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Spherical Harmonics and Spherical Averages of Fourier Transforms.

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ABSTRACT - We give estimates for spherical averages of Fourier transforms of functions which are linear combinations of products of radial functions and spherical harmonics. This generalizes the case of radial functions.

1. Introduction.

We shall here study Fourier transforms in $\mathbb{R}^n$ and we shall always assume $n \geq 2$. Let θ denote the area measure on S^{n-1} and set

$$\sigma(f)(R) = \int_{S^{n-1}} |\hat{f}(R\xi)|^2 \, d\theta(\xi), \qquad R > 1,$$

where $\hat{f}$ denotes the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$. We are interested in estimates of the type

$$\sigma(f)(R) \leq CR^{-\beta} \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \frac{1}{|\xi|^{n-\alpha}} \, d\xi, \qquad R > 1. \tag{1}$$

For $0 < \alpha \leq n$ we shall consider the statement

$$\begin{cases} \text{there exists a constant } C = C_{\alpha,\beta} \text{ such that (1)} \\ \text{holds for all } f \in C_0^\infty(\mathbb{R}^n) \text{ with supp } f \subset B_1 \text{ and } f \geq 0. \end{cases} \tag{2}$$

Here B_1 denotes the unit ball in $\mathbb{R}^n$.

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We set $\beta_+(\alpha) = \sup\{\beta; (2) \text{ holds}\}$. The number $\beta_+(\alpha)$ has been studied in Mattila [2], Sjölin [4], Bourgain [1], and Wolff [9]. In the case $n = 2$ it is known that

$$\beta_+(\alpha) = \begin{cases} \alpha, & 0 < \alpha \leqslant 1/2 \\ 1/2, & 1/2 < \alpha \leqslant 1 \\ \alpha/2, & 1 < \alpha \leqslant 2 \end{cases}$$

(see [2] and [9]). For $n \geqslant 3$ one knows that $\beta_+(\alpha) = \alpha$ for $0 < \alpha \leqslant (n-1)/2$, $\max((n-1)/2, \alpha - 1) \leqslant \beta_+(\alpha) \leqslant \min(\alpha, \alpha/2 + n/2 - 1)$ for $(n-1)/2 < \alpha < n$, and $\beta_+(n) = n - 1$ (see [2] and [4]).

Results of this type have applications in geometric measure theory in the study of distance sets.

In Sjölin and Soria [6], [7], θ is replaced by general measures and in these papers one also studies the case when the condition $f \geqslant 0$ in (2) is removed.

The case when f is also assumed to be radial is studied in Sjölin [5]. We shall here generalize the case of radial functions. We recall that $L^2(\mathbb{R}^n) = \sum_{k=0}^{\infty} \oplus H_k$, where H_k is the space of all linear combinations of functions of the form fP, where f ranges over the radial functions and P over the solid spherical harmonics of degree k, so that fP belongs to $L^2(\mathbb{R}^n)$ (see Stein and Weiss [8], p. 151).

Now fix $k \geqslant 0$ and let $P_1, P_2, \ldots, P_{a_k}$ be an orthonormal basis for the space of solid spherical harmonics of degree k (where we use the inner product in $L^2(S^{n-1})$). The elements in H_k can be written in the form

$$f(x) = \sum_{j=1}^{a_k} f_j(r)\, P_j(x) \qquad (\text{here } r = |x|) \tag{3}$$

and

$$\int_{\mathbb{R}^n} |f(x)|^2\, dx = \sum_1^{a_k} \int_0^\infty |f_j(r)|^2 r^{n+2k-1}\, dr.$$

We let $\mathcal{R}$ denote the class of all functions g on $[0, \infty)$, which satisfy the following conditions:

$$\begin{aligned} &g(r) \geqslant 0 \quad &&\text{for } r \geqslant 0\,, \\ &g \text{ is } C^{\infty} \quad &&\text{on } (0,\ \infty)\,, \\ &g(r) = 0 \quad &&\text{for } r > 1\,, \end{aligned}$$

and

$$\text{there exists } \varepsilon > 0 \text{ such that } g(r) = 0 \text{ for } 0 \leqslant r \leqslant \varepsilon\,.$$

We say that $f \in \mathcal{S}_k$, $k = 0, 1, 2, \ldots$, if f is given by (3) with all $f_j \in \mathcal{R}$.

