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Harnack's Inequalities for Solutions to the Mean Curvature Equation and to the Capillarity Problem.

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ABSTRACT - We impose suitable conditions to obtain Harnack inequalities for solutions to the capillarity problems in terms merely of the prescribed boundary contact angle, the prescribed mean curvature and the dimension. Moreover, for solutions to mean curvature e quation in a ball $B_R(x_0)$, Harnack's inequalities are shown to hold in $B_{\lambda R}(x_0)$ in terms merely of the mean curvature, λ and the dimension. Furthermore, Harnack's inequalities for neighborhoods of the boundary points will be established. We emphasize that the constant concerned are all explicitly obtaned.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$. Let $H(x, u(x))$ be a given Lipschitz-continuous function in $\Omega \times \mathbb{R}$. We consider solutions to the mean curvature equation of surfaces of prescribed mean curvature

$$\operatorname{div} Tu = nH(x, u(x)) \quad \text{in } \Omega , \tag{0.1}$$

where

$$Tu = \frac{Du}{\sqrt{1 + |Du|^2}} .$$

A solution of the capillarity problem can be looked at as a solution of the

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equation (0.1) subject to the «contact angle» boundary condition

$$(0.2) \qquad Tu\cdot\nu = \cos\theta\,,$$

where ν is the outward pointing unit normal of $\partial\Omega$. Thus, geometrically we are considering a function u on $\overline{\Omega}$ whose graph has the prescribed mean curvature H and which meets the boundary cylinder in the prescribe d angle θ.

One main purpose of this paper is to obtain Harnack's inequalities for solutions to the capillarity problems in terms merely of *the dimension* n, *the boundary contact angle* θ and *the mean curvature* H. These results are formulated as Theorems 2-3. Moreover, for solutions to the mean curvature equation (0.1) in a b all $B_R(x)$, Harnack's inequalities are shown to hold in $B_{\lambda R}(x)$, $0<\lambda<1$, in terms merely of *the mean curvature* H, λ and n. This is formualted as Theorem 4. Furthermore, Harnack's inequalities for neighborhoods of the boundary points will be established and formulated as Theorems 5-6.

We recall a Harnack's inequality due to Serrin[25] which can be stated as follows:

«Suppose $u(x)\in C^2(\Omega)$ is a non-negative solution of (0.1) *in a two-dimensional ball $B_R(x_0)$ for $H\equiv 1$ and suppose that $u(x_0)=m$. Then there exist functions $\varrho(m)>0$ and $\Phi(m;r)<\infty$ in $r<\varrho(m)$, such that $|u(x)|<\Phi(m;|x|)$ in $B_R(x_0)$. There holds $\lim\limits_{r\to\varrho}\Phi(m;r)=\infty$, while $\varrho(m)\searrow 0$ as $|m|\to\infty$.»*

In Finn [3], a simpler proof of this result is given by employing the notion of *generalized solutions*, together with *either* constructing barriers to apply comparison principles *or* showing gradient estimates of a special type. This new pr oof yields considerably improved and qualitatively different information. Indeed, the following is obtained in [3], in which it is remarkable that *the one sided bound* essential for the classical Harnack's inequality does not appear:

«There exist a universal constant R_0 and a constant $\widehat{R}$ determined entirely by R for $R>R_0$, such that in Serrin's result $\varrho(m)\geqslant\widehat{R}$ if $R>R_0$. Furthermore, there exist functions $A_0^-(R;r)$ and $A_0^+(R;r)$ such that if $u\in C^2(\Omega)$ is a solution to (0.1) *in $B_R(x_0)$ with $u(x_0)=m$, then*

$$A_0^-(R;|x|)<u(x)-m-1+\sqrt{1-r^2}<A_0^+(R;|x|),$$

where $\lim_{R\to 1} A_0^-(R;r) = \lim_{R\to 1} A_0^+(R;r) = 0$, *for all* $r<1$ *and* $A_0^+(R;r) < \infty$, *for all* $0<\varepsilon<R$.

If $R>R_0$, $m>0$, *then* $\varrho(m)=\infty$. *Furthermore, a function* $A_1(R)$ *exists such that* $A_1(R)\searrow 0$ *as* $R\to 1$ *and*

$$u - m - 1 + \sqrt{1-r^2} < A_1(R). \text{ »}$$

In case $n=2$ and H satisfies monotonicity condition instead of being constant, Finn and Lu [6] obtained gradient estimates of a type analogous to that employed in [3], which immediately yields the following, in which *the one sided bound* does not appear either.

«*Assume* $H'(u)\geqslant 0$, $H(-\infty)\neq H(+\infty)$. *Then there exist a positive constant* $\varrho^+(u_0;R)\leqslant R$ *and a continuous function* $A_*^+(u_0;R;\varrho)$ *with* $A_*^+(u_0;R;0)=u_0$ *such that if* $u(x)$ *satisfies* (0.1) *in two -dimensional ball* $B_R(x_0)$ *and* $u(x_0)=u_0$, *then* $u\leqslant A_*^+$ *throughout* $B_{\varrho^+}(x_0)$.

There also exist a positive function $\varrho^-(u_0;R)\leqslant R$ *and a continuous function* $A_*^-(u_0;R;\varrho)$ *with* $A_*^-(u_0;R;0)=u_0$ *such that* $u\geqslant A_*^-$ *throughout* $B_{\varrho^-}(x_0)$.

If $H(+\infty)=+\infty$ *and* $H(-\infty)=-\infty$, *then the functions* $A_*^+-u_0$ *and* $u_0-A_*^-$ *do not depend on* u_0, *and additionally* $\varrho^+=\varrho^-=R$.»

Indeed, the gradient estimates resorted to in [3] and [6] take the following form and is proved in [5], [15] and [6], respectively.

«*Let* $R>R_0=0.5654062332\ldots$ *Let* $\Omega_R\subset\mathbb{R}^2$ *be a "moon" domain bounded by two circular arcs* Γ_1 *and* Γ_2 *of the respective radius* R *and* $\frac{1}{2}$ *such that* $2|\Omega_R| = |\Gamma_1| - |\Gamma_2|$, *where* $|\cdot|$ *denotes either the Hausdorff* 2*-measure or* 1*-measure. Let* $\widehat{R}(R)$ *be the radius of the largest disk concentric to* $B_R(x_0)$ *such that* $B_{\widehat{R}}(x_0)\subset\Omega_R$. *There exists a positive function* $A(R;\varepsilon)<\infty$ *such that for any solution* $u(x)$ *of (0.1) in* $B_R(x_0)$ *and any* $\varepsilon<0$, *there holds* $|\nabla u(x)|<A(R;\varepsilon)$ *in* $B_{\widehat{R}-\varepsilon}(x_0)$. *Furthermore, the value* $\widehat{R}$ *cannot be improved.*»

«*Assume* $H'(u)\geqslant 0$, $H(-\infty)\neq H(+\infty)$. *Let* $u(x)$ *be a solution of* (0.1) *over a two-dimensional disk* $B_R(x_0)$. *Then* $|\nabla u(x_0)|$ *is bounded, depending only on* R *and on* $u(x_0)$. *If* $H(-\infty)=-\in\infty$ *and* $H(+\infty)=$ $=+\infty$, *then the bound depends only on* R.»

The remarkable feature of this type of gradient estimate is that it depends on neither boundary data nor bounds of any sorts, in contrast to results in [1] [7] [8] [9] [14] [17] [26] [27], for example. Progress aimed at obtaining gradient estimates of this type are made in [16] [17] [18].

The above-mentioned Harnack's inequalities and gradient estimates, however, has the disadvantage that, while it guarantees the existence of upper or lower bounds, the explicit value of the bounds are not known. In this paper, Harnack's inequalities are s hown to hold under some imposed condition, in particular, under the one-sided bounded condition; however, the constant concerned are all explicitly obtained.

In [28], Harnack,s inequalities are obtained for nonnegative bounded solutions $u \in W^{2,n}(\Omega)$ of (0.1) in which $H(x, u)$ satisfies structure conditions of different feature than those required in this paper (cf. (0.12) below). The Harnack's inequalities in [28] takes the form $\sup_{B_{\sigma R}} u \leqslant C \inf_{B_\sigma} u$ for any ball $B_R \subset \Omega$ and $0 < \sigma < 1$, in which the constant C can be explicitly calculated in terms of n, σ, R, the upper bound of u in B_R and the quantities involved in the structure conditions.

0. Introduction.

0.1. *Preliminary Harnack's inequalities.*

Of basic importance is the following Preliminary Harnack's Inequality, which estimate the growth of a solution in a small ball B in terms of the ratio of the measure of level sets inside this ball B t o the whole ball B. Namely,

PROPOSITION 1 (the Preliminary Harnack's Inequality). *Let u be a $C^2(\Omega)$ function over a domain $\Omega \subset \mathbb{R}^n$, with subgraph*

$$U = \{(x, t) \in \Omega \times \mathbb{R},\ t < u(x)\}.$$

For points $\hat{z} = (\hat{x}, \hat{t}) \in \Omega \times \mathbb{R}$ and for $r > 0$, we set

$$U_r(\hat{z}) = C_r(\hat{z}) \cap U, \qquad and \qquad U_r'(\hat{z}) = C_r(\hat{z}) U\,,$$

with

$$C_r(\hat{z}) = \{(x, t) \colon |x - \hat{x}\,| < r,\ |t - \hat{t}\,| < r\}.$$

Suppose there exist positive constants α_* and R^* depending only on n,

$\inf_{\Omega \times \mathbb{R}} H$ *and* $\sup_{\Omega \times \mathbb{R}} H$ *such that*

$$|U_r(\hat{z})| \geqslant \alpha_* r^{n+1} \text{ for all } r \leqslant \min(R^*, \operatorname{dist}(\hat{z}, \partial(\Omega \times \mathbb{R}))), \quad \text{if } |U_r(\hat{z})| > 0 \text{ for all } r > 0, \tag{0.3}$$

and

$$|U_r'(\hat{z})| \geqslant \alpha_* r^{n+1} \text{ for all } r \leqslant \min(R^*, \operatorname{dist}(\hat{z}, \partial(\Omega \times \mathbb{R}))) \quad \text{if } |U_r'(\hat{z})| > 0 \text{ for all } r > 0, \tag{0.4}$$

Let us set, for $\beta > 0$,

$$D_\beta = \{x : x \in \Omega,\ |Du| \geqslant \beta\}, \qquad \widehat{R}^* = \max\left(R - 2R^*,\ \frac{3}{4}R\right),$$

and

$$\Lambda^* = \Lambda^*_{R,\beta} = \min(1, \beta)\max\left(\frac{R^*}{R},\ \frac{1}{8}\right).$$

If the ball $B_R(x_0)$ *has the radius* $R \leqslant \min(\widehat{R}^*, \operatorname{dist}(x_0, \partial\Omega))$, *then there exist two positive constants* ξ_{n,α_*} *and* $C_{\alpha_*,\beta}$ *determined completely by* α_*, β *and* n *such that, for any* $x_1 \in B_{\widehat{R}^*}(x_0)$ *with* $B_{2\Lambda^* R}(x_1) \subset D_\beta$, *we have*

$$u(x_0) - m_{\Omega_0} \leqslant \xi_{n,\alpha_*}\omega_n(u(x_1) - m_{\Omega_0}) + {} + (2 + \Lambda^*_{R,\beta}\xi_{n,\alpha_*})\,\omega_n R + \xi_{n,\alpha_*} C_{\alpha_*,\beta} R^{1-n} \int_{B_R(x_0)} |Du|\,dx, \tag{0.5}$$

and

$$M_{\Omega_0} - u(x_0) \leqslant \xi_{n,\alpha_*}\omega_n(M_{\Omega_0} - u(x_1)) + {} + (2 + \Lambda^*_{R,\beta}\xi_{n,\alpha_*})\,\omega_n R + \xi_{n,\alpha_*} C_{\alpha_*,\beta} R^{1-n} \int_{B_R(x_0)} |Du|\,dx, \tag{0.6}$$

where we set $M_{\Omega_0} = \sup_{\Omega_0} u$ *and* $m_{\Omega_0} = \inf_{\Omega_0} u$, *for any domain* Ω_0 *such that* $B_R(x_0) \subseteq \Omega_0 \subseteq \Omega$. *In fact, we are allowed to take*

$$\xi_{n,\alpha_*} = \frac{2^{n+2}}{\alpha_*}, \tag{0.7}$$

and

$$(0.8) \qquad C_{\alpha_*, \beta} = 2^{2n+1-\frac{1}{n}} \Lambda^*_{R,\beta} \left(\frac{\omega_n}{\alpha_*} \right)^{\frac{1}{n}}.$$

In Theorem 1 and throughout this paper, we denote by $|\,.\,|$ either the Hausdorff $(n+1)$-measure or the Hausdorff n-measure and denote by $B_s(x)$, $s > 0$, $x \in \Omega$, a ball centered at x and of radius s.

