \dots, \boldsymbol{A}_r)$ be the substitution $x_i \rightarrow \boldsymbol{A}_i$ ($i = 1, \dots, r$). We say that $\mathcal{S}$ is a standard substitution if

i) $\partial(x_i) = \partial(\boldsymbol{A}_i)$ for each x_i occurring in m;

ii) $\boldsymbol{A}_i \in \mathcal{B}$ for each i.

Since $\operatorname{char} \mathbb{K} = 0$, the graded polynomial identities of $T_{\mathbb{Z}_n \times \mathbb{Z}_2}(M_{k,l}(E) \otimes E)$ are determined by the multilinear ones, i.e. by the spaces

$$V_r^{\mathbb{Z}_n \times \mathbb{Z}_2} \cap T_{\mathbb{Z}_n \times \mathbb{Z}_2}(M_{k,l}(E) \otimes E) \ \text{ for all } r \in \mathbb{N}.$$

Actually, it suffices to prove that a multilinear polynomial is zero under all standard substitutions in order to prove that it is a graded polynomial identity. In the next considerations, the following Lemma is useful. Its proof can be found in ([V], Lemma 1), and we shall omit it here.

LEMMA 2.4. *Let $\boldsymbol{e}_{i_1 j_1}, \boldsymbol{e}_{ij}, \boldsymbol{e}_{i_2 j_2} \in M_n(\mathbb{K})$ be elementary matrices with $\mathbb{Z}_n$-degrees*

$$\partial_{\mathbb{Z}_n}(\boldsymbol{e}_{i_1 j_1}) = \partial_{\mathbb{Z}_n}(\boldsymbol{e}_{i_2 j_2}) = -\partial_{\mathbb{Z}_n}(\boldsymbol{e}_{ij}).$$

Then

$$e_{i_1j_1}e_{ij}e_{i_2j_2}\neq 0 \quad \textit{if and only if} \quad i_1=j=i_2 \ \textit{and}\ j_1=i=j_2.$$

If this is the case, it holds: $e_{i_1j_1}e_{ij}e_{i_2j_2}=e_{i_2j_2}e_{ij}e_{i_1j_1}$.

DEFINITION 2.5. Let $\mathfrak{I}$ be the following set of multilinear polynomials:

$$[x_1^{(0,0)}, x_2^{(0,0)}] \qquad [x_1^{(0,1)}, x_2^{(0,0)}] \qquad x_1^{(0,1)}\circ x_2^{(0,1)}$$

$$x_1^{(t,0)}x^{(-t,0)}x_2^{(t,0)}-x_2^{(t,0)}x^{(-t,0)}x_1^{(t,0)} \quad x_1^{(t,1)}x^{(-t,0)}x_2^{(t,0)}-x_2^{(t,0)}x^{(-t,0)}x_1^{(t,1)}$$

$$x_1^{(t,0)}x^{(-t,1)}x_2^{(t,0)}-x_2^{(t,0)}x^{(-t,1)}x_1^{(t,0)} \quad x_1^{(t,1)}x^{(-t,0)}x_2^{(t,1)}+x_2^{(t,1)}x^{(-t,0)}x_1^{(t,1)}$$

$$x_1^{(t,1)}x^{(-t,1)}x_2^{(t,0)}+x_2^{(t,0)}x^{(-t,1)}x_1^{(t,1)} \quad x_1^{(t,1)}x^{(-t,1)}x_2^{(t,1)}+x_2^{(t,1)}x^{(-t,1)}x_1^{(t,1)}$$

where t varies in $\mathbb{Z}_n$ and $a\circ b$ denotes the Jordan product $a\circ b = ab+ba$. We shall denote by I the $T_{\mathbb{Z}_n\times\mathbb{Z}_2}$-ideal generated by $\mathfrak{I}$.