For $0 < \alpha \leqslant n$ we shall consider the statement:

(4) there exists $C = C_{\alpha, \beta, k}$ such that (1) holds for all $f \in \mathcal{S}_k$.

We then set $\beta(\alpha) = \beta_k(\alpha) = \beta_{n,k}(\alpha) = \sup\{\beta; (4) \text{ holds}\}$.

We have the following result.

THEOREM. *For $k = 0, 1, 2, \ldots$, we have $\beta_k(\alpha) = \alpha$ for $0 < \alpha \leqslant n - 1$, and $\beta_k(\alpha) = n - 1$ for $n - 1 < \alpha \leqslant n$.*

We shall first give a proof of the theorem which works directly for all $k \geqslant 0$. Another possibility is to first treat the case $k = 0$ (i.e. the case of radial functions), and then use the case $k = 0$ to study the case $k \geqslant 1$. We shall also say something about this second approach.

2. Proofs.

If f is a function on $[0,\ \infty)$ we shall also use the notation f for the corresponding radial function in $\mathbb{R}^n$. We also let $\mathcal{F}_n$ denote the Fourier transformation in $\mathbb{R}^n$.

PROOF OF THE THEOREM. Assume that f is given by (3) with all f_j belonging to $\mathcal{R}$. It then follows from [8], p. 158, that

$$\widehat{f}(x) = \sum_{1}^{a_k} F_j(r)\, P_j(x) \qquad (\text{here } r = |x|)\,,$$

where

$$F_j(r) = c_k r^{1-n/2-k} \int_0^\infty f_j(s)\, J_{n/2+k-1}(rs)\, s^{n/2+k}\, ds\,, \qquad r > 0\,,$$

and J_m denotes Bessel functions.

For $|\xi| = 1$ we obtain

$$\widehat{f}(R\xi) = \sum_j F_j(R)\, P_j(R\xi) = R^k \sum_j F_j(R)\, P_j(\xi)$$

and hence

$$\sigma(f)(R) = R^{2k} \int_{S^{n-1}} \left| \sum_j F_j(R)\, P_j(\xi) \right|^2 d\theta(\xi) = R^{2k} \sum_j |F_j(R)|^2.$$

We also have

$$\int_{\mathcal{R}^n} |\widehat{f}(\xi)|^2\, |\xi|^{\alpha-n}\, d\xi = \int_0^\infty \left(\int_{S^{n-1}} |\widehat{f}(r\xi')|^2\, d\theta(\xi') \right) r^{\alpha-1}\, dr$$

$$= \int_0^\infty r^{2k} \left(\sum_j |F_j(r)|^2 \right) r^{\alpha-1}\, dr = \sum_j \int_0^\infty |F_j(r)|^2\, r^{2k+\alpha-1}\, dr$$

$$= \sum_j \int_{\mathbb{R}^n} |F_j(r)|^2\, |\xi|^{2k+\alpha-n}\, d\xi$$

(where $r = |\xi|$ in the last integral).