In Section 1, this Preliminary Harnack's Inequality will be proved by adapting the reasoning on pages 312-313 of Giusti [12], together with an application of the following *modified* version of *Poincarè inequality.*

Proposition 2 (a modified version of Poincarè inequality). *Suppose $w \in W^{1,p}(\Omega)$ for some $p \geqslant 1$ and convex Ω, with*

$$|\{x : x \in \Omega,\, w(x) \leqslant 0\}| \geqslant \alpha_1 |\Omega|$$

If $p > 1$, then we have

$$\|w\|_p \leqslant C_{\alpha_1} \|Dw\|_p,$$

with

$$C_{\alpha_1} = (1 - (1 - \alpha_1)^{\frac{p-1}{p}})^{-1} \left(\frac{\omega_n}{|\Omega|} \right)^{1-\frac{1}{n}} (\text{diam } \Omega)^n.$$

If $p = 1$ and if we have, in addition,

$$|\{x : x \in \Omega,\, w(x) \leqslant 0\} \geqslant \alpha_2 |\Omega|,$$

then

$$\|w\|_1 \leqslant C_{\alpha_1, \alpha_2} \|Dw\|_1,$$

with

$$(0.9) \quad C_{\alpha_1,\alpha_2} = \max(\alpha_1, \alpha_2) \left(\left(\frac{1}{\alpha_1} \right)^{1-\frac{1}{n}} + \left(\frac{1}{\alpha_2} \right)^{1-\frac{1}{n}} \right) \left(\frac{\omega_n}{|\Omega|} \right)^{1-\frac{1}{n}} (\text{diam } \Omega)^n.$$

A proof of this inequality is given in [18]. We remark that inequalities of this type are indicaded to hold, for example, in [20] and [29] for a class of domains with much less restrictions than that of *convexity* imposed here. However, in results of [20] and [29], the constants C_{α_1} and C_{α_1, α_2} are not given explicitly.

For sufficiently small r, the number α_* in **Proposition 1** can be estimated in terms of the mean curvature H and n. In Appendix, we will resort to the estimates obtained in Giusti [12] for generalized solutions of the equation (0.1), taking ad vantage of the fact that generalized solutions for the equation (0.1) are allowed to take infinite values $+\infty$ and/or $-\infty$ in subdomains of Ω of positive n-Hausdorff measure. We shall obtain

PROPOSOTION 3 (estimates for the number α_* in (0.3) and (0.4)). *Let u be a generalized solution to* (0.1) *in Ω with the subgraph U. Let $U_r(\hat{z})$, $U_r'(\hat{z})$ be as in Theorem* 1. *If*

$$|U_r(\hat{z})| > 0 \quad \text{and} \quad |U_r'(\hat{z})| > 0 \ \text{for all} \ r > 0,$$

then, setting

$$\alpha_* = \frac{1}{4(n+1)\,k_{(n+1)}}, \tag{0.10}$$

with $k_{(m)}$ being the isoperimetric constant in $\mathbb{R}^m$, $m \geqslant 1$, and setting

$$R_-^* = \begin{cases} \left(\dfrac{1}{2k_{(n)}\,\omega_n\,|\inf\limits_{\Omega\times\mathbb{R}} H(x,t)|} \right)^{\frac{1}{n}}, & \text{if } \inf\limits_{\Omega\times\mathbb{R}} H(x,t) < 0, \\ \infty & \text{if } \inf\limits_{\Omega\times\mathbb{R}} H(x,t) \geqslant 0, \end{cases}$$

$$R_+^* = \begin{cases} \left(\dfrac{1}{2k_{(n)}\,\omega_n\,|\sup\limits_{\Omega\times\mathbb{R}} H(x,t)|} \right)^{\frac{1}{n}}, & \text{if } \sup\limits_{\Omega\times\mathbb{R}} H(x,t) \leqslant 0, \\ \infty & \text{if } \inf\limits_{\Omega\times\mathbb{R}} H(x,t) > 0, \end{cases} \tag{0.11}$$

we have

$$|U_r(\hat{z})| \geqslant \alpha_* r^{n+1} \quad \text{for all} \quad r \leqslant \min(R_-^*, \operatorname{dist}(\hat{z}, \partial(\Omega\times\mathbb{R})))$$

and

$$|U_r'(\hat{z})| \geqslant \alpha_* r^{n+1} \quad \text{for all} \quad r \leqslant \min(R_+^*, \operatorname{dist}(\hat{z}, \partial(\Omega\times\mathbb{R}))).$$

Inserting the value of the number α_* in (0.10) into Proposition 1, we obtain

THEOREM 1 (the Preliminary Harnack's Inequality*). *Let* $u \in$ $\in C^2(\Omega)$ *be a solution of* (0.1) *in* Ω. *Let* $B_R(x_0)$, $B_{\widehat{R}^*}(x_0)$, $\Lambda^*_{R,\beta}$ *and* D_β *be as in Theorem* 1. *If* $x_1 \subset B_{\widehat{R}^*}(x_0)$ *with* $B_{2\Lambda^* R}(x_1) \subset D_\beta$, *then* (0.7) *and* (0.8) *hold with* α_* *in* (0.10), $R^* = \min(R^*_-, R^*_+)$ *and*

$$\xi_{n,\alpha_*} = 2^{n+4}(n+1)\,k_{(n+1)}.$$

0.2. *Harnack's inequalities for solutions to the capillarity problem.*

Assume that $u \in C^2(\Omega) \cap C^{0,1}(\Omega)$ is a solution to the capillarity problem (0.1) and (0.2). Furthermore, suppose that $\partial\Omega$ is of class C^2 and that the functions

$$H \in C^{0,1}(\Omega \times \mathbb{R}) \quad \text{and} \quad \cos\theta \in C^{0,1}(\partial\Omega)$$

satisfy the conditions

$$|\cos\theta| \leqslant \widehat{\gamma}, \quad \text{and} \quad \inf_{x \in \Omega} H \geqslant 0, \tag{0.12}$$

for some positive constant $\widehat{\gamma}$, $0 \leqslant \widehat{\gamma} < 1$.

First of all, let us extend $\cos\theta$ and ν into the whole domain Ω such that $\cos\theta$ belonging to $C^{0,1}(\overline{\Omega})$ still satisfies (0.12) and such that the vector field ν is unifomly Lipschitz continuous in Ω and bounded in absolute value by the number 1. The extensions are possible in view of the smoothness of $\partial\Omega$.

An integration of (0.1) and (0.2) yields

$$\int_\Omega Tu \cdot D\eta\, dx + n\int_\Omega H\eta\, dx - \int_{\partial\Omega} \cos\theta\eta\, d\mathcal{H}_{n-1} = 0, \tag{0.13}$$

for all $\eta \in C^1(\overline{\Omega})$. Henceforth, we may say that $u \in C^1(\overline{\Omega}) \cap W^{1,1}(\Omega)$ is a solution of (0.1) and (0.2) in the weak sense in Ω if it satisfies (0.1) in Ω and (0.13) for all $\eta \in C^1(\overline{\Omega})$.

To handle the third term on the left hand side of (0.13), we recall the following result in Lemma 1.1 of Giusti [12] and its proof.

LEMMA 1 (Giusti [12]). *Let* $\partial\Omega$ *be of class* C^2 *and*

$$d(x) = \operatorname{dist}(x, \partial\Omega),$$

for $x \in \Omega$. *For* $\varepsilon > 0$ *which is so small that the function* $d(x)$ *is of class* C^2 *in*

$$\Sigma_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leqslant \varepsilon\}, \tag{0.14}$$

there exists a constant $C_{\varepsilon,\Omega}$ *determined completely by* ε *and* $\partial\Omega$ *such that*

$$\int_{\partial\Omega} |w|\, d\mathcal{H}_{n-1} \leqslant \int_{\Sigma_\varepsilon} |Dw| + C_{\varepsilon,\Omega}\int_{\Sigma_\varepsilon} |w|\, dx, \tag{0.15}$$

for every $w \in BV(\Omega)$. *In fact, if we let* η_ε *be a* C^∞ *function with*

$$\begin{cases} 0 \leqslant \eta_\varepsilon \leqslant 1, & \\ \eta_\varepsilon = 1 & \text{on } \partial\Omega, \\ \eta_\varepsilon = 0 & \text{in } \Omega\setminus\Sigma_\varepsilon, \end{cases} \tag{0.16}$$

then we can take

$$C_{\varepsilon,\Omega} = \sup_{\Omega} |\operatorname{div}(\eta_\varepsilon Dd)|\,. \tag{0.17}$$

Indeed, the following results are well known (cf. e.g. pages 354-357 of [10], pages 420-422 of [24]).

LEMMA 2. *Let* $\partial\Omega$ *be of class* C^2 *whose principal curvatures are bounded in absolute value by* $\mathcal{K}_{\partial\Omega}$. *Then* $d(x) = \operatorname{dist}(x, \partial\Omega)$ *is of class* C^2 *in* Σ_ε, *for* $\varepsilon \leqslant \dfrac{1}{\mathcal{K}_{\partial\Omega}}$, *where* Σ_ε *is given in* (0.13).

Furthermore, for points $\bar{x}$ *in* Σ_ε, $\varepsilon \leqslant \dfrac{1}{\mathcal{K}_{\partial\Omega}}$, *define* $\bar{y} = y(\bar{x})$ *to be the (unique) points on* $\partial\Omega$ *nearest to* $\bar{x}$. *Consider the special coordinate frame in which the* x_n*-axis is oriented along the inward normal to* $\partial\Omega$ *at* $\bar{y}$ *and the coordinates* $x_1, \ldots, x_{n-1}$ *lie along the principal directions of* $\partial\Omega$ *at the point* $\bar{y}$. *In this special coordinates, we have at* $\bar{x}$

$$Dd = (0, \ldots, 0, 1) \tag{0.18}$$

and

$$D^2 d = \operatorname{diagonal}\left[\frac{k_1}{1-k_1 d}, \ldots, \frac{k_{n-1}}{1-k_{n-1} d}, 0\right], \tag{0.19}$$

where $k_1, \ldots, k_{n-1}$ *are the principal curvatures of* $\partial\Omega$ *at* $\bar{y}$.

Inserting (0.18) and (0.19) into (0.17), we obtain the following

LEMMA 3. *Let* $\partial\Omega$ *be of class* C^2 *whose principal curvatures are bounded in absolute value by* $\mathcal{K}_{\partial\Omega}$. *Then, for* $\varepsilon \leqslant \dfrac{1}{2\,\mathcal{K}_{\partial\Omega}}$ *and for each* δ,

$0 < \delta \leqslant 1$, *we can take in* (0.13)

$$C_{\varepsilon,\Omega} \leqslant |D\eta_\varepsilon| + 2(n-1)\,\mathcal{K}_{\partial\Omega} \leqslant \left(\frac{1+\delta}{\varepsilon}\right) + 2(n-1)\,\mathcal{K}_{\partial\Omega}. \tag{0.20}$$

Combining (0.12) and (0.15) with $C_{\varepsilon,\Omega}$ being as in (0.20), we obtain, for $\varepsilon \leqslant \dfrac{1}{2\,\mathcal{K}_{\partial\Omega}}$,

$$\left| \int_{\partial\Omega} \cos\theta\eta\, d\mathcal{H}_{n-1} \right| \leqslant \widehat{\gamma} \int_{\Sigma_\varepsilon} |D\eta| + \widehat{\gamma}\, C_{\varepsilon,\Omega} \int_{\Sigma_\varepsilon} |\eta|\,dx \leqslant \tag{0.21}$$

$$\leqslant \widehat{\gamma} \int_{\Sigma_\varepsilon} |D\eta| + \widehat{\gamma}\left(\frac{1+\delta}{\varepsilon} + 2(n-1)\,\mathcal{K}_{\partial\Omega}\right) \int_{\Sigma_\varepsilon} |\eta|\,dx,$$

for all $\eta \in C^1(\overline{\Omega})$.