PROPOSITION 2.6.

$$I\subseteq T_{\mathbb{Z}_n\times\mathbb{Z}_2}(M_{k,l}(E)\otimes E).$$

PROOF. It is enough to test polynomials listed in Definition 2 under standard substitutions, and verify they are zero for each such substitutions. The generic standard substitution for 3-degree polynomials will be

$$A_1 = a_1^{\varepsilon(i_1,j_1)}e_{i_1j_1}\otimes b_1^{\lambda_1+\varepsilon(i_1,j_1)}$$

$$A_2 = a_2^{\varepsilon(i_2,j_2)}e_{i_2j_2}\otimes b_2^{\lambda_2+\varepsilon(i_2,j_2)}$$

$$A = a^{\varepsilon(i,j)}e_{ij}\otimes b^{\lambda+\varepsilon(i,j)}$$

for $\overline{j_1-i_1}=\overline{j_2-i_2}=\overline{i-j}=t\in\mathbb{Z}_n$. By Lemma 2.4, the products $e_{i_1j_1}e_{ij}e_{i_2j_2}$ and $e_{i_2j_2}e_{ij}e_{i_1j_1}$ are both zero unless

$$i_1=i_2=j \quad \text{and} \quad j_1=j_2=i$$

and in this case they are equal (to e_{ji}) and $\varepsilon(i_1,j_1)=\varepsilon(i_2,j_2)=\varepsilon(i,j)$ $=:\varepsilon$. So the substitutions we have to test are of kind

$$A_1=a_1^{\varepsilon}e_{ji}\otimes b_1^{\lambda_1+\varepsilon} \quad A_2=a_2^{\varepsilon}e_{ji}\otimes b_2^{\lambda_2+\varepsilon} \quad A=a^{\varepsilon}e_{ij}\otimes b^{\lambda+\varepsilon} \ \ (\text{for } \overline{i-j}=t).$$

Here we verify just one of them, the other ones can be treated in the same way. For instance, consider $x_1^{(t,1)} x^{(-t,1)} x_2^{(t,0)} + x_2^{(t,0)} x^{(-t,1)} x_1^{(t,1)}$: $\lambda_1 = \lambda = 1$, $\lambda_2 = 0$. If $\varepsilon = 0$, then

$$\boldsymbol{A}_2 \boldsymbol{A} \boldsymbol{A}_1 = a_2^0 \, a^0 \, a_1^0 \boldsymbol{e}_{ji} \otimes b_2^0 \, b^1 \, b_1^1 = -\boldsymbol{A}_1 \boldsymbol{A} \boldsymbol{A}_2$$

and if $\varepsilon = 1$, then

$$\boldsymbol{A}_2 \boldsymbol{A} \boldsymbol{A}_1 = a_2^1 \, a^1 \, a_1^1 \boldsymbol{e}_{ji} \otimes b_2^1 \, b^0 \, b_1^0 = -\boldsymbol{A}_1 \boldsymbol{A} \boldsymbol{A}_2 .$$

The same arguments and the use of Lemma 2.2 yield that the polynomials of second degree in the list are graded identities for $M_{k,l}(E) \otimes E$ as well. ■

In the rest of the section, let $m := x_1 \ldots x_r$ be a multilinear graded monomial of length r. If $\sigma \in S_r$, we denote by m_σ the monomial $x_{\sigma(1)} \ldots x_{\sigma(r)}$. If $\mathcal{S}$ is any (graded) substitution, we denote by $m|_{\mathcal{S}}$ the value of m under the substitution $\mathcal{S}$.

REMARK 2.7. For each $\sigma \in S_r$ there exists a graded standard substitution $\mathcal{S}$ such that

$$m_\sigma |_{\mathcal{S}} \neq 0 .$$

This is easy to prove, for instance using induction on the length r.

DEFINITION 2.8. For $1 \leqslant p \leqslant q \leqslant r$, call

$$m_\sigma^{[p,q]} := x_{\sigma(p)} \ldots x_{\sigma(q)} .$$

REMARK 2.9. Let $\mathcal{S}$ be a standard substitution, and fix from now on

$$\mathcal{S}: x_s \to \boldsymbol{A}_s := a_s^{\varepsilon(i_s, j_s)} \boldsymbol{e}_{i_s j_s} \otimes b_s^{\lambda_s + \varepsilon(i_s, j_s)} \qquad (s = 1, \ldots, r),$$

where $\partial(x_s) = (\overline{j_s - i_s}, \lambda_s) = \partial(\boldsymbol{A}_s)$. If

$$m_\sigma |_{\mathcal{S}} = \boldsymbol{A}_{\sigma(1)} \ldots \boldsymbol{A}_{\sigma(r)} \neq 0$$

then there exists $\boldsymbol{A} \in \mathcal{B}$, $c \in \{1, -1\}$ such that $m_\sigma |_{\mathcal{S}} = c\boldsymbol{A}$. Moreover, $m_\sigma |_{\mathcal{S}} \neq 0$ if and only if $\forall p, q$ $1 \leqslant p \leqslant q \leqslant r$ it is $m_\sigma^{[p,q]} |_{\mathcal{S}} \neq 0$, and in this case it holds

$$\partial(m_\sigma^{[p,q]}) = (\overline{j_{\sigma(q)} - i_{\sigma(p)}}, \lambda_{\sigma(p)} + \ldots + \lambda_{\sigma(q)}) .$$