It follows that the statement (4) is equivalent to the statement: if $f \in \mathbb{R}$ and

$$F(r) = c_k r^{1-n/2-k} \int_0^\infty f(s)\, J_{n/2+k-1}(rs)\, s^{n/2+k}\, ds\,, \qquad r > 0\,, \tag{5}$$

then

$$R^{2k} |F(R)|^2 \leqslant C_k R^{-\beta} \int_{R^n} |F(r)|^2\, |\xi|^{2k+\alpha-n}\, d\xi\,, \qquad R > 1\,. \tag{6}$$

Now assume that $f \in \mathcal{R}$ and that F is given by (5). It is then clear that

$$F(r) = c_k\, \mathcal{F}_{n+2k} f(r) \tag{7}$$

and since $f \in C_0^\infty(\mathbb{R}^{n+2k})$ it follows that $F \in \mathcal{S}(\mathbb{R}^{n+2k})$, where $\mathcal{S}$ denotes the Schwartz class. Assume that $2 - n - 2k \leqslant \beta \leqslant 1$. Then $n/2 + k - 1 \geqslant -\beta/2$ and it follows that

$$|J_{n/2+k-1}(s)| \leqslant Cs^{-\beta/2}, \qquad s > 0$$

(cf. [8], p. 158).

Inserting this estimate in (5) we obtain

$$\begin{aligned}|F(r)| &\leqslant Cr^{1-n/2-k}\int_0^1 f(s)(rs)^{-\beta/2}s^{n/2+k}\,ds\\ &= Cr^{1-n/2-k}\int_0^\infty f(s)\,\varphi(s)(rs)^{-\beta/2}s^{n/2+k}\,ds\,,\end{aligned}$$

where $\varphi \in C_0^\infty(0, \infty)$, $\quad \varphi \geqslant 0$ and

$$\varphi(s) = \begin{cases} 1, & 0 < s \leqslant 1 \\ 0, & s \geqslant 2 \end{cases}.$$

The Fourier inversion formula implies that

$$f(s) = c_k s^{1-n/2-k}\int_0^\infty F(t)\,J_{n/2+k-1}(st)\,t^{n/2+k}\,dt$$

and hence

$$\begin{aligned}|F(r)| &\leqslant Cr^{1-n/2-k}\int_0^\infty\left(s^{1-n/2-k}\int_0^\infty F(t)\,J_{n/2+k-1}(st)\,t^{n/2+k}\,dt\right)\\ &\quad\cdot\varphi(s)(rs)^{-\beta/2}s^{n/2+k}\,ds\\ &= Cr^{1-n/2-k-\beta/2}\int_0^\infty F(t)\,t^{n/2+k}\left(\int_0^\infty \varphi(s)\,s^{1-\beta/2}J_{n/2+k-1}(st)\,ds\right)dt\,.\end{aligned}$$

We conclude that

$$r^k\,|F(r)| \leqslant Cr^{1-n/2-\beta/2}\int_0^\infty F(t)\,t^{n/2+k}I(t)\,dt\,, \tag{8}$$

where

$$I(t)=\int_0^\infty \varphi(s)\, s^{1-\beta/2} J_{n/2+k-1}(st)\, ds\,, \qquad t>0\,.$$

We shall now estimate $I(t)$. Setting $\gamma=\beta/2+n/2+k-1$ we obtain

$$\begin{aligned} I(t) &= t^{n/2+k-1}\, t^{1-n/2-k}\int_0^\infty \varphi(s)\, J_{n/2+k-1}(st)\, s^{n/2+k} s^{-n/2-k+1-\beta/2}\, ds \\ &= c_k t^{n/2+k-1}\mathcal{F}_{n+2k}(\varphi(s)\, s^{1-\beta/2-n/2-k})(t) \\ &= c_k t^{n/2+k-1}\mathcal{F}_{n+2k}(\varphi(s)\, s^{-\gamma})(t)\,, \qquad t>0. \end{aligned}$$

First assume $2-n-2k<\beta\leqslant 1$. Then $\gamma\leqslant 1/2+n/2+k-1<n+2k$ and $\gamma>1-n/2-k+n/2+k-1=0$. It follows that $\mathcal{F}_{n+2k}(\varphi(s)\, s^{-\gamma})= = c_k(\mathcal{F}_{n+2k}\varphi) * s^{-n-2k+\gamma}$, where the convolution is taken in $\mathbb{R}^{n+2k}$. Since $\mathcal{F}_{n+2k}\varphi\in \mathcal{S}(\mathbb{R}^{n+2k})$ we obtain

$$|\mathcal{F}_{n+2k}(\varphi(s)\, s^{-\gamma})(t)|\leqslant C(1+t)^{-n-2k+\gamma}, \qquad t>0\,. \tag{9}$$

In the remaining case $\beta=2-n-2k$ we have $\gamma=0$ and it is clear that (9) holds also in this case.

