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Endoprimal Abelian Groups of Torsion-Free Rank 1.

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ABSTRACT - This paper completely solves the endoprimality problem of (mixed) abelian groups of torsion-free rank 1.

1. Introduction.

In an abelian group A, the functions of the form

$$f(x_1, \dots, x_n) = k_1 x_1 + \dots + k_n x_n$$

with integers $k_1, \dots, k_n$ are called the *term functions*. Clearly, term functions permute with all endomorphisms of A, i.e., if $f(x_1, \dots, x_n)$ is a term function and ϕ is an endomorphism of A then

$$\phi(f(a_1, \dots, a_n)) = f(\phi(a_1), \dots, \phi(a_n))$$

for any $a_1, \dots, a_n \in A$. More generally, we call a function f of finite arity in an abelian group A an *endofunction* if it permutes with all endomorphisms of A. An abelian group is called *endoprimal* if all its endofunctions are term functions, in other words, if the term functions are the

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only functions in the group which permute with all endomorphisms.

The notion of endoprimality comes from universal algebra, where it has emerged in two ways: as a generalisation of an important property (called primality) of the two-element Boolean algebra and in the course of investigations into a general duality theory. The investigation of abelian groups with respect to endoprimality was started by Davey and Pitkethly [1] who, beside results for other kinds of algebraic structures, described those bounded abelian groups which are endoprimal.

As is well known, classification results for abelian groups, like for most algebraic structures, can only be obtained under severe restrictions. Obviously, endoprimality is a strong condition also for abelian groups and thus it seems promising to find characterisations for these groups. In [5] the present authors pointed at the connections of endoprimality with (direct) decomposition properties of abelian groups. Among others, they proved that a torsion group is endoprimal if and only if it is of finite exponent m containing $\mathbb{Z}_m \oplus \mathbb{Z}_m$ as a subgroup, and that any group of the form $B \oplus \mathbb{Z}$ with B unbounded is endoprimal, and also characterised endoprimal rank-2 torsion-free groups by means of a decomposition property. Investigations of endoprimality in torsion-free groups have been continued in [4] and [6]. In [4], endoprimal torsion-free separable groups are characterised, but also arbitrarily large indecomposable endoprimal groups are presented. In [6], endoprimal rank-3 torsion-free groups are described, and even among these there are indecomposable ones.

In the present paper we take up a further line from [5]. There the endoprimality problem was solved for the mixed groups of torsion-free rank 1 with splitting torsion part. To formulate that result and for further use, let us fix some notations.

In what follows, all groups are abelian and $T = T(A)$ always stands for the torsion part of a group A. Moreover, for any prime p, $T_p = T_p(A)$ is the p-component of T and T_p^* is the sum all T_q, $q \neq p$. The fact that B is a direct summand of A is denoted by $B \sqsubset A$. By J_p we denote the ring of p-adic integers.

So we have:

THEOREM 1.1 ([5], Theorem 5.9). *Let A be a group and P be the set of those primes p for which A/T is p-divisible. Assume that A has torsion-free rank* 1 *and $T \sqsubset A$. Then A is endoprimal if and only if T is unbounded and, for every $p \in P$, the subgroup T_p is not reduced.*

Now we are going to settle the mixed torsion-free rank 1 case completely and prove the following theorem as the main result of the paper. Notice that we also get endoprimal groups in which none of the primary parts split off; in other words, they are as close to being indecomposable as a mixed group of torsion-free rank 1 can be.

THEOREM 1.2. *Let A be a group and P be the set of those primes p for which A/T is p-divisible. Assume A has torsion-free rank* 1. *Then A is endoprimal if and only if T is unbounded and, for every $p \in P$, T_p is either not reduced or it is not a direct summand of A.*

2. Unary endofunctions.

The proof of the first three lemmas is straightforward.

LEMMA 2.1. *Every endofunction of a group A preserves the kernels and images of all endomorphisms of A.* ■

LEMMA 2.2 ([5], Proposition 2.5). *Let $A = A_1 \oplus A_2$ and f be an n-ary endofunction of A. Then f_i, the restriction of f to A_i, is an endofunction of A_i, $i = 1, 2$, and*

$$f(x_1 + y_1, \dots, x_n + y_n) = f_1(x_1, \dots, x_n) + f_2(y_1, \dots, y_n)$$

holds for arbitrary $x_i \in A_1$, $y_i \in A_2$, $i = 1, \dots, n$. ■

LEMMA 2.3. *Let A be a group and f a unary endofunction in A. If b and c are endomorphic images of the same element $a \in A$ then $f(b + c) = f(b) + f(c)$.* ■