In fact it is:

$$\begin{aligned}\partial(m_\sigma^{[p,q]}) &= \partial(x_{\sigma(p)}) + \ldots + \partial(x_{\sigma(q)}) = \\ &= (\overline{j_{\sigma(p)} - i_{\sigma(p)}}, \lambda_{\sigma(p)}) + \ldots + (\overline{j_{\sigma(q)} - i_{\sigma(q)}}, \lambda_{\sigma(q)}) = \\ &= (\overline{j_{\sigma(q)} - i_{\sigma(p)}}, \lambda_{\sigma(p)} + \ldots + \lambda_{\sigma(q)})\end{aligned}$$

by Lemma 2.2.

3. Technical results.

The considerations in this (and the next) section are similar to those in [V]. We start with rearranging a lemma. The symbols used are the same listed in the previous section. We recall that I denotes the $T_{\mathbb{Z}_n \times \mathbb{Z}_2}$-ideal generated by the polynomials listed in Definition 2.5.

LEMMA 3.1. *Suppose that for a graded standard substitution* $\mathcal{S}$ *it results*

$$m_\sigma|_{\mathcal{S}} = \pm m|_{\mathcal{S}} \neq 0 .$$

Then there exists $c \in \{1, -1\}$ *such that*

$$m_\sigma \equiv c x_1 m'(x_2, \ldots, x_r) \bmod I .$$

PROOF. First of all, note that $m_\sigma|_{\mathcal{S}} = \pm m|_{\mathcal{S}} \neq 0$ implies that $i_1 = i_{\sigma(1)}$. Of course, we may suppose $\sigma(1) \neq 1$, so $1 < \sigma^{-1}(1)$. We may write the integers in $[1, \sigma^{-1}(1)]$ in the form $\sigma^{-1}(j+1)$ for $j = 0, \ldots, r-1$; then call

$$t := \min\{ j \leqslant r-1 \mid 1 \leqslant \sigma^{-1}(j+1) < \sigma^{-1}(1)\} .$$

By its definitions, t satisfies $1 \leqslant \sigma^{-1}(t+1) < \sigma^{-1}(1) \leqslant \sigma^{-1}(t)$; set

$$p := \sigma^{-1}(t+1) \qquad q := \sigma^{-1}(1) \qquad u := \sigma^{-1}(t)$$

and consider the two possibilities: $p = 1$ or $p > 1$. For convenience, define

$$\lambda_\sigma^{[a,b]} := \lambda_{\sigma(a)} + \ldots + \lambda_{\sigma(b)} .$$

First, suppose $p = 1$. By Lemma 2.9, it results

$$\partial(m_\sigma^{[1,\,q-1]}) = (\overline{j_{\sigma(q-1)} - i_{\sigma(1)}},\, \lambda_\sigma^{[1,\,q-1]})$$

$$\partial(m_\sigma^{[q,\,u]}) = (j_{\sigma(u)} - i_{\sigma(q)},\, \lambda_\sigma^{[q,\,u]})$$

and both the words are not zero under the substitution $\mathcal{S}$; by Lemma 2.9 this yields

$$j_{\sigma(q-1)} - i_{\sigma(1)} = i_{\sigma(q)} - i_1 = i_1 - i_1 = 0$$
$$j_{\sigma(u)} - i_{\sigma(q)} = j_t - i_1 = i_{t+1} - i_1 = i_{\sigma(p)} - i_1 = i_{\sigma(1)} - i_1 = i_1 - i_1 = 0\,.$$

With respect to the parities of $\lambda_\sigma^{[1,\,q-1]}$ and $\lambda_\sigma^{[q,\,u]}$ there is $c \in \{1,\,-1\}$ such that

$$x_1^{(0,\,\lambda_\sigma^{[1,\,q-1]})} x_2^{(0,\,\lambda_\sigma^{[q,\,u]})} \equiv c x_2^{(0,\,\lambda_\sigma^{[q,\,u]})} x_1^{(0,\,\lambda_\sigma^{[1,\,q-1]})} \bmod I\,.$$