We have $n+2k-\gamma=n+2k-\beta/2-n/2-k+1=n/2+k-\beta/2+1$, and hence

$$|I(t)|\leqslant Ct^{n/2+k-1}(1+t)^{-n/2-k+\beta/2-1}, \qquad t>0\,.$$

Thus $|I(t)|\leqslant Ct^{n/2+k-1}$ for $0<t\leqslant 1$, and $|I(t)|\leqslant Ct^{\beta/2-2}$ for $t>1$. Invoking (8) we then get

$$r^k|F(r)|\leqslant Cr^{1-n/2-\beta/2}\int_0^\infty |F(t)|\,\psi(t)\, dt\,, \qquad r>0\,, \tag{10}$$

where $\psi(t)=t^{n+2k-1}$ for $0<t\leqslant 1$, and $\psi(t)=t^{n/2+k+\beta/2-2}$ for $t>1$.

Using the Schwarz inequality we obtain

$$\int_0^\infty |F|\,\psi\, dt\leqslant\left(\int_0^\infty |F(t)|^2 t^{2k+\alpha-1}\, dt\right)^{1/2} \left(\int_0^\infty \psi(t)^2 t^{-2k-\alpha+1}\, dt\right)^{1/2}. \tag{11}$$

We have

$$\int_0^1 \psi(t)^2 t^{-2k-\alpha+1}\,dt = \int_0^1 t^{2n+4k-2-2k-\alpha+1}\,dt$$

$$= \int_0^1 t^{2n+2k-\alpha-1}\,dt < \infty$$

since $2n+2k-\alpha \geqslant n$. On the other hand we also have

$$\int_1^\infty \psi(t)^2 t^{-2k-\alpha+1}\,dt = \int_1^\infty t^{n+2k+\beta-4-2k-\alpha+1}\,dt$$

$$= \int_1^\infty t^{n+\beta-\alpha-3}\,dt\,,$$

which is finite if $n+\beta-\alpha-3 < -1$ i.e. $\beta < 2+\alpha-n$.

Invoking (10) and (11) we conclude that

$$r^{2k}\,|F(r)|^2 \leqslant C r^{2-n-\beta}\int_0^\infty |F(t)|^2 t^{2k+\alpha-1}\,dt\,, \qquad r>0\,,$$

if $2-n-2k \leqslant \beta \leqslant 1$ and $\beta < 2+\alpha-n$. Setting $M=\{\beta;\, 2-n-2k \leqslant \beta \leqslant 1 \text{ and } \beta < 2+\alpha-n\}$ we obtain

$$\beta(\alpha) \geqslant \sup_{\beta\in M}(\beta+n-2) = n-2+\sup M\,. \tag{12}$$

Then assume $0<\alpha\leqslant n-1$. We have $2-n-2k<2+\alpha-n\leqslant 1$, and it follows that $\sup M = 2+\alpha-n$ and thus $\beta(\alpha)\geqslant n-2+2+\alpha-n=\alpha$ in this case.

Then assume $n-1<\alpha\leqslant n$. In this case $2+\alpha-n>1$ and it follows that $\sup M=1$. Invoking (12) we obtain $\beta(\alpha)\geqslant n-1$.