LEMMA 2.4. *If f is any unary endofunction on a group A then $f|_T$ is an endomorphism of T. Moreover, for every p-component T_p of T, there exists $\xi \in J_p$ such that $f(x) = \xi x$ for every $x \in T_p$.*

PROOF. We first consider the action of f on the p-component T_p. We start with the case when T_p is not reduced, that is, it has a quasicyclic subgroup D. Obviously, D is a direct summand of A; let $A = B \oplus D$. Then by 2.2 there exist endofunctions f_B and f_D of the groups B and D, respectively, such that for every $x \in B$, $y \in D$ we have $f(x + y) = f_B(x) + f_D(y)$. It is an easy exercise to show that the unary endofunctions of D are precisely its endomorphisms, that is, multiplications by p-adic integers. Let ξ

be the p-adic integer such that $f(x) = \xi x$ for every $x \in D$. Now, if D_1 is any other quasicyclic subgroup of T_p then there exists $\phi \in \operatorname{End} A$ that maps D isomorphically onto D_1. This yields that $f(x) = \xi x$ holds for all $x \in D_1$, but then also for any $x \in E$ where E is the divisible part of T_p.

Let $A = B_1 \oplus E$, $B_2 = B_1 \cap T_p$, thus B_2 is a reduced p-group. If B_2 is bounded then it has a cyclic direct summand C of A of highest order. Now there exists $\phi \in \operatorname{End} A$ that embeds C into D, implying $f(x) = \xi x$ for every $x \in C$. Also, every cyclic subgroup of B_2 is an endomorphic image of C, hence $f(x) = \xi x$ holds for every $x \in B_2$. Now, if $x \in B_2$ and $y \in E$ are arbitrary then the projections of A to B_1 and E take $x + y$ to x and y, respectively, thus by Lemma 2.3 we obtain

$$f(x + y) = f(x) + f(y) = \xi x + \xi y = \xi(x + y)\,.$$

Now assume that B_2 is unbounded. A reduced unbounded p-group has cyclic direct summands of arbitrarily high orders (see e.g. [3], Exercise 27.1). These direct summands, being pure subgroups of T_p, are pure subgroups of A. Then, as bounded pure subgroups, they all are direct summands of A. Thus, A has cyclic direct summands C_i of arbitrarily high orders p^{n_i}. Since any of these subgroups can be embedded by means of an endomorphism into D, it follows that $f(x) = \xi x$ holds for every x in any C_i. Since any cyclic subgroup of T_p is an endomorphic image of some C_i, we have $f(x) = \xi x$ for all $x \in T_p$.

The case when T_p is reduced can be handled similarly. If T_p is bounded then it has a cyclic direct summand C of highest order and C is a direct summand of A as well. This yields that the restriction of f to C is an endofunction of C, thus there is an integer n such that $f(x) = nx$ for any $x \in C$. Also, for every cyclic subgroup C_1 of T_p there exists $\phi \in \operatorname{End} A$ that maps C onto C_1. This yields that $f(x) = nx$ holds for any $x \in T_p$.

To finish with T_p we need to handle the case when T_p is reduced but not bounded. As above, we find cyclic direct summands C_i of A contained in T_p of strictly increasing orders and on each of them the function f can be computed by $f(x) = n_i x$ where n_i is an integer. Because for every i, j with $i < j$ there is an endomorphism of A which embeds C_i into C_j, all these n_i's can be replaced by a single p-adic integer ξ. Thus, $f(x) = \xi x$ holds for every i and every $x \in C_i$. Since any cyclic subgroup of T_p is an endomorphic image of some C_i, the same formula holds for any $x \in T_p$.

By this we have, in particular, that f acts as an endomorphism in every T_p.

Next we prove the equality $f(x + y) = f(x) + f(y)$ for the case when

$x, y \in T$ belong to different p-components. By the foregoing we have $f(x) = mx$ and $f(y) = ny$ for some integers m, n. We first find $k, l \in \mathbb{Z}$ such that

$$kx = x, \quad ky = 0 = lx, \quad ly = y\,.$$

The orders of $f(x)$ and $f(y)$ divide the orders of x and y, respectively, hence

$$k(f(x+y)) = f(k(x+y)) = f(x) = mx = k(mx+ny) = k(f(x)+f(y))$$

and similarly $l(f(x+y)) = l(f(x)+f(y))$. Since obviously k and l can be chosen coprime, we have $f(x+y) = f(x)+f(y)$.

Now, using an easy induction argument one can prove the formula

(1) $$f(x_1 + \ldots + x_n) = f(x_1) + \ldots + f(x_n)$$

for arbitrary $x_1, \ldots, x_n$ belonging to pairwise different p-components of T.