In Section 2.1, we shall obtain Harnack's inequality in the following formulation by inserting (0.21) into (0.13) either with $\eta = (u - m_{\Omega_0})$ or with $\eta = (M_{\Omega_0} - u)$, and subsequently inserting what results in into (0.5) and (0.6).

THEOREM 2 (the first Harnack's inequality). *Let* $u \in C^2(\Omega) \cap W^{1,1}(\Omega)$ *be a solution to* (0.1) *and* (0.2) *in the weak sense in* Ω. *Suppose that* $\partial\Omega \in C^2$ *such that* (0.12) *holds in which* $\widehat{\gamma}$ *satisfies*

$$2\,\widehat{\gamma} < \frac{\left(\inf_{x \in B_R(x_0)} H\right)}{\mathcal{K}_{\partial\Omega}}. \tag{0.22}$$

Let us set

$$\mathcal{A}_1 = (2 + \xi_{n,\alpha_*} \Lambda^*_{R,\beta}) \frac{\omega_n}{1-\widehat{\gamma}} R + \xi_{n,\alpha_*} C_{\alpha_*,\beta} \frac{|\Omega|}{(1-\widehat{\gamma})\,R^{n-1}}$$

and

$$\mathcal{A}_2 = (2 + \xi_{n,\alpha_*} \Lambda^*_{R,\beta})\, \omega_n R + \xi_{n,\alpha_*} C_{\alpha_*,\beta} \frac{|\Omega|}{R^{n-1}}.$$

Then, for any $x_1 \in B_{\widehat{R}^*}(x_0)$ *with* $B_{2\Lambda^* R}(x_1) \subset D_\beta$, ($\widehat{R}^*$ *and* Λ^* *being given* *in Proposition* 1), *we have*

$$(0.23)\qquad u(x_0) - m_{\Omega_0} \leqslant \frac{\xi_{n,\alpha*}\omega_n}{1-\widehat{\gamma}}(u(x_1) - m_{\Omega_0}) + \mathcal{A}_1.$$

and

$$(0.24)\qquad M_{\Omega_0} - u(x_0) \leqslant \frac{\xi_{n,\alpha*}\omega_n}{1-\widehat{\gamma}}(M_{\Omega_0} - u(x_1)) + \mathcal{A}_1,$$

where we set $m_{\Omega_0} = \inf_{\Omega_0} u$ *and* $M_{\Omega_0} = \sup_{\Omega_0} u$, *for any domain* Ω_0 *such that* $B_R(x_0) \subseteq \Omega_0 \subseteq \Omega$. *Furthermore, for any* $x_1 \in B_{\widehat{R}_0^*}(x_0)$ *with* $B_{2\Lambda^* R}(x_1) \subset D_\beta$, $\widehat{R}_0^* = \max\left(\frac{R}{2} - 2R^*, \frac{3}{8}R\right)$, *we have*

$$(0.25)\qquad u(x_0) - m_{\Omega_0} \leqslant \xi_{n,\alpha*}\omega_n(u(x_1) - m_{\Omega_0}) + \mathcal{A}_2,$$

$$(0.26)\qquad M_{\Omega_0} - u(x_0) \leqslant \xi_{n,\alpha*}\omega_n(M_{\Omega_0} - u(x_1)) + \mathcal{A}_2,$$

and for any $x_1, x_2 \in B_{\widehat{R}_1^*}(x_0)$ *with* $B_{2\Lambda^* R}(x_1) \subset D_\beta$, $\widehat{R}_1^* = \max\left(\frac{R}{4} - 2R^*\right), \frac{3}{16}R)$, *we have*

$$(0.27)\qquad \begin{aligned} u(x_1) - m_{\Omega_0} &\leqslant \xi_{n,\alpha*}\omega_n(u(x_2) - m_{\Omega_0}) + \\ &+ (2 + \xi_{n,\alpha*}\Lambda^*_{R,\beta})\,\omega_n \operatorname{dist}(x_1, x_2) + \xi_{n,\alpha*} C_{\alpha*,\beta}\omega_n \operatorname{dist}(x_1, x_2), \end{aligned}$$

$$(0.28)\qquad \begin{aligned} M_{\Omega_0} - u(x_1) &\leqslant \xi_{n,\alpha*}\omega_n(M_{\Omega_0} - u(x_2)) + \\ &+ (2 + \xi_{n,\alpha*}\Lambda^*_{R,\beta})\,\omega_n \operatorname{dist}(x_1, x_2) + \xi_{n,\alpha*} C_{\alpha*,\beta}\omega_n \operatorname{dist}(x_1, x_2). \end{aligned}$$

In the special case where Ω is the ball $B_R(x_0)$, we have

$$(0.29)\qquad k_i = \frac{1}{R}, \quad i = 1, \dots, n-1, \quad \text{and} \quad \mathcal{H}_{\partial\Omega} = \frac{1}{R},$$

and from (0.12) we obtain immediately

$$(0.30)\qquad \left| \int_{\partial B_R(x_0)} \cos\theta\eta\, d\mathcal{H}_{n-1} \right| \leqslant \widehat{\gamma} n\omega_n R^{n-1} \sup_{B_R} |\eta|.$$

In Sections 2.2 and 2.3, the inequality (0.30) is inserted into (0.5) and (0.6) to obtain the following.

Corollary 1 (the second Harnack's inequality). *Let* $u \in C^2(\Omega) \cap W^{1,1}(\Omega)$ *be a solution to* (0.1) *and* (0.2) *in the weak sense in* $B_R(x_0)$.

Suppose that (0.12) *holds in which* $\widehat{\gamma}$ *satisfies*

$$2\,\widehat{\gamma} < R \inf_{x \in B_R(x_0)} H\,. \tag{0.22*}$$

Then, for any $x_1 \in B_{\widehat{R}^*}(x_0)$ *with* $B_{2\Lambda^* R}(x_1) \subset D_\beta$, *if we set*

$$M_R = \sup_{B_R(x_0)} u \qquad \text{and} \qquad m_R = \inf_{B_R(x_0)} u\,,$$

then there hold <u>either</u>

$$u(x_0) - m_R \leqslant \xi_{n,\alpha_*}\,\omega_n (u(x_1) - m_R) + \mathcal{A}_{2,R} \tag{0.31}$$

<u>or</u>

$$M_R - u(x_0) \leqslant \xi_{n,\alpha_*}\,\omega_n (M_R - u(x_1)) + \mathcal{A}_{2,R}\,, \tag{0.32}$$

where

$$\mathcal{A}_{2,R} = \big(2 + \xi_{n,\alpha_*} \Lambda^*_{R,\beta}\big)\, \omega_n R + \xi_{n,\alpha_*} C_{\alpha_*,\beta}\, \omega_n R\,. \tag{0.33}$$

COROLLARY 2 (the third Harnack's inequality). *Let* $u \in C^2(\Omega) \cap \cap W^{1,1}(\Omega)$ *be a solution to* (0.1) *and* (0.2) *in the weak sense in* $B_R(x_0)$. *Suppose that* (0.12) *holds. Furthermore, suppose that* $\widehat{\gamma} R^{n-1}$ *is so small that, for some* $0 < \tau \leqslant \frac{1}{2}$,

$$n\omega_n \widehat{\gamma} R^{n-1} \leqslant \frac{\tau}{\xi_{n,\alpha_*} C_{\alpha_*,\beta}}\,. \tag{0.34}$$

Then, for any $x_1 \in B_{\widehat{R}^*}(x_0)$ *with* $B_{2\Lambda^* R}(x_1) \subset D_\beta$, *there holds* <u>either</u>

$$u(x_0) - m_R \leqslant \frac{1}{(1-2\tau)}\, \xi_{n,\alpha_*}\,\omega_n (u(x_1) - m_R) + \frac{1}{(1-2\tau)}\, \mathcal{A}_{2,R}\,, \tag{0.35}$$

<u>or</u>

$$M_R - u(x_0) \leqslant \frac{1}{(1-2\tau)}\, \xi_{n,\alpha_*}\,\omega_n (M_R - u(x_1)) + \frac{1}{(1-2\tau)}\, \mathcal{A}_{2,R}\,, \tag{0.36}$$

where M_R, m_R *are given in Theorem* 2 *and* $\mathcal{A}_{2,R}$ *is given in* (0.33).

We note that the mean curvature H is not involved in the condition (0.34).

0.3. *Harnack's inequalities for solutions to the mean curvature equation.*

For solutions to the mean curvature equation (0.1) in $B_R(x_0)$, we shall establish in Section 3 Harnack's inequalities in $B_{\lambda R}(x_0)$ in the following formulation in which the constants are completely determined by λ, H and n. We emphasize that *no boundary condition* is involved in these Harnack's inequalities.

THEOREM 3 (the fourth Harnack's inequality). *Let* $u \in C^2(\Omega)$ *be a solution to* (0.1) *in* $B_R(x_0)$. *Suppose that* $(\inf_{x \in B_R(x_0)} H) \geqslant 0$. *For any* λ, $0 < \lambda < 1$, *and for any point* $x_1 \in B_{R_\lambda^*}(x_0)$, $R_\lambda^* = \max\left(\lambda R - 2R^*, \frac{3\lambda R}{4}\right)$, *with* $B_{2\Lambda^* R}(x_1) \subset D_\beta$, *we have*, <u>either</u>

$$(0.37)\ u(x_0) - m_R \leqslant \xi_{n,\alpha_*}(1 + 2C_\lambda C_\lambda^* C_{\varepsilon,\Omega}\,\omega_n)(u(x_1) - m_R) + \mathcal{A}_{2,R},$$

<u>or</u>

$$(0.38)\ M_R - u(x_0) \leqslant \xi_{n,\alpha_*}(1 + 2C_\lambda C_\lambda^* C_{\varepsilon,\Omega}\,\omega_n)(M_R - u(x_1)) + \mathcal{A}_{2,R},$$

where we set

$$(0.39) \qquad C_\lambda = \lambda^{n-1} + \lambda^{n-2} + \ldots + \lambda + 1 = \frac{1-\lambda^n}{1-\lambda},$$

and

$$(0.40) \qquad C_\lambda^* = 1 - (\inf_{x \in B_R(x_0)} H)\, R\left(\frac{n}{n+1}\right)(1-\lambda)\left(1 + \frac{\lambda^n}{C_\lambda}\right) - n(\inf_{x \in B_R(x_0)} H)\, R\,\frac{\lambda^{n+1}}{C_\lambda}.$$

If there holds, for some τ, $0 < \tau \leqslant \frac{1}{2}$,

$$(0.41) \qquad C_\lambda C_\lambda^* \,\omega_n R^{n-1} < \frac{\tau}{\xi_{n,\alpha_*} C_{\varepsilon,\Omega}},$$

then we have <u>either</u>

$$(0.42)\ u(x_0) - m_R \leqslant \frac{1}{(1-2\tau)}\,\xi_{n,\alpha_*}(u(x_1) - m_R) + \frac{1}{(1-2\tau)}\,\mathcal{A}_{2,R},$$

or

$$(0.43)\quad M_R - u(x_0) \leqslant \frac{1}{(1-2\tau)}\, \xi_{n,\alpha_*}(M_R - u(x_1)) + \frac{1}{(1-2\tau)}\, \mathcal{a}_{2,R}.$$

0.4. *Boundary Harnack's inequality.*

Appealing to the following results in [12], Harnack's inequalities for neighborhoods of boundary points can be established by the reasoning in Section 2 and Section 3.1 without essential change. Thus, we formulate the first four Harnack's inequalities w ithout giving a proof. In Section 3.2, we shall briefly indicate the reasoning leading to the fourth boundary Harnack's inequality. A proof of Proposition 4 will be given in Appendix.