Hence we get

$$m_\sigma \equiv c m_\sigma^{[q,\,u]} m_\sigma^{[1,\,q-1]} m_\sigma^{[u+1,\,r]} \bmod I\,,$$

and $m_\sigma^{[q,\,u]}$ starts with x_1. Now consider the case $p > 0$; with consideration similar to the previous case, it is

$$\partial(m_\sigma^{[1,\,p-1]}) = (\overline{j_{\sigma(p-1)} - i_{\sigma(1)}},\, \lambda_{\sigma(1)} + \ldots + \lambda_{\sigma(p-1)})$$

$$\partial(m_\sigma^{[p,\,q-1]}) = (\overline{j_{\sigma(q-1)} - i_{\sigma(p)}},\, \lambda_{\sigma(p)} + \ldots + \lambda_{\sigma(q-1)})$$

$$\partial(m_\sigma^{[q,\,u]}) = (\overline{j_{\sigma(u)} - i_{\sigma(q)}},\, \lambda_{\sigma(q)} + \ldots + \lambda_{\sigma(u)})\,.$$

Call

$$d := \begin{cases} i_{t+1} - i_1 & \text{if } i_{t+1} - i_1 \geqslant 1 \\ i_{t+1} - i_i + n & \text{if } i_{t+1} - i_1 < 1\,. \end{cases}$$

Then it holds that:

$$j_{\sigma(p-1)} - i_{\sigma(1)} = i_{\sigma(p)} - i_1 = i_{t+1} - i_1 \equiv d \bmod n$$

$$j_{\sigma(q-1)} - i_{\sigma(p)} = i_{\sigma(q)} - i_{\sigma(p)} = i_1 - i_{t+1} \equiv -d \bmod n$$

$$j_{\sigma(u)} - i_{\sigma(q)} = j_t - i_1 = i_{t+1} - i_1 \equiv d \bmod n\,.$$

As before, there is $c \in \{1,\,-1\}$ such that

$$x_1^{(d,\,\lambda_\sigma^{[1,\,p-1]})} x^{(-d,\,\lambda_\sigma^{[p,\,q-1]})} x_2^{(d,\,\lambda_\sigma^{[q,\,u]})} \equiv c x_2^{(d,\,\lambda_\sigma^{[q,\,u]})} x^{(-d,\,\lambda_\sigma^{[p,\,q-1]})} x_1^{(d,\,\lambda_\sigma^{[1,\,p-1]})}$$

modulo I; then

$$m_\sigma \equiv c m_\sigma^{[q,u]} m_\sigma^{[p,q-1]} m_\sigma^{[1,p-1]} m_\sigma^{[u+1,r]} \operatorname{mod} I$$

and this monomial starts with x_1. ■

LEMMA 3.2. *With the same notation as in the previous Lemma, if for a standard substitution* $\mathcal{S}$ *it holds*

$$m_\sigma|_{\mathcal{S}} = cm|_{\mathcal{S}} \neq 0 ,$$

for a certain $c \in \{1, -1\}$, *then*

$$m_\sigma \equiv cm \operatorname{mod} I .$$

PROOF. Let s be the greatest positive integer such that

$$m_\sigma \equiv c_0 x_1 \dots x_s m'(x_{s+1}, \dots, x_r) \operatorname{mod} I$$

for a certain $c_0 \in \{1, -1\}$. By Lemma 3.1, the number s does exist and it is at least 1. We want to show that $s = r$. Suppose on the contrary that $1 \leqslant s < r$, so that $s \leqslant r-2$. It holds

$$x_1 \dots x_s m'(x_{s+1}, \dots, x_r)|_{\mathcal{S}} = \pm m_\sigma|_{\mathcal{S}} = \pm m|_{\mathcal{S}} \neq 0 .$$