Thus we have obtained lower bounds for $\beta(\alpha)$. We shall now obtain upper bounds, and we first assume $0<\alpha\leqslant n-1$. Also assume that (6) holds for all F given by (5) with $f\in\mathcal{R}$. We shall prove that then $\beta\leqslant\alpha$. First choose $f\in\mathcal{R}$ with $f\not\equiv 0$. Then there exists $b>0$ such that $F(b)\neq 0$. Also set $f_a(s)=f(as)$, $a>1$, and

$$F_a(r) = c_k r^{1-n/2-k}\int_0^\infty f(as)\,J_{n/2+k-1}(rs)\,s^{n/2+k}\,ds\,, \qquad r>0\,.$$

Performing a change of variable $as = t$ we obtain

$$F_a(r) = c_k r^{1-n/2-k} \int_0^\infty f(t)\, J_{n/2+k-1}(rt/a)\, t^{n/2+k}\, dt\, a^{-n/2-k-1}$$

$$= a^{-n-2k} F(r/a),$$

and (6) yields

$$r^{2k} a^{-2n-4k} |F(r/a)|^2 \leqslant C r^{-\beta} \int_0^\infty |F(r/a)|^2 r^{2k+\alpha-1}\, dr\, a^{-2n-4k}.$$

Performing a change of variable we then get

$$r^{2k} |F(r/a)|^2 \leqslant C r^{-\beta} \int_0^\infty |F(s)|^2 s^{2k+\alpha-1}\, ds\, a^{2k+\alpha} \tag{13}$$

$$= C r^{-\beta} a^{2k+\alpha}$$

for all $a > 1$ and $r > 1$, where C depends on f but not on a or r. We now choose $a = r/b$, where r is large, and it follows from (13) that

$$r^{2k} |F(b)|^2 \leqslant C r^{-\beta} r^{2k+\alpha} b^{-2k-\alpha}.$$

We conclude that $r^\beta \leqslant C r^\alpha$ and it follows that $\beta \leqslant \alpha$. Hence $\beta(\alpha) \leqslant \alpha$ for $0 < \alpha \leqslant n-1$ and we have proved that $\beta(\alpha) = \alpha$ in this case.

It remains to study the case $n - 1 < \alpha \leqslant n$. Assume as above that (6) holds for all $f \in \mathcal{R}$.

Let $\mathcal{L}_+$ denote the class of all $f \in L^1[0, \infty)$ with $f \geqslant 0$ and satisfying $f(r) = 0$ for $r \geqslant 7/8$ and $f(r) = 0$ for $0 \leqslant r \leqslant \varepsilon$ for some $\varepsilon > 0$. It is then easy to see that (6) holds also for all $f \in \mathcal{L}_+$. In fact, this follows from approximation of $f \in \mathcal{L}_+$ with $f * \varphi_\varepsilon$, where the convolution is taken in $\mathbb{R}^{n+2k}$ and φ_ε is an approximate identity in $\mathbb{R}^{n+2k}$.

Then choose $\varphi \in C_0^\infty(0, \infty)$ with $\operatorname{supp} \varphi \subset (1/2, 7/8)$, $\varphi \geqslant 0$, and $\varphi(3/4) = 1$. Also set

$$f(s) = f_R(s) = e^{-iRs} \varphi(s), \qquad s > 0,$$

where R is large. Then

$$f = f_1 - f_2 + i f_3 - i f_4,$$

where $f_j \in \mathcal{L}_+$ and $f_j \leqslant |f|$ for $j = 1, 2, 3, 4$. Let F_j correspond to f_j in the same way as F corresponds to f in (5). Then $F_j = c_k \mathcal{F}_{n+2k} f_j$ and since (6) holds for F_j we obtain

$$R^{2k} |F(R)|^2 \leqslant CR^{2k} \sum_j |F_j|^2 \leqslant CR^{-\beta} \sum_j \int_0^\infty |F_j(r)|^2 r^{2k+\alpha-1} dr$$

$$= CR^{-\beta} \sum_j \int_{\mathbb{R}^{n+2k}} |F_j(r)|^2 |\xi|^{\alpha-n} d\xi .$$