PROPOSITION 4. *Let u be a function in $C^2(\Omega) \cap W^{1,1}(\Omega)$ in a domain $\Omega \subset \mathbb{R}^n$ with the subgraph U. Let $U_r(\hat{z})$, $U_r'(\hat{z})$ be as in Theorem* 1. *Suppose the first inequality in* (0.12) *holds and suppose that $\partial\Omega$ is of c lass C^2 with $C_{\varepsilon,\Omega}$ given in* (0.17). *If*

$$|U_r(\hat{z})| > 0 \quad \text{and} \quad |U_r'(\hat{z})| > 0 \quad \text{for all } r > 0\,,$$

then there exist positive constants R_-^{**}, R_+^{**} and α_{**} determined completely by n, $\inf_{\Omega\times\mathbb{R}} H$, $\sup_{\Omega\times\mathbb{R}} H$, $\hat{\gamma}$ and $C_{\varepsilon,\Omega}$ such that

$$|U_r(\hat{z})| > \alpha_{**} r^{n+1} \quad \text{for every} \quad r \leqslant R_-^{**},$$

and

$$|U_r'(\hat{z})| > \alpha_{**} r^{n+1} \quad \text{for every} \quad r \leqslant R_+^{**}.$$

In particular, we can take

$$(0.44)\qquad \alpha_{**} = \frac{1-\hat{\gamma}}{16(n+1)\,k_{(n+1)}}\,,$$

$$(0.45)\quad R_-^{**} = \begin{cases} \dfrac{C_{\hat{\gamma}}}{C_{\varepsilon,\Omega} k_{(n+1)}}\,, & \text{if } \inf_{\Omega\times\mathbb{R}} H(x,t) \geqslant 0,\\[2ex] \min\left(\dfrac{C_{\hat{\gamma}}^-}{C_{\varepsilon,\Omega} k_{(n+1)} (2\omega_n)^{1/(n+1)}}\,, \widetilde{R}_-^{**}\right), & \text{if } \inf_{\Omega\times\mathbb{R}} H(x,t) < 0, \end{cases}$$

and

$$(0.46)\quad R_+^{**} = \begin{cases} \dfrac{C_{\widehat{\gamma}}}{C_{\varepsilon,\,\Omega}\,k_{(n+1)}}, & \text{if } \sup\limits_{\Omega\times\mathbb{R}} H(x,t) \leqslant 0, \\ \min\left(\dfrac{C_{\widehat{\gamma}}^+}{C_{\varepsilon,\,\Omega}(k_{(n+1)})(2\omega_n)^{1/(n+1)}}, \widetilde{R}_+^{**}\right), & \text{if } \sup\limits_{\Omega\times\mathbb{R}} H(x,t) > 0, \end{cases}$$

in which we set

$$C_{\widehat{\gamma}} = \min\left(\frac{1}{2}, \frac{1-\widehat{\gamma}}{3\,\widehat{\gamma}+1}\right),$$

with

$$\widetilde{R}_-^{**} = \left(\frac{1-\widehat{\gamma}}{4n(k_{(n+1)})\,\omega_n\,|\inf\limits_{\Omega\times\mathbb{R}} H|}\right)^{n+1},$$

$$\widetilde{R}_+^{**} = \left(\frac{1-\widehat{\gamma}}{4n(k_{(n+1)})\,\omega_n\,|\sup\limits_{\Omega\times\mathbb{R}} H|}\right)^{n+1},$$

and

$$C_{\widehat{\gamma}}^- = \min\left(\frac{1}{2}, \frac{1-\widehat{\gamma}-2n(k_{(n+1)})\,|\inf\limits_{\Omega\times\mathbb{R}} H|\,\omega_n(\widetilde{R}_-^{**})^n}{3\,\widehat{\gamma}+1}\right),$$

$$C_{\widehat{\gamma}}^+ = \min\left(\frac{1}{2}, \frac{1-\widehat{\gamma}-2n(k_{(n+1)})\,|\sup\limits_{\Omega\times\mathbb{R}} H|\,\omega_n(\widetilde{R}_+^{**})^n}{3\,\widehat{\gamma}+1}\right).$$

THEOREM 4 (the preliminary boundary Harnack's inequality). *Let* $u \in C^2(\Omega)\cap W^{1,1}(\Omega)$ *be a solution to* (0.1) *and* (0.2) *in the weak sense in* Ω.

Let us set α_{**} *as in* (0.44), $R^{**} = \min(R_-^{**}, R_+^{**})$, with R_-^{**} and R_+^{**} given as in (0.45) *and* (0.46) *and set*

$$\widehat{R}^{**} = \max\left(R - 2R^{**}, \frac{3}{4}R\right),$$

and

$$\Lambda^{**}_{R,\beta} = \Lambda^{**} = \min(1, \beta)\max\left(\frac{R^{**}}{R}, \frac{1}{8}\right).$$

If the ball has the radius $R \leqslant R^{**}$, then there exist two positive constants $\xi_{n,\alpha_{**}}$ and $C_{\alpha_{**},\beta}$ determined completely by $\alpha_{**}, \beta, \widehat{\gamma}$ and n such that, for any $x_1 \in B_{\widehat{R}^{**}}(x_0) \cap \Omega$ with $B_{2\Lambda^{**}R}(x_1) \cap \Omega \subset D_\beta$, we have

$$u(x_0) - m_{\Omega_0} \leqslant \xi_{n,\alpha_{**}} \omega_n (u(x_1) - m_{\Omega_0}) + \\ + (2 + C_{\alpha_{**},\beta}\xi_{n,\alpha_{**}})\omega_n R + \xi_{n,\alpha_{**}} C_{\alpha_{**},\beta} R^{n-1} \int_{\Omega_0} |Du|\,dx,$$

and

$$M_{\Omega_0} - u(x_0) \leqslant \xi_{n,\alpha_{**}} \omega_n (M_{\Omega_0} - u(x_1)) + \\ + (2 + C_{\alpha_{**},\beta}\xi_{n,\alpha_{**}})\omega_n R + \xi_{n,\alpha_{**}} C_{\alpha_{**},\beta} R^{n-1} \int_{\Omega_0} |Du|\,dx,$$

where we set $M_{\Omega_0} = \sup_{\Omega_0} u$ and $m_{\Omega_0} = \inf_{\Omega_0} u$, for any subset Ω_0 of Ω with $(B_R(x_0) \cap \overline{\Omega}) \subseteq \overline{\Omega}_0 \subseteq \overline{\Omega}$. In fact, we are allowed to take

$$\xi_{n,\alpha_{**}} = \frac{2^{n+2}}{\alpha_{**}},$$

and

$$C_{\alpha_{**},\beta} = 2^{n+1+\frac{1}{n}} \alpha_{**} \left(\frac{\omega_n}{\alpha_{**}}\right)^{\frac{1}{n}}.$$

THEOREM 5 (the first boundary Harnack's inequality). *Let $u \in C^2(\Omega) \cap W^{1,1}(\Omega)$ be a solution to* (0.1) *and* (0.2) *in the weak sense in Ω. Suppose $\partial\Omega \in C^2$ whose principal curvatures are bounded in absolute value by $\mathcal{K}_{\partial\Omega}$. For $x_0 \in \partial\Omega$, suppose that the first inequality in* (0.12) *holds for some number $\widehat{\gamma}$, $0 < \widehat{\gamma} < 1$, in $\partial\Omega \cap B_R(x_0)$ and $(\inf_{x \in (B_{\widehat{R}}(x_0) \cap \Omega)} H) \geqslant 0$. Suppose that* (0.22) *holds. Suppose further that*

$$Tu \cdot \nu_R \leqslant 0, \tag{0.47}$$

throughout $\partial B_{2\Lambda^{**}R}(x_1) \cap \Omega \cap \Omega$, where ν_R is the outward unit normal with respect to $B_{2\Lambda^{**}R}(x_1) \cap \Omega \cap \Omega$. Let us set $\mathcal{A}_1$ and $\mathcal{A}_2$ as in Theorem 2.

Then, for any $x_1 \in B_{\widehat{R}^{**}}(x_0) \cap \Omega$ *with* $B_{2\Lambda^{**}R}(x_1) \cap \Omega \subset D_\beta$, *we have*

$$u(x_0) - m_{\Omega_0} \leqslant \frac{\xi_{n,\alpha_{**}} \omega_n}{1-\widehat{\gamma}} (u(x_1) - m_{\Omega_0}) + \mathcal{C}_1,$$

and

$$M_{\Omega_0} - u(x_0) \leqslant \frac{\xi_{n,\alpha_{**}} \omega_n}{1-\widehat{\gamma}} (M_{\Omega_0} - u(x_1)) + \mathcal{C}_1,$$

for any subset Ω_0 *of* Ω *with* $(B_R(x_0) \cap \overline{\Omega}) \subseteq \overline{\Omega}_0 \subseteq \overline{\Omega}$. *Furthermore, for any* $x_1 \in B_{\widehat{R}_0^{**}}(x_0) \cap \Omega$, $\widehat{R}_0^{**} = \max\left(\frac{R}{2} - 2R_{**}, \frac{3}{8}R\right)$, *with* $B_{2\Lambda^{**}R}(x_1) \cap \cap \Omega \subset D_\beta$, *we have*

$$u(x_0) - m_{\Omega_0} \leqslant \xi_{n,\alpha_{**}} \omega_n (u(x_1) - m_{\Omega_0}) + \mathcal{C}_2,$$

$$M_{\Omega_0} - u(x_0) \leqslant \xi_{n,\alpha_{**}} \omega_n (M_{\Omega_0} - u(x_1)) + \mathcal{C}_2,$$

and for any $x_1, x_2 \in B_{\widehat{R}_1^{**}}(x_0) \cap \Omega$, $\widehat{R}_1^{**} = \max\left(\frac{R}{4} - 2R^{**}, \frac{3}{16}R\right)$, *with* $B_{2\Lambda^{**}R}(x_1) \cap \Omega \subset D_\beta$, *we have*

$$u(x_1) - m_{\Omega_0} \leqslant \xi_{n,\alpha_{**}} \omega_n (u(x_2) - m_{\Omega_0}) +$$
$$(1 + \xi_{n,\alpha_{**}} \Lambda^{**})\, \omega_n \operatorname{dist}(x_1, x_2) + \xi_{n,\alpha_{**}} C_{\alpha_{**},\beta} \omega_n \operatorname{dist}(x_1, x_2),$$

$$M_{\Omega_0} - u(x_1) \leqslant \xi_{n,\alpha_{**}} \omega_n (M_{\Omega_0} - u(x_2)) +$$
$$+ (1 + \xi_{n,\alpha_{**}} \Lambda^{**}) \omega_n \operatorname{dist}(x_1, x_2) + \xi_{n,\alpha_{**}} C_{\alpha_{**},\beta} \omega_n \operatorname{dist}(x_1, x_2).$$

COROLLARY 3 (the second boundary Harnack's inequality). *Let* $u \in \in C^2(\Omega) \cap W^{1,1}(\Omega)$ *be a solution to* (0.1) *and* (0.2) *in* $B_{\tilde{R}}(x_0)$. *For* $x_0 \in \in \partial B_{\tilde{R}}(x_0)$, *suppose that the first inequality in* (0.12) *holds in* $\partial B_{\tilde{R}}(x_0) \cap \cap B_R(x_0)$ *and* $(\inf_{x \in (B_{\tilde{R}}(x_0) \cap B_R(x_0))} H) \geqslant 0$. *Suppose that* (0.22) *holds. Suppose further that* (0.47) *holds along* $\partial B_R(x_0) \cap B_{\tilde{R}}(x_0)$. *Then, for any* $x_1 \in \in B_{\widehat{R}^{**}}(x_0) \cap B_{\tilde{R}}(x_0)$ *with* $B_{2\Lambda^{**}R}(x_1) \cap B_{\tilde{R}}(x_0) \subset D_\beta$, *if we set*

$$\widetilde{M}_R = \sup_{B_R(x_0) \cap B_{\tilde{R}}(x_0)} u \quad \textit{and} \quad \widetilde{m}_R = \inf_{B_R(x_0) \cap B_{\tilde{R}}(x_0)} u\,,$$

then there hold <u>either</u>

$$u(x_0) - \widetilde{m}_R \leqslant \xi_{n,\alpha_{**}} \omega_n (u(x_1) - \widetilde{m}_R) + \mathcal{C}_{2,R}$$

or

$$\widetilde{M}_R - u(x_0) \leqslant \xi_{n,\alpha_{**}}\omega_n(\widetilde{M}_R - u(x_1)) + \mathcal{a}_{2,R}$$

where $\mathcal{a}_{2,R}$ *is given in* (0.33).