Now compare $m'|_{\mathcal{S}}$ and $x_{s+1} \dots x_r|_{\mathcal{S}}$. If we consider only the elementary matrices which occur in $\mathcal{S}$, it has to be true that

$$\boldsymbol{e}_{i_1 j_1} \dots \boldsymbol{e}_{i_s j_s} m'(\boldsymbol{e}_{i_{s+1} j_{s+1}}, \dots, \boldsymbol{e}_{i_r j_r}) = \boldsymbol{e}_{i_1 j_1} \dots \boldsymbol{e}_{i_s j_s} (\boldsymbol{e}_{i_{s+1} j_{s+1}} \dots \boldsymbol{e}_{i_r j_r}) \neq 0$$

so

$$\boldsymbol{e}_{i_1 j_s} m'(\boldsymbol{e}_{i_{s+1} j_{s+1}}, \dots, \boldsymbol{e}_{i_r j_r}) = \boldsymbol{e}_{i_1 j_s} \boldsymbol{e}_{i_{s+1} j_r} \neq 0 .$$

Then $m'(\boldsymbol{e}_{i_{s+1} j_{s+1}}, \dots, \boldsymbol{e}_{i_r j_r})$ has to be an elementary matrix, say $\boldsymbol{e}_{pq}$, and this leads to $p = j_s$ and $q = j_r$, so we get

$$m'(\boldsymbol{e}_{i_{s+1} j_{s+1}}, \dots, \boldsymbol{e}_{i_r j_r}) = \boldsymbol{e}_{i_{s+1} j_{s+1}} \dots \boldsymbol{e}_{i_r j_r} \neq 0 .$$

Therefore the restriction $\mathcal{S}'$ of $\mathcal{S}$ to $t = s+1, \dots, r$ is such that

$$m'(x_{s+1}, \dots, x_r)|_{\mathcal{S}'} = \pm (x_{s+1} \dots x_r)|_{\mathcal{S}'} \neq 0$$

and by Lemma 3.1 this yields that there exists $c' \in \{1, -1\}$ such that

$$m' \equiv c' x_{s+1} m''(x_{s+2}, \dots, x_r) \operatorname{mod} I .$$

Then

$$m_\sigma \equiv c_0 c' x_1 \dots x_s x_{s+1} m''(x_{s+2}, \dots, x_r) \bmod I$$

which contradicts the definition of s. Now it follows easily that $c_0 = c$. ■

COROLLARY 3.3. *Let σ, τ be in S_r, and suppose that for a standard substitution $\mathcal{S}$ it results*

$$m_\sigma |_{\mathcal{S}} = c m_\tau |_{\mathcal{S}} \neq 0$$

for a certain $c \in \{1, -1\}$. Then

$$m_\sigma \equiv c m_\tau \bmod I .$$

4. The main results.

THEOREM 4.1. *Let $n = k + l$. Then the set $\mathfrak{I}$ described in Definition 2.5 generates $T_{\mathbb{Z}_n \times \mathbb{Z}_2}(M_{k,l}(E) \otimes E)$, that is*

$$I = T_{\mathbb{Z}_n \times \mathbb{Z}_2}(M_{k,l}(E) \otimes E).$$

PROOF. By Proposition 2.6 we have to prove only that every multilinear graded identity for $M_{k,l}(E) \otimes E$ is in I. Suppose on the contrary that there exists a polynomial

$$f = f(x_1, \dots, x_r) \in V_r^{\mathbb{Z}_n \times \mathbb{Z}_2} \cap T_{\mathbb{Z}_n \times \mathbb{Z}_2}(M_{k,l}(E) \otimes E)$$

which is not in I. Then we may write

$$f \equiv \sum_{s=1}^{t} d_{\sigma_s} m_{\sigma_s} \bmod I$$

for some monomials $m_{\sigma_s} \in V_r^{\mathbb{Z}_n \times \mathbb{Z}_2}$, $\sigma_s \in S_r$, non-zero scalars $0 \neq d_s \in \mathbb{K}$ and $s = 1, \dots, t$. Take t minimal with this property (of course, t should be at least 2 by Remark 2.7): we want to prove that $t = 0$.