In the case $n-1 < \alpha < n$ we invoke Lemma 12.12 in Mattila [5], p. 162, and then get

$$R^{2k} |F(R)|^2 \leqslant CR^{-\beta} \sum_j \int_{\mathbb{R}^{n+2k}} \int_{\mathbb{R}^{n+2k}} |x-y|^{-\alpha-2k} f_j(x) f_j(y)\, dx\, dy$$

$$\leqslant CR^{-\beta} \int_{\mathbb{R}^{n+2k}} \int_{\mathbb{R}^{n+2k}} |x-y|^{-\alpha-2k} |f(x)|\, |f(y)| dx\, dy .$$

Hence

$$R^{2k} |F(R)|^2 \leqslant CR^{-\beta}, \tag{14}$$

where C depends on φ but not on R. In the case $\alpha = n$ (14) follows from an application of the Plancherel theorem.

We have

$$R^k F(R) = cR^{1-n/2} \int_0^\infty e^{-iRs} \varphi(s) J_{n/2+k-1}(Rs)\, s^{n/2+k} ds$$

and we shall use the asymptotic formula

$$J_{n/2+k-1}(t) = c_1 e^{it} t^{-1/2} + c_2 e^{-it} t^{-1/2} + \mathcal{O}(t^{-3/2}), \qquad t \to \infty$$

(see [8], p. 158). We obtain

$$R^k F(R) = cR^{1-n/2} \int_0^1 \left[c_1 \frac{e^{iRs}}{(Rs)^{1/2}} + c_2 \frac{e^{-iRs}}{(Rs)^{1/2}} + \mathcal{O}((Rs)^{-3/2}) \right]$$

$$\cdot s^{n/2+k} e^{-iRs} \varphi(s)\, ds$$

$$= cc_1 R^{1/2-n/2} \int_0^1 s^{n/2+k-1/2} \varphi(s)\, ds$$

$$+ cc_2 R^{1/2 - n/2} \int_0^1 e^{-2iRs} s^{n/2 + k - 1/2} \varphi(s)\, ds$$

$$+ \mathcal{O}(R^{-1/2 - n/2}) \geqslant cR^{1/2 - n/2}$$

and hence

$$R^{2k} |F(R)|^2 \geqslant cR^{1-n}$$

for large values of R.

The formula (14) then yields

$$R^{1-n} \leqslant CR^{-\beta}$$

i.e.

$$R^{\beta} \leqslant CR^{n-1}$$

and we conclude that $\beta \leqslant n - 1$. Hence $\beta(\alpha) \leqslant n - 1$ for $n - 1 < \alpha \leqslant n$ and it follows that $\beta(\alpha) = n - 1$ in this case. The proof of the theorem is complete.

We shall finally discuss how results for radial functions (i.e. the case $k = 0$) can be used to study the case $k \geqslant 1$. Therefore assume $n \geqslant 2$, $k \geqslant 1$ and $0 < \alpha \leqslant n$. If $f \in \mathcal{R}$ and F is given by (5), then the estimate

$$|F(R)|^2 \leqslant CR^{-\beta} \int_0^\infty |F(r)|^2 r^{\alpha + 2k - 1}\, dr$$

is equivalent to the estimate

$$R^{2k} |F(R)|^2 \leqslant CR^{-(\beta - 2k)} \int_0^\infty |F(r)|^2 r^{\alpha + 2k - 1}\, dr\,.$$

It follows that

$$\beta_{n,k}(\alpha) = \beta_{n+2k,0}(\alpha + 2k) - 2k\,. \tag{15}$$

Assume then that we know that $\beta_{n,0}(\alpha) = \min(\alpha, n - 1)$ for all $n \geqslant 2$ and $0 < \alpha \leqslant n$. For $k \geqslant 1$, $n \geqslant 2$ and $0 < \alpha \leqslant n$ (15) then yields

$$\beta_{n,k}(\alpha) = \min(\alpha + 2k, n + 2k - 1) - 2k = \min(\alpha, n - 1),$$

which is the desired formula.

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