COROLLARY 4 (the third boundary Harnack's inequality). *Let* $u \in \in C^2(\Omega) \cap W^{1,1}(\Omega)$ *be a solution to* (0.1) *and* (0.2) *in the weak sense in* $B_{\tilde{R}}(x_0)$. *For* $x_0 \in \partial B_{\tilde{R}}(x_0)$, *suppose that the first inequality in* (0.12) *holds in* $\partial B_{\tilde{R}}(x_0) \cap B_R(x_0)$ *and* $(\inf_{x \in (B_{\tilde{R}}(x_0) \cap B_R(x_0))} H) \geqslant 0$. *Suppose* (0.34) *holds for some* τ, $0 < \tau \leqslant \frac{1}{2}$.

(1) *Suppose further that* (0.47) *holds along* $\partial B_R(x_0) \cap B_{\tilde{R}}(x_0)$. *Then, for any* $x_1 \in B_{\widehat{R}^{**}}(x_0) \cap B_{\tilde{R}}(x_0)$ *with* $B_{2\Lambda^{**}R}(x_1) \cap B_{\tilde{R}}(x_0) \subset D_\beta$, *there holds* either

$$u(x_0) - \widetilde{m}_R \leqslant \frac{1}{(1-2\tau)}\xi_{n,\alpha_{**}}\omega_n(u(x_1) - \widetilde{m}_R) + \frac{1}{(1-2\tau)}\mathcal{a}_{2,R},$$

or

$$\widetilde{M}_R - u(x_0) \leqslant \frac{1}{(1-2\tau)}\xi_{n,\alpha_{**}}\omega_n(\widetilde{M}_R - u(x_1)) + \frac{1}{(1-2\tau)}\mathcal{a}_{2,R}.$$

(2) *If* (0.47) *fails to hold throughout* $\partial B_{2\Lambda^* R}(x_1) \cap \Omega$, *but if*

$$n\omega_n \widehat{\gamma} R^{n-1} + n\omega_n \widetilde{R}^{n-1} < \frac{\tau}{\xi_{n,\alpha_*} C_{\alpha_*,\beta}},$$

for some τ, $0 < \tau < \frac{1}{2}$, *then we have*

$$u(x_0) - \widetilde{m}_R \leqslant \frac{1}{(1-2\tau)}\xi_{n,\alpha_{**}}\omega_n(u(x_1) - \widetilde{m}_R) + \frac{1}{(1-2\tau)}\mathcal{a}_{2,R},$$

and

$$\widetilde{M}_R - u(x_0) \leqslant \frac{1}{(1-2\tau)}\xi_{n,\alpha_{**}}\omega_n(\widetilde{M}_R - u(x_1)) + \frac{1}{(1-2\tau)}\mathcal{a}_{2,R}.$$

THEOREM 6 (the fourth boundary Harnack's inequality). *Let* $u \in \in C^2(\Omega)$ *be a solution to* (0.1) *in* $B_{\tilde{R}}(x_0)$. *For* $x_0 \in \partial B_{\tilde{R}}(x_0)$, *suppose that* $(\inf_{x \in (B_{\tilde{R}}(x_0) \cap B_R(x_0))} H) \geqslant 0$. *For any* λ, $0 < \lambda < 1$, *and for any point* $x_1 \in$

$\in B_{R_\lambda^{**}}(x_0) \cap B_{\tilde{R}}(x_0)$, $R_\lambda^{**} = \max\left(\lambda R - 2R^{**}, \frac{3\lambda R}{4}\right)$, *with* $B_{2\Lambda^{**}R}(x_1) \cap$ $\cap B_{\tilde{R}}(x_0) \subset D_\beta$, *we have,* <u>either</u>

$$u(x_0) - \widetilde{m}_R \leqslant \xi_{n,\alpha_*}(1 + 2C_\lambda C_{\varepsilon,\Omega}\omega_n)(u(x_1) - \widetilde{m}_R) + \mathcal{C}_{2,R}, \tag{0.48}$$

<u>or</u>

$$\widetilde{M}_R - u(x_0) \leqslant \xi_{n,\alpha_*}(1 + 2C_\lambda C_{\varepsilon,\Omega}\omega_n)(\widetilde{M}_R - u(x_1)) + \mathcal{C}_{2,R}. \tag{0.49}$$

If there holds, for some τ, $\tau \leqslant \frac{1}{2}$,

$$C_\lambda \omega_n R^{n-1} < \frac{\tau}{\xi_{n,\alpha_*} C_{\varepsilon,\Omega}}, \tag{0.50}$$

then we have <u>either</u>

$$u(x_0) - \widetilde{m}_R \leqslant \frac{1}{(1-2\tau)}\xi_{n,\alpha_*}(u(x_1) - \widetilde{m}_R) + \frac{1}{(1-2\tau)}\mathcal{C}_{2,R}, \tag{0.51}$$

<u>or</u>

$$\widetilde{M}_R - u(x_0) \leqslant \frac{1}{(1-2\tau)}\xi_{n,\alpha_*}(\widetilde{M}_R - u(x_1)) + \frac{1}{(1-2\tau)}\mathcal{C}_{2,R}. \tag{0.52}$$

1. Proof of the preliminary Harnack's inequality.

In this section, we shall prove the Preliminary Harnack's Inequality (Proposition 1), adapting the reasoning on pages 312-313 of Giusti [12], together with an application of Proposition 2. The reasoning suggested by Giusti [12] enables us to estimate the l eft hand side of (0.5) and (0.6) in terms of the L^1-norm of u and a subsequent application of Proposition 2 yields this estimate in terms of the L^1-norm of $|Du|$.

1.1. Suppose that $u \in C^2(\Omega)$ satisfies (0.3) and (0.4). For any domain Ω_0 such that $B_R(x_0) \subset \Omega_0 \subset \Omega$, let us set

$$M_{\Omega_0} = \sup_{\Omega_0} u, \qquad \text{and} \qquad m_{\Omega_0} = \inf_{\Omega_0} u\,.$$

Let

$$z_j = (x_0, m_{\Omega_0} + 2jR),$$

for $j \in \mathbb{N}$. Then

$$z_j \in U,$$

for

$$j \leqslant j_1 = \left[\frac{u(x_0) - m_{\Omega_0}}{2R} \right],$$

where $[s]$ denotes the largest integer less than s for $s > 0$. From (0.3), we have

$$|U_{R/2}(z_j)| \geqslant \alpha_* \left(\frac{R}{2} \right)^{n+1},$$

for $1 \leqslant j \leqslant j_1$ and therefore

$$\int_{B_R(x_0)} (u - m_{\Omega_0})\, dx \geqslant \sum_{j=1}^{j_1} |U_{R/2}(z_j)| \geqslant j_1 \alpha_* \left(\frac{R}{2} \right)^{n+1}.$$

Hence

$$\begin{aligned} M_{\Omega_0} &= u(x_0) + (M_{\Omega_0} - u(x_0)) \\ &\leqslant 2(j_1 + 1) R + m_{\Omega_0} + (M_{\Omega_0} - u(x_0)) \\ &\leqslant \frac{2^{n+2}}{\alpha_* R^n} \int_{B_R(x_0)} (u - m_{\Omega_0})\, dx + 2R + m_{\Omega_0} + (M_{\Omega_0} - u(x_0)); \end{aligned}$$

that is,

$$u(x_0) - m_{\Omega_0} \leqslant \frac{2^{n+2}}{\alpha_* R^n} \int_{B_R(x_0)} (u - m_{\Omega_0})\, dx + 2R. \tag{1.1}$$

To estimate the integral on the right hand side of (1.1) under the hypotheses that $x_1 \in B_{\widehat{R}^*}(x_0)$ and $B_{2\Lambda^* R}(x_1) \subset D_\beta$, let $\widehat{x}_1 \in \partial B_{2\Lambda^* R}(x_1)$ be a point at which

$$u(\widehat{x}_1) - u(x_1) \geqslant 2\Lambda^*_{R,\beta} R\beta.$$

Let

$$z_1 = (x_1, u(x_1))$$

and

$$\widehat{z}_1 = (\widehat{x}_1,\, u(\widehat{x}_1))\,.$$

From (0.3) and (0.4), we have

$$|\, U'_{\Lambda^*_{R,\beta} R}(z_1)\,| \geqslant \alpha_*(\Lambda^*_{R,\beta} R)^{n+1},$$

and

$$|\, U_{\Lambda^*_{R,\beta} R}(\widehat{z}_1)\,| \geqslant \alpha_*(\Lambda^*_{R,\beta} R)^{n+1}.$$

These yield

$$\{x : x \in B_R(x_0),\, u(x) \leqslant u(x_1) + \Lambda^*_{R,\beta} R\} \geqslant \alpha_*(\Lambda^*_{R,\beta} R)^n,$$

and

$$\{x : x \in B_R(x_0),\, u(x) \geqslant u(x_1) + \Lambda^*_{R,\beta} R\} \geqslant \alpha_*(\Lambda^*_{R,\beta} R)^n.$$

Hence, by Proposition 2, we have

$$\int_{B_R(x_0)} (u - m_{\Omega_0})\, dx \leqslant \tag{1.2}$$

$$\leqslant C_{\alpha_*,\beta} R \int_{B_R(x_0)} |Du|\, dx + \big((u(x_1) - m_{\Omega_0}) + \Lambda^*_{R,\beta} R\big)\, |B_R(x_0)\,|\,,$$

with $C_{\alpha_*,\beta}$ being as in (0.8) by setting $\alpha_1 = \alpha_2 = \dfrac{\alpha_*(\Lambda^*_{R,\beta})^n}{\omega_n}$ in (0.9).

Inserting this into (1.1), we obtain (0.5) with the value ξ_{n,α_*} given in (0.7).

1.2. Analogously, we let

$$z_j^+ = (x_0,\, M_{\Omega_0} - 2Rj)\,,$$

for $j \in \mathbb{N}$. Then

$$z_j^+ \in U' = Q \backslash U$$

for

$$j \leqslant j_1^+ = \left[\frac{M_{\Omega_0} - u(x_0)}{2R}\right].$$

We obtain from (0.4) that

$$|U'_{R/2}(z_j^+)| \geqslant \alpha_*\left(\frac{R}{2}\right)^{n+1},$$

and therefore

$$\int\limits_{B_R(x_0)} (M_{\Omega_0} - u)\, dx \geqslant \sum_{j=1}^{j_1^+} |U'_{R/2}(z_j^+)| \geqslant j_1^+ \alpha_*\left(\frac{R}{2}\right)^{n+1},$$

which yields

$$\begin{aligned} -m_{\Omega_0} &= u(x_0) + (u(x_0) - m_{\Omega_0}) \\ &\leqslant -M_{\Omega_0} + 2(j_1^+ + 1)\, R + (u(x_0) - m_{\Omega_0}) \\ &\leqslant \frac{2^{n+2}}{\alpha_* R^n} \int\limits_{B_R(x_0)} (M_{\Omega_0} - u)\, dx + 2R - M_{\Omega_0} + (u(x_0) - m_{\Omega_0}). \end{aligned}$$

That is,

$$M_{\Omega_0} - u(x_0) \leqslant \frac{2^{n+2}}{\alpha_* R^n} \int\limits_{B_R(x_0)} (M_{\Omega_0} - u)\, dx + 2R. \tag{1.4}$$

Under the hypotheses that $x_1 \in B_{\widehat{R}^*}(x_0)$ and $B_{2\Lambda^* R}(x_1) \subset D_\beta$, let $x_1^+ \in \partial B_{2\Lambda^* R}(x_1)$ be a point at which

$$u(x_1) - u(\widehat{x}_1^+) \geqslant 2\Lambda^*_{R,\beta} R\beta,$$

where the point $x_1 \in B_{\widehat{R}^*}(x_0)$ is chosen as in **1.1**. Then setting

$$\widehat{z}_1^+ = (x_1^+, u(\widehat{x}_1^+)),$$

we obtain from (0.3) and (0.4) that

$$|U_{\Lambda^*_{R,\beta} R}(z_1)| \geqslant \alpha_*(\Lambda^*_{R,\beta} R)^{n+1}$$

and

$$|U'_{\Lambda^*_{R,\beta} R}(\widehat{z}_1^+)| \geqslant \alpha_*(\Lambda^*_{R,\beta} R)^{n+1},$$

which and Proposition 2 yield

$$\int_{B_R(x_0)} (M_{\Omega_0} - u)\, dx \leqslant C_{\alpha *, \beta} R \int_{B_R(x_0)} |Du|\, dx +$$

$$+ \big((M_{\Omega_0} - u(x_1)) + \Lambda^*_{R,\beta} R\big) |B_R(x_0)| \,.$$

This and (1.4) yield (0.6).