By Remark 2.7 there exists a graded standard substitution $\mathcal{S}$ such that $m_{\sigma_1} |_{\mathcal{S}} \neq 0$. Since f is an identity for $M_{k,l}(E) \otimes E$ it is $f|_{\mathcal{S}} = 0$. Hence

$$d_{\sigma_1} m_{\sigma_1} |_{\mathcal{S}} = - \sum_{s=2}^{t} d_{\sigma_s} m_{\sigma_s} |_{\mathcal{S}}.$$

As in Remark 2.9, there exists $\boldsymbol{A} \in \mathcal{B}$ such that

$$0 \neq m_{\sigma_1}|_{\mathcal{S}} = c_1 \boldsymbol{A} \text{ for some } c_1 \in \{1, -1\}.$$

Hence there must be $p \in \{2, \dots, t\}$ such that

$$0 \neq m_{\sigma_p}|_{\mathcal{S}} = c_2 \boldsymbol{A} \text{ for some } c_2 \in \{1, -1\}.$$

Then

$$0 \neq m_{\sigma_1}|_{\mathcal{S}} = c_1 c_2 m_{\sigma_p}|_{\mathcal{S}},$$

and applying Corollary 3.3 we get

$$m_{\sigma_p} \equiv c m_{\sigma_1} \bmod I \text{ for } c = c_1 c_2.$$

In the end, it is

$$f \equiv (d_{\sigma_1} + c d_{\sigma_p})\, m_{\sigma_1} + \sum_{s=2,\, s \neq p}^{t} d_{\sigma_s} m_{\sigma_s} \bmod I$$

contradicting the minimality of t. ∎

Now we recall the main result in [DVN]:

THEOREM 4.2. *$T_{\mathbb{Z}_n \times \mathbb{Z}_2}(M_n(E))$ is generated by the polynomials in $\mathfrak{I}$.*

As a corollary of Theorems 4.1 and 4.2 we get

COROLLARY 4.3. *The algebras $M_{k,l}(E) \otimes E$ and $M_n(E)$ are PI-equivalent as $\mathbb{Z}_n \times \mathbb{Z}_2$-graded algebras.*

Then it follows

COROLLARY 4.4. *For $n = k + l$, the algebras $M_{k,l}(E) \otimes E$ and $M_n(E)$ are PI-equivalent.*

PROOF What we have to show is that the multilinear parts of the ordinary T-ideals $T(M_{k,l}(E) \otimes E)$ and $T(M_n(E))$ are equal. Note that each of them is a subset of the corresponding $T_{\mathbb{Z}_n \times \mathbb{Z}_2}$-ideal.

So take $f \in V_r \cap T(M_n(E))$. Then it suffices to prove that $f|_{\mathcal{S}} = 0$ for any ordinary standard substitution, i.e. for every substitution

$$\mathcal{S}: x_i \to \boldsymbol{A}_i, \ \ (i = 1, \dots, r)$$

such that $\boldsymbol{A}_1, \dots, \boldsymbol{A}_r \in \mathcal{B}$.

Let $\mathcal{S}$ be a ordinary standard substitution with elements in $\mathcal{B}$, and define

$$\tilde{f} := f(x_1^{\partial(\boldsymbol{A}_1)}, \ldots, x_r^{\partial(\boldsymbol{A}_r)}) \in V_r^{\mathbb{Z}_n \times \mathbb{Z}_2}.$$

$\tilde{f}$ is a multilinear graded element in $T_{\mathbb{Z}_n \times \mathbb{Z}_2}(M_n(E)) = T_{\mathbb{Z}_n \times \mathbb{Z}_2}(M_{k,l}(E) \otimes E)$, and the substitution $\mathcal{S}$ is admissible for this polynomial. Hence $f|_{\mathcal{S}} = \tilde{f}\,|_{\mathcal{S}} = 0$ and this means that $f \in T(M_{k,l}(E) \otimes E)$. Reversing the roles of $T(M_n(E))$ and $T(M_{k,l}(E) \otimes E)$ leads to the reverse inclusion. ■

REFERENCES

[B] A. Berele, *Supertraces and matrices over Grassmann algebras*, Advances in Math., **108** (1) (1994), pp. 77-90.

[DVN] O. M. Di Vincenzo - V. Nardozza, *Graded polynomial identities for tensor products by the Grassmann Algebra*, Comm. Algebra (2002) (in press).

[K] A. R. Kemer, *Varieties and Z_2-graded algebras*, Math. USSR Izv., **25** (1985), pp. 359-374.

[K1] A. R. Kemer, *Ideals of identities of associative algebras*, AMS Trans. of Math. Monographs, **87** (1991).

[R] A. Regev, *Tensor product of matrix algebras over the Grassmann algebra*, J. Algebra, **133** (1990), pp. 351-369.

[V] S. Y. Vasilovsky, *Z_n-graded polynomial identities of the full matrix algebra of order n*, Proc. Amer. Math. Soc., **127** (12) (1999), pp. 3517-3524.

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