2. Proof of Harnack's inequalities for solutions to the capillarity problem.

2.1. *Proof of theorem* 2.

Setting $\varepsilon_0 = \dfrac{1}{2\,\mathcal{K}_{\partial\Omega}}$, we obtain from (0.21) that

$$\left| \int_{\partial\Omega} \cos\theta\eta \, d\mathcal{H}_{n-1} \right| \leqslant \widehat{\gamma} \int_{\Sigma_{\varepsilon 0}} |D\eta|\, dx + 2\,\widehat{\gamma}\,\mathcal{K}_{\partial\Omega}(n+\delta) \int_{\Sigma_{\varepsilon 0}} |\eta|\, dx \,,$$

for each δ, $0 < \delta \leqslant 1$ and for each $\eta \in C^1(\overline{\Omega})$. By this and (0.13), if u is a solution in (0.1) and (0.2) in the weak sense, we obtain

$$\int_{\Omega} \frac{|Du|}{\sqrt{1+|Du|^2}} \cdot D\eta \, dx \leqslant \widehat{\gamma} \int_{\Sigma_{\varepsilon 0}} |D\eta|\, dx + \tag{2.1}$$

$$+ 2n\widehat{\gamma}\,\mathcal{K}_{\partial\Omega} \int_{\Sigma_{\varepsilon 0}} |\eta|\, dx - n \int_{\Omega} H\eta\, dx \,,$$

for each $\eta \in C^1(\overline{\Omega})$. Taking $\eta = u - m_{\Omega_0} \geqslant 0$, we obtain from

$$\frac{|Du|}{\sqrt{1+|Du|^2}} = \sqrt{1+|Du|^2} - \frac{1}{\sqrt{1+|Du|^2}} > |Du| - 1$$

and (2.1) that

$$\int_{\Omega\setminus\Sigma_{\varepsilon 0}} |Du|\, dx + (1-\widehat{\gamma}) \int_{\Sigma_{\varepsilon 0}} |Du|\, dx < \tag{2.2}$$

$$< |\Omega| + 2n\widehat{\gamma}\,\mathcal{K}_{\partial\Omega} \int_{\Sigma_{\varepsilon 0}} (u - m_{\Omega_0})\, dx - n \int_{\Omega} H(u - m_{\Omega_0})\, dx \,.$$

Taking $\eta = M_{\Omega_0} - u$ in (2.1) instead, we obtain

$$\int_{\Omega \setminus \Sigma_{\varepsilon 0}} |Du|\,dx + (1-\widehat{\gamma}) \int_{\Sigma_{\varepsilon 0}} |Du|\,dx < \tag{2.3}$$

$$< |\Omega| + 2n\widehat{\gamma}\,\mathcal{K}_{\partial\Omega} \int_{\Sigma_{\varepsilon 0}} (M_{\Omega_0} - u)\,dx - n \int_{\Omega} H(M_{\Omega_0} - u)\,dx\,.$$

In case (0.22) holds, we obtain from (2.2) and (2.3) that

$$\int_{\Omega \setminus \Sigma_{\varepsilon 0}} |Du|\,dx < |\Omega|\,. \tag{2.4}$$

and

$$\int_{\Omega} |Du|\,dx < \frac{1}{1-\widehat{\gamma}}\,|\Omega|\,. \tag{2.5}$$

Inserting (2.5) into (0.5) and (0.6), we obtain respectively (0.23) and (0.24). By using (2.4) instead of (2.5) and replacing R by $R/2$ in (0.5) and (0.6), we obtain (0.25), (0.26), (0.27) and (0.28).

2.2. *Proof of corollary* 1.

Suppose that u is a solution to (0.1) and (0.2) in the weak sense in $B_R(x_0)$ and that (0.22*) holds . We insert the inequality (0.30) into (0.13) and take $\eta = u - m_R$ and $\eta = M_R - u$ in (0.12) to obtain, respecti vely

$$\int_{B_R(x_0)} |Du|\,dx \leqslant |B_R(x_0)| + \tag{2.6}$$

$$+ n\omega_n \widehat{\gamma} R^{n-1}(M_R - m_R) - n \int_{B_R(x_0)} H(u - m_R)\,dx\,,$$

and

$$\int_{B_R(x_0)} |Du|\,dx \leqslant |B_R(x_0)| + \tag{2.7}$$

$$+ n\omega_n \widehat{\gamma} R^{n-1}(M_R - m_R) - n \int_{B_R(x_0)} H(M_R - u)\,dx\,.$$

Since $M_R - m_R = (M_R - u(x_0)) + (u(x_0) - m_R)$, we have <u>either</u>

$$\int_{B_R(x_0)} H(u - m_R)\, dx \geqslant \frac{1}{2}(\inf_{x \in B_R(x_0)} H)(M_R - m_R)\, |B_R(x_0)|\,,$$

<u>or</u>

$$\int_{B_R(x_0)} H(M_R - u)\, dx \geqslant \frac{1}{2}(\inf_{x \in b_R(x_0)} H)(M_R - m_R)\, |B_R(x_0)|\,.$$

Inserting these and (0.22*) into (2.6) and (2.7) yields estimates of $\int_{B_R(x_0)} |Du|\, dx$, which we subsequently insert into <u>either</u> (0.5) <u>or</u> (0.6) to establish Corollary 1.

2.3. *Proof of corollary* 2.

Suppose that u is a solution to (0.1) and (0.2) in the weak sense in $B_R(x_0)$ and that (0.34) holds. We obtain from (2.6), (0.12) and (0.34)

$$\int_{B_R(x_0)} |Du|\, dx \leqslant |B_R(x_0)| + \frac{\tau(M_R - m_R)}{\xi_{n,\alpha_*} C_{\alpha_*,\beta}}\,.$$

Inserting this into (0.5) and (0.6), we obtain respectively

$$u(x_0) - m_R \leqslant \tau(M_R - m_R) + \xi_{n,\alpha_*} \omega_n (u(x_1) - m_R) + \mathcal{A}_{2,R} \tag{2.8}$$

and

$$M_R - u(x_0) \leqslant \tau(M_R - m_R) + \xi_{n,\alpha_*} \omega_n (M_R - u(x_1)) + \mathcal{A}_{2,R}, \tag{2.9}$$

where $\mathcal{A}_{2,R}$ is given in (0.33). If $u(x_0) - m_R \leqslant M_R - u(x_0)$, we have $M_R - m_R \leqslant 2(u(x_0) - m_R)$, which and (2.8) yield (0.35). If $M_R - u(x_0) \leqslant u(x_0) - m_R$, then we have $M_R - m_R \leqslant 2(M_R - u(x_0))$, which and (2.9) yield (0.36).

3. Proof of Harnack's inequality for solutions to the mean curvature equation and proof of a boundary Harnack's inequality.

Choose $\eta_\lambda = \eta_\lambda(\varrho) \in C^1(B_R(x_0))$, $\varrho = \operatorname{dist}(x_0, x_1)$, with

$$\left\{\begin{array}{ll} 0 \leqslant \eta_\lambda \leqslant 1, & \\ \eta_\lambda = 1 & \text{in } B_{\lambda R}(x_0), \\ \eta_\lambda = 0 & \text{on } \partial B_R(x_0), \end{array}\right. \tag{3.1}$$

for some λ, $0 < \lambda < 1$, such that

$$\frac{1}{(1-\lambda)R} \leqslant |D\eta_\lambda(\varrho)| \leqslant \frac{1+\delta_0}{(1-\lambda)R}, \tag{3.2}$$

for some δ_0, $0 < \delta_0 < 1$ and for $\lambda R \leqslant \varrho \leqslant R$. Thus

$$1 - \frac{\varrho}{R} \leqslant \eta_\lambda(\varrho) \leqslant 1 - \frac{(1+\delta_0)\varrho}{R}, \tag{3.3}$$

for $\lambda \leqslant \varrho \leqslant R$.

Suppsose that $u \in C^2(\Omega)$ is a solution to (0.1) in Ω. Taking $\eta = \eta_\lambda(u - m_R)$ and $\eta = \eta_\lambda(M_R - u)$ in (0.12), we obtain

$$\int_{B_R(x_0)} \frac{|Du|}{\sqrt{1+|Du|^2}} \cdot D\eta\, dx + n \int_{B_R(x_0)} H\eta\, dx = 0,$$

which yields

$$\begin{aligned} &\int_{B_{\lambda R}(x_0)} \frac{|Du|^2}{\sqrt{1+|Du|^2}}\, dx \leqslant \\ &\leqslant \int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(u - m_R)\, dx - n \int_{B_R(x_0)} H\eta_\lambda(u - m_R)\, dx = \\ &= \int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(u - m_R)\, dx - n \int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} H\eta_\lambda(u - m_R)\, dx - \\ &\qquad - n \int_{B_{\lambda R}(x_1)} H(u - m_R)\, dx, \end{aligned} \tag{3.4}$$

and

$$\text{(3.5)} \qquad \int_{B_{\lambda R}(x_0)} \frac{|Du|^2}{\sqrt{1+|Du|^2}}\,dx \leqslant$$

$$\leqslant \int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(M_R-u)\,dx - n\int_{B_{\lambda R}(x_0)} H\eta_\lambda(M_R-u)\,dx =$$

$$= \int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(M_R-u)\,dx - n\int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} H\eta_\lambda(M_R-u)\,dx -$$

$$-n\int_{B_{\lambda R}(x_0)} H(M_R-u)\,dx\,.$$

By (3.1) and (3.2), we have

$$\int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(M_R-m_R)\,dx \leqslant n\omega_n(M_R-m_R)\left(\frac{1+\delta_0}{(1-\lambda)R}\right)\int_{\lambda R}^{R} \varrho^{n-1}\,d\varrho =$$

$$= \left(\frac{1+\delta_0}{1-\lambda}\right)(1-\lambda^n)(M_R-m_R)\,\omega_n R^{n-1}.$$

Since δ_0 can be arbitrarily small, we have

$$\text{(3.6)} \qquad \int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(M_R-m_R)\,dx \leqslant \left(\frac{1-\lambda^n}{1-\lambda}\right)(M_R-m_R)\,\omega_n R^{n-1}.$$

3.1. *Proof of theorem* 3.

We also have

$$n\int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} H\eta_\lambda(M_R-m_R)\,dx \geqslant$$

$$\geqslant n^2\big(\inf_{x\in B_R(x_0)} H\big)\,\omega_n(M_R-m_R)\left[\int_{\lambda R}^{R}\left(1-\frac{(1+\delta_0)\,\varrho}{R}\right)\varrho^{n-1}\,d\varrho\right]$$

$$\geqslant n^2\big(\inf_{x\in B_R(x_0)} H\big)(M_R-m_R)\left(\frac{1-\lambda^n}{n}-\frac{1-\lambda^{n+1}}{n+1}\right)\omega_n R^n.$$

Since δ_0 can be arbitrarily small, we obtain

$$(3.7)\qquad n\int\limits_{B_R(x_0)\setminus B_{\lambda R}(x_0)} H\eta_\lambda (M_R-m_R)\,dx \geqslant$$

$$\geqslant n^2(\inf_{x\in B_R(x_0)} H)(M_R-m_R)\left(\frac{1-\lambda^n}{n}-\frac{1-\lambda^{n+1}}{n+1}\right)\omega_n R^{n-1}.$$

Moreover, we have

$$(3.8)\qquad n\int\limits_{B_{\lambda R}(x_0)} H(M_R-m_R)\,dx \geqslant n(\inf_{x\in B_R(x_0)} H)(M_R-m_R)\,\lambda^n\omega_n R^n.$$

From (3.6), (3.7),(3.8) and (0.39), we obtain

$$\int\limits_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(M_R-m_R)\,dx -$$

$$-n\int\limits_{B_R(x_0)\setminus B_{\lambda R}(x_0)} H\eta_\lambda (M_R-m_R)\,dx - n\int\limits_{B_{\lambda R}(x_0)} H(M_R-m_R)\,dx \leqslant$$

$$\leqslant (M_R-m_R)\,\omega_n R^{n-1}\cdot$$

$$\cdot\left[\left(\frac{1-\lambda^n}{1-\lambda}\right)-n^2(\inf_{x\in B_R(x_0)} H)\,R\left(\frac{1-\lambda^n}{n}-\frac{1-\lambda^{n+1}}{n+1}+\lambda^n\right)\right]=$$

$$=(M_R-m_R)\,\omega_n R^{n-1}\cdot$$

$$\cdot\left(\frac{1-\lambda^n}{1-\lambda}\right)\left[1-n(\inf_{x\in B_R(x_0)} H)\,R(1-\lambda)\left(1-\frac{n}{n+1}\,\frac{1-\lambda^{n+1}}{1-\lambda^n}+\frac{\lambda^n}{1-\lambda^n}\right)\right]=$$

$$=(M_R-m_R)\,\omega_n R^{n-1}C_\lambda\cdot$$

$$\cdot\left[1-(\inf_{x\in B_R(x_0)} H)\,R\left(\frac{n}{n+1}\right)(1-\lambda)\left(1-\frac{n\lambda^n}{C_\lambda}\right)-(\inf_{x\in B_R(x_0)} H)\,R\left(\frac{n\lambda^n}{C_\lambda}\right)\right]=$$

$$=(M_R-m_R)\,\omega_n R^{n-1}C_\lambda\cdot$$

$$\cdot\left[1-(\inf_{x\in B_R(x_0)} H)R\left(\frac{n}{n+1}\right)(1-\lambda)\left(1+\frac{\lambda^n}{C_\lambda}\right)-n(\inf\nolimits_{x\in B_R(x_0)} H)R\frac{\lambda^{n+1}}{C_\lambda}\right].$$

Since there holds either $M_R - m_R \leqslant 2(u(x_1) - m_R)$ or $M_R - m_R \leqslant$ $\leqslant 2(M_R - u(x_1))$, we have either

$$\int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(u - m_R)\,dx -$$

$$- n \int_{B_R(x_0)} H\eta_\lambda(u - m_R)\,dx \leqslant 2(u(x_1) - m_R)\,C_\lambda C_\lambda^* \omega_n R^{n-1},$$

or

$$\int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(M_R - u)\,dx -$$

$$- n \int_{B_R(x_0)} H\eta_\lambda(M_R - u)\,dx \leqslant 2(M_R - u(x_1))\,C_\lambda C_\lambda^* \omega_n R^{n-1}.$$

where C_λ and C_λ^* are given respectively in (0.39) and (0.40). Inserting these into (3.4) or (3.5), and subsequently insert what results in into (0.5) or (0.6), we obtain (0.37) and (0.38).

Since there also holds either $M_R - m_R \leqslant 2(u(x_0) - m_R)$ or $M_R -$ $- m_R \leqslant 2(M_R - u(x_0))$, we have, either

$$\int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(u - m_R)\,dx -$$

$$- n \int_{B_R(x_0)} H\eta_\lambda(u - m_R)\,dx \leqslant 2(u(x_0) - m_R)\,C_\lambda C_\lambda^* \omega_n R^{n-1},$$

or

$$\int_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(M_R - u)\,dx -$$

$$- n \int_{B_R(x_0)} H\eta_\lambda(M_R - u)\,dx \leqslant 2(M_R - u(x_0))\,C_\lambda C_\lambda^* \omega_n R^{n-1}.$$

Inserting these into (3.4) or (3.5), and subsequently insert what results in into (0.5) or (0.6), we obtain (0.42) and (0.43) under the hypothesis of (0.41).

3.2. *Proof of theorem* 6.

Since there holds either $M_R - m_R \leqslant 2(u(x_1) - m_R)$ or $M_R - m_R \leqslant 2(M_R - u(x_1))$, we obtain from (3.6) that either

$$\int\limits_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(u(x_0) - m_R)\,dx \leqslant$$

$$\leqslant 2\left(\frac{1-\lambda^n}{1-\lambda}\right)(u(x_1) - m_R)\,\omega_n R^{n-1} = 2C_\lambda(u(x_1) - m_R)\,\omega_n R^{n-1},$$

or

$$\int\limits_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(M_R - u(x_0))\,dx \leqslant$$

$$\leqslant 2\left(\frac{1-\lambda^n}{1-\lambda}\right)(u(x_1) - m_R)\,\omega_n R^{n-1} = 2C_\lambda(M_R - u(x_1))\,\omega_n R^{n-1},$$

Inserting these into (3.4) or (3.5), and subsequently insert what results in into (0.5) or (0.6), we obtain (0.48) and (0.49).

Since there also holds either $M_R - m_R \leqslant 2(u(x_0) - m_R)$ or $M_R - m_R \leqslant 2(M_R - u(x_0))$, we have, either

$$\int\limits_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(u(x_0) - m_R)\,dx \leqslant$$

$$\leqslant 2\left(\frac{1-\lambda^n}{1-\lambda}\right)(u(x_0) - m_R)\,\omega_n R^{n-1} = 2C_\lambda(u(x_0) - m_R)\,\omega_n R^{n-1},$$

or

$$\int\limits_{B_R(x_0)\setminus B_{\lambda R}(x_0)} |D\eta_\lambda|(M_R - u(x_0))\,dx \leqslant$$

$$\leqslant 2\left(\frac{1-\lambda^n}{1-\lambda}\right)(M_R - u(x_0))\,\omega_n R^{n-1} = 2C_\lambda(M_R - u(x_0))\,\omega_n R^{n-1}.$$

Inserting these into (3.4) or (3.5), and subsequently insert what results in into (0.5) or (0.6), we obtain (0.51) and (0.52) under the hypothesis of (0.50).

Appendix. Proof of Proposition 3 and Proposition 4..

The equation (0.1) is the Euler equation of the functional

$$\mathcal{F}_*(v) = \int_\Omega \sqrt{1 + |Dv|^2}dx + n\int_\Omega \int_0^v H(x, t)\, dt\, dx\,.$$

And corresponding to the Dirichlet problem with boundary data ψ and the capillarity problem with boundary contact angle θ are the problems of minimizing the respective functionals

$$\mathcal{F}_*(v) + \int_{\partial\Omega} |v - \psi|\, d\mathcal{H}_{n-1}$$

and

$$\mathcal{F}_*(v) + \int_{\partial\Omega} (\cos\theta)\, v\, d\mathcal{H}_{n-1}$$

among all $v \in \mathrm{BV}(\Omega)$, where $\mathcal{H}_k$ is the k-dimensional Hausdorff measure.

Alternatively, we consider the problem of minimizing the functional

$$\mathcal{F}(v) = \int_\Omega \sqrt{1 + |Du|^2}dx + \int_\Omega \int_0^v H(x, t)\, dx\, dt + \int_{\partial\Omega} \kappa(x, v)\, d\mathcal{H}_{n-1},$$

with

$$\kappa(x, v) = \int_0^v \gamma(x, t)\, dt\,.$$

For the capillarity problem, we have

$$\gamma(x, t) = \cos\theta$$

and

$$\kappa(x, t) = \int_0^v \cos\theta\, dt\,.$$

For the Dirichlet problem, we have

$$\gamma(x, t) = 1 - 2\phi_\psi(x, t)$$

and

$$\kappa(x, u) = |u - f(x)| - |f(x)|.$$

Here and throughout this section, ϕ_V is the characteristic function of the subgraph V of v:

$$\phi_V(x, t) = \begin{cases} 1, & \text{if } t < v(x), \\ 0, & \text{if } t \geqslant v(x). \end{cases}$$

M. Miranda [22] introduced the notion of *generalized solutions* for the minimal surface equation and used it successively both in the Dirichlet problem in infinite domains [22] and in the problem of removable singularities [23]. E. Giusti in [11] and [12] used the same notion of generailized solutions respectively in the problem of maximal domains for the mean curvature equation and boundary value problems for the mean curvature equation.

The idea of *generalized solutions* originates from the observation that a function $u : \Omega \mapsto \mathbb{R}$ is a solution of (0.1) in Ω if and only if its subgraph

$$U = \{(x, t) \in \Omega \times \mathbb{R}, t < u(x)\}$$

minimizes the functional

$$\mathbb{F}_*(U) = \int_{\Omega \times \mathbb{R}} |D\phi_U| + n \int_{\Omega \times \mathbb{R}} H\phi_U \, dx \, dt$$

locally in $\Omega \times \mathbb{R}$, in the sense that for every set V coinciding with U outside some compact set $K \subset \Omega \times \mathbb{R}$, we have

$$\int_K |D\phi_U| + n \int_K H\phi_U \, dx \, dt \leqslant \int_K |D\phi_V| + n \int_K H\phi_V \, dx \, dt.$$

Moreover, a function $u \in \mathrm{BV}(\Omega)$ minimizes $\mathcal{F}$ in Ω if and only if its subgraph minimizes the functional

$$\mathbb{F}(U) = \int_{\Omega \times \mathbb{R}} |D\phi_U| + n \int_{\Omega \times \mathbb{R}} H\phi_U \, dx \, dt + \int_{\partial\Omega \times \mathbb{R}} \gamma \phi_U \, d\mathcal{H}_n.$$

Minimization is here to be understood in the following sense: for $T > 0$, set

$$Q_T = \Omega \times [-T, T], \qquad \delta Q_T = \partial\Omega \times [-T, T],$$

and for $U \subset Q$,

$$\mathbb{F}_T(U) = \int_{Q_T} |D\phi_U| + n \int_{Q_T} H\phi_U \, dx\, dt - \int_{\partial Q_T} \gamma \phi_U \, d\mathcal{H}_n .$$

We say that U *minimizes* $\mathbb{F}_T$ in Q_T if

$$\mathbb{F}_T(U) \leqslant \mathbb{F}_T(S)$$

for every Caccioppoli set $S \subset Q_T$. We say that U *minimizes* $\mathbb{F}$ in $\Omega \times \mathbb{R}$ if U minimizes $\mathbb{F}_T$ in Q_T for every $T > 0$.

DEFINITION (Miranda[22]).

(1) *A function* $u : \Omega \mapsto [-\infty, \infty]$ *is a* generalized solution *of the equation* (0.1) *in* Ω *if its subgraph* U *minimizes the functional* $\mathbb{F}_*$ *locally in* $\Omega \times \mathbb{R}$.

(2) *A function* $u : \Omega \mapsto [-\infty, \infty]$ *is a* generalized solution *for the function* $\mathcal{F}$ *if its subgraph* U *minimizes* $\mathbb{F}$ *in* Q.

We note that a generalized solution can take the values $\pm\infty$ on a set of positive n-dimensional Hausdorff measure. However, it follows from Miranda [21] that if a generalized solution $u(x)$ can be modified on a set of zero n-dimensional Hausdo rff measure to be locally bounded, then $u(x)$ is a classical solution of (0.1) in Ω.

Proposition 5 below is derived in the proof of Theorem 1.1 of Giusti [12]. The special case in Proposition 6 below where $\mu = 1$ and $\partial\Omega \in C^2$ are shown by the proof of Theorem 3.2 of [12]. Proposition 6 is fully established in Lemma 7.6 of Finn [3].

It is easy to see that Proposition 3 and Proposition 4 follow immediately from Proposition 5 and Proposition 6, together with a subsequent consideration of U' instead of U.

PROPOSITION 5. *Let* U *minimize* $\mathbb{F}_*$ *locally in* $Q = \Omega \times \mathbb{R}$. *If* $z_0 = (x_0, t_0)$ *is a point in* Q *and if for all* $r > 0$ *we have*

$$|U_r(z_0)| > 0 ,$$

then, there exist positive constants C_0 *and* R_0, *depending only on* n *and* $\inf_Q H$ *such that*

$$|U_r(z_0)| \geqslant C_0 r^{n+1}, \tag{A.1}$$

for every $r \leqslant \min(R_0, \operatorname{dist}(z_0, \partial Q))$, *where we set*

$$U_r(z_0) = U \cap C_r(z_0),$$

with

$$C_r(z_0) = \{z = (x, t) \colon |x - x_0| < r,\ |t - t_0| < r\}.$$

In particular, we can take

$$C_0 = \frac{1}{4(n+1)\, k_{(n+1)}}, \tag{A.2}$$

and

$$R_0 = \begin{cases} \left(\dfrac{1}{2nk_{(n)}\, \omega_n \, |\inf_Q H(x, t)|} \right)^{1/n} & \text{if } \inf_Q H(x, t) < 0, \\ \infty, & \text{if } \inf_Q H(x, t) \geqslant 0, \end{cases} \tag{A.3}$$

where we denote $k_{(m)}$ *the isoperimetric constant in* $\mathbb{R}^m$, $m \geqslant 1$.

PROPOSITION 6. *Suppose that there exist constants* $Q_0 > 0$ *and* $\widehat{\gamma}$, $0 \leqslant \widehat{\gamma} < 1$, *such that*

$$\gamma(x, t) \geqslant -\widehat{\gamma}, \qquad \text{for all } x \in \partial\Omega \text{ and } t > \theta_0.$$

Suppose further that for some constant μ, *with* $\mu\widehat{\gamma} < 1$ *and* C_Ω *depending only on* Ω, *an inequality*

$$\int_{\partial\Omega} |v|\, dx \leqslant \mu \int_\Omega |Dv|\, dx + C_\Omega \int_\Omega |v|\, dx, \tag{A.4}$$

holds for all $v \in \mathrm{BV}(\Omega)$. *Let* U *minimizes* $\mathbb{F}$ *in* $Q = \Omega \times \mathbb{R}$, *and let* $z_0 = (x_0, t_0)$, $t_0 > \theta_0 + 1$, *be a point of* $\overline{Q}$ *such that for every positive* r

$$|U_r| > 0,$$

where U_r *is defined as in* Proposition 1. *Then there exist constants* $R_1 > 0$ *and* $C_1 > 0$ *determined completely by* n, $\inf_Q H(x, t)$, $\widehat{\gamma}$, μ *and* C_Ω *such that*

$$|U_r| \geqslant C_1 r^{n+1}, \qquad \text{for every } r \leqslant R_1. \tag{A.5}$$

In particular, we can take

$$C_1 = \frac{1-\widehat{\gamma}}{16(n+1)\,k_{(n+1)}}\,. \tag{A.6}$$

and if $\inf_Q H(x, t) \geqslant 0$,

$$R_1 = \left(\frac{C^*_{\widehat{\gamma}}}{C_\Omega k_{(n+1)}(2\omega_n)^{1/(n+1)}} \right), \tag{A.7}$$

where we set

$$C^*_{\widehat{\gamma}} = \min\left(\frac{1}{2}\,,\ \frac{1-(2\mu-1)\,\widehat{\gamma}}{3\,\widehat{\gamma}+1} \right);$$

if $\inf_Q H(x, t) < 0$, *we firstly take* $\widehat{R}_1$ *so small that*

$$\widehat{R}_1 \leqslant \left(\frac{1-(2\mu-1)\,\widehat{\gamma}}{2(\mu+1)\,nk_{(n)}\,\omega_n \,|\inf_Q H|} \;\Big|\,\right)^{1/n},$$

and then take

$$R_1 = \min\left(\frac{C^{**}_{\widehat{\gamma}}}{C_\Omega k_{(n+1)}(2\omega_n)^{1/(n+1)}}\,,\ \widehat{R}_1 \right), \tag{A.8}$$

where we set

$$C^{**}_{\widehat{\gamma}} = \min\left(\frac{1}{2}\,,\ \frac{1-(2\mu-1)\,\widehat{\gamma}-(\mu+1)\,nk_{(n)}\,|\inf_Q H|\,\omega_n(\widehat{R}_1)^n}{3\,\widehat{\gamma}+1} \right).$$

We notice that Lemma 1.1 in Giusti [11] established (A.4) for $\mu = 1$ in the special case that $\partial\Omega \in C^2$ and we have formulated this result as Lemma 1 in **0.2** of our present work.

An inequality of the form (A.4) appears first in Emmer [2], with

$$\mu = \sqrt{1+L^2}$$

for any Lipschitz domain with Lipschitz constant L. (See also [19, page 203]). On pages 141-143 of Finn [3], this result is extended to include domains in which one or more corners with inward opening angle appear. As pointed out on page 197 of [3], this extended result permits inward cusps and even boundary segments that may physically coincide but are adjacent to different parts of Ω. However, it is pointed out on page 143 of [3] that an outward cusp or a vertex of an outwa rd corner is not permitted.

We end this section with a sketch of the reasoning in [3] and [12] which leads to Propositions 5 and 6, mainly for the purpose of unifying the notation designations in [3] and [12]. Indeed, from comparing the values of the functionals $\mathbb{F}_*$ and $\mathbb{F}$ taken by U with those taken by $U \backslash C_r$, we obtain

$$\int_{C_r} |D\phi_U| + n\int_{C_r} H\phi_U \, dx\, dt \leqslant \int_{\partial C_r} \phi_U \, d\mathcal{H}_n, \qquad \text{if } r < \operatorname{dist}(z_0, \partial Q),$$

and

$$\int_{Q\cap C_r} |D\phi_U| + n \int_{Q\cap C_r} H\phi_U \, dx\, dt + \\ + \int_{\partial Q\cap C_r} \gamma\phi_U \, d\mathcal{H}_n \leqslant \int_{\partial C_r} \phi_U \, d\mathcal{H}_n, \qquad \text{if } r \geqslant \operatorname{dist}(z_0, \partial Q).$$

Since

$$\int_Q |D\phi_{U_r}| = \int_{C_r} |D\phi_U| + \int_{\partial C_r} \phi_U \, d\mathcal{H}_n$$

for almost all r, the previous two inequalities lead respectively to

$$\text{(A.9)} \qquad \int |\mathrm{D}\phi_{\mathrm{U_r}}| + \mathrm{n}\int \mathrm{H}\phi_{\mathrm{U_r}} \, \mathrm{dx}\, \mathrm{dt} \leqslant 2 \int_{\partial \mathrm{C_r}} \phi_{\mathrm{U}} \, \mathrm{d}\mathcal{H}_{\mathrm{n}}, \qquad \text{if } r < \operatorname{dist}(z_0, \partial Q),$$

and

$$\text{(A.10)} \qquad \int_{\mathrm{Q}} |\mathrm{D}\phi_{\mathrm{U_r}}| + \mathrm{n}\int_{\mathrm{Q}} \mathrm{H}\phi_{\mathrm{U_r}} \, \mathrm{dx}\, \mathrm{dt} + \\ + \int_{\partial Q} \gamma\phi_{U_r} \, d\mathcal{H}_n \leqslant 2 \int_{\partial C_r} \phi_U \, d\mathcal{H}_n, \qquad \text{if } r \geqslant \operatorname{dist}(z_0, \partial Q).$$

Setting

$$H_0^-(x) = \min(\inf_t H(x, t), 0),$$

the curvature term can be estimated as follows:

$$\text{(A.11)} \qquad \int H\phi_{U_r}\,dx\,dt \geqslant \int_Q H_0^- \,\phi_{U_r}\,dx\,dt$$

$$\geqslant -\|H_0^-\|_{n,\,B_r(x_0)} \int_{t_0-r}^{t_0+r} |C_r|^{1-\frac{1}{n}}\,dt, \quad \text{by Hölder's inequality}$$

$$\geqslant -k_{(n)}\|H_0^-\|_{n,\,B_r(x_0)} \int_{t_0-r}^{t_0+r} \left(\int |D\phi_{C_r}|\right) dt, \quad \text{by the isoperimetric inequality}$$

$$\geqslant -k_{(n)}\|H_0^-\|_{n,\,B_r(x_0)} \int |D\phi_{U_r}|\,.$$

Inserting this into (A.9), we obtain, if $r < \operatorname{dist}(z_0,\,\partial Q)$,

$$(A.12) \qquad (1 - nk_{(n)}\|H_0^-\|_{n,\,B_r(x_0)}) \int |D\phi_{U_r}| \leqslant 2 \int_{\partial C_r} \phi_U\,d\mathcal{H}_n,$$

which yields

$$\frac{d}{dr}|U_r| = \int_{\partial C_r} \phi_U\,d\mathcal{H}_n$$

$$\geqslant \frac{1}{2}(1 - nk_{(n)}\|H_0^-\|_{n,\,B_r(x_0)}) \int |D\phi_{U_r}|$$

$$\geqslant \frac{1 - nk_{(n)}\|H_0^-\|_{n,\,B_R(x_0)}}{2k_{(n+1)}} |U_r|^{\frac{n}{n+1}},$$

again by the isoperimetric inequality,

$$\geqslant \frac{1}{4k_{(n+1)}} |U_r|^{\frac{n}{n+1}},$$

if we choose r so small that $\|H_0^-\|_{n,\,B_R(x_0)} \leqslant \dfrac{1}{2nk_{(n)}}$

This leads to the estimate (A.1) with C_0 taken as in (A.2), whenever $r <$ $< \min(R_0,\, \operatorname{dist}(z_0,\, \partial Q))$, with R_0 given in (A.3).

In case $r \geqslant \operatorname{dist}(z_0,\, \partial Q)$, we have to handle the third term on the right hand side of (A.10). By (A.4) and the isoperimetric inequality,

$$\begin{aligned}\int_{\partial Q} \phi_{U_r} d\mathcal{H}_n &\leqslant \mu \int_Q |D\phi_{U_r}| + C_\Omega |U_r| \\ &\leqslant \mu \int_Q |D\phi_{U_r}| + C_\Omega k_{(n+1)} |U_r|^{\frac{1}{n+1}} \int |D\phi_{U_r}| .\end{aligned}$$

Since we have

$$\int |D\phi_{U_r}| d\mathcal{H}_n = \int_Q |D\phi_{U_r}| + \int_{\partial Q} \phi_{U_r} d\mathcal{H}_n,$$

the last inequality leads to

$$\text{(A.13)} \qquad \int_{\partial Q} \phi_{U_r} d\mathcal{H}_n \leqslant \frac{\mu + C_\Omega k_{(n+1)} |U_r|^{\frac{1}{n+1}}}{1 - C_\Omega k_{(n+1)} |U_r|^{\frac{1}{n+1}}} \int_Q |D\phi_{U_r}| \leqslant$$

$$\leqslant \frac{\mu + C_\Omega k_{(n+1)} |C_r|^{\frac{1}{n+1}}}{1 - C_\Omega k_{(n+1)} |C_r|^{\frac{1}{n+1}}} \int_Q |D\phi_{U_r}|,$$

if r is so small that

$$\text{(A.14)} \qquad C_\Omega k_{(n+1)} |C_r|^{\frac{1}{n+1}} \leqslant \frac{1}{2} .$$

This and the last identity yield

$$\text{(A.15)} \qquad \int |D\phi_{U_r}| d\mathcal{H}_n \leqslant \frac{\mu + 1}{1 - C_\Omega k_{(n+1)} |C_r|^{\frac{1}{n+1}}} \int_Q |D\phi_{U_r}| .$$

From (A.11), (A.13) and (A.15), we obtain

$$\text{(A.16)} \qquad n \int_Q H\phi_{U_r} dx\, dt + \int_{\partial Q} \gamma\phi_{U_r} d\mathcal{H}_n \geqslant$$

$$\geqslant -\left\{ \widehat{\gamma} \frac{\mu + C_\Omega k_{(n+1)} |C_r|^{\frac{1}{n+1}}}{1 - C_\Omega k_{(n+1)} |C_r|^{\frac{1}{n+1}}} + \frac{(\mu+1)\, n k_{(n)} \|H_0^-\|_{n, B_r(x_0)}}{1 - C_\Omega k_{(n+1)} |C_r|^{\frac{1}{n+1}}} \right\} \int_Q |D\phi_{U_r}| .$$

Choosing r so small that (A.7) is satisfied if $\inf_Q H \geqslant 0$ and (A.8) is satisfied if $\inf_Q H < 0$, we know that (A.14) is satisfied and the right hand side of (A.16) is bounded below by $\left(-\frac{(1+\widehat{\gamma})}{2}\right)\int\limits_Q |D\phi_{U_r}|$. Inserting this into (A.10), we obtain

$$\text{(A.17)} \qquad \int\limits_{\partial C_r} \phi_U \, d\mathcal{H}_n \geqslant \frac{1-\widehat{\gamma}}{4} \int\limits_Q |D\phi_{U_r}|.$$

From this, (A.14), (A.15) and the isoperimetric inequality, we obtain

$$\frac{d}{dr}|U_r| = \int\limits_{\partial C_r} \phi_U \, d\mathcal{H}_n \geqslant \frac{1-\widehat{\gamma}}{16} \int |D\phi_{U_r}| \geqslant \frac{1-\widehat{\gamma}}{16} \frac{1}{k_{(n+1)}} |U_r|^{\frac{n}{n+1}}.$$

This leads to the estimate (A.5) with C_1 taken as in (A.6) and completes the proof of Proposition 6.

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