To conclude the proof, we prove the equality $f(x+y) = f(x)+f(y)$ for arbitrary elements $x, y \in T$. Assume that $x, y \in T_{p_1} + \ldots + T_{p_n}$ and $x = x_1 + \ldots + x_n$, $y = y_1 + \ldots + y_n$ where $x_i, y_i \in T_{p_i}$, $i = 1, \ldots, n$. Then, using the formula (1) and the fact that f induces endomorphisms on all p-components, we have

$$\begin{aligned} f(x+y) &= f((x_1 + \ldots + x_n) + (y_1 + \ldots + y_n)) \\ &= f((x_1 + y_1) + \ldots + (x_n + y_n)) \\ &= f(x_1 + y_1) + \ldots f(x_n + y_n) \\ &= (f(x_1) + f(y_1)) + \ldots + (f(x_n) + f(y_n)) \\ &= f(x_1 + \ldots + x_n) + f(y_1 + \ldots + y_n) \\ &= f(x) + f(y)\,. \qquad \blacksquare \end{aligned}$$

LEMMA 2.5. *Let A be an abelian group with torsion part T, and assume that* $\operatorname{rank}(A/T) = 1$. *Then every unary endofunction of A is an endomorphism.*

PROOF. Consider an arbitrary unary endofunction f of A. By Lemma 2.4, f preserves T and the restriction of f to T is an endomorphism of T. Next we show that, for any $a \in A \setminus T$ and $t \in T$, $f(a+t) = f(a) + f(t)$. First observe that the general case of this formula can be derived by an induction argument from the special one with $t \in T_p$. Thus, assume $t \in T_p$. Moreover, for the beginning assume that t is contained in a direct summand C of A. Let $A = B \oplus C$, $a = b + t_1$, $b \in B$, $t_1 \in C$. Then it follows from Lemma 2.2 that $f(a) = f(b) + f(t_1)$ and $f(a + t) = f(b) + f(t_1 + t)$. Hence, using Lemma 2.4 we have

$$f(a + t) = f(b) + f(t_1 + t) = f(b) + f(t_1) + f(t) = f(a) + f(t).$$

Consequently, the formula $f(a + t) = f(a) + f(t)$ holds whenever T_p is a direct summand of A.

It remains to consider the case when $T_p \not\subseteq A$. In this case T_p contains cyclic direct summands of A of arbitrarily high orders. We take one of order at least the order of t. Let it be $C = \langle c \rangle$. Let $A = A_1 \oplus C$, $a = b + u$ where $b \in B$, $u \in C$. Now both a and t are endomorphic images of $b + c$, and Lemma 2.3 gives $f(a + t) = f(a) + f(t)$.

Finally, if $a_1, a_2 \in A \setminus T$ then there exist $a \in A \setminus T$, $m_1, m_2 \in \mathbb{Z}$, t_1, $t_2 \in T$ such that $a_1 = m_1 a + t_1$, $a_2 = m_2 a + t_2$. Hence

$$\begin{aligned} f(a_1 + a_2) &= f((m_1 + m_2)a + (t_1 + t_2)) \\ &= f((m_1 + m_2)a) + f(t_1 + t_2) \\ &= (m_1 + m_2)f(a) + f(t_1) + f(t_2) \\ &= (m_1 f(a) + f(t_1)) + (m_2 f(a) + f(t_2)) \\ &= (f(m_1 a) + f(t_1) + (f(m_2 a) + f(t_2)) \\ &= f(m_1 a + t_1) + f(m_2 a + t_2) \\ &= f(a_1) + f(a_2). \qquad \blacksquare \end{aligned}$$

Call a group *1-endoprimal* if those of its unary functions which permute with all endomorphisms are exactly the unary term functions, that is, the functions of the form kx with some integer k. Now we have:

COROLLARY 2.1. *An abelian group of torsion-free rank* 1 *is* 1-*endoprimal if and only if it is not p-divisible for any prime p and its endomorphism ring has trivial centre.* $\blacksquare$

3. Proof of the Main Theorem.

The following statement is a direct consequence of [7], Proposition 2.2 or even of [2], Theorem 1.2.

LEMMA 3.1. *Let A be a group of torsion-free rank* 1. *If $T = T_p$ and A/T is not p-divisible then $T \sqsubset A$.* ■

For easy reference, we state without proof the following simple fact.

LEMMA 3.2. *For any group A and prime number p:*

$$T/T_p^* \sqsubset A/T_p^* \Rightarrow T_p \sqsubset A . \qquad ■$$

Lemmas 3.1 and 3.2 directly imply

LEMMA 3.3. *Let A be a group of torsion-free rank* 1. *If A/T is not p-divisible then $T_p \sqsubset A$.* ■

The most unexpected conclusion from Theorem 1.2 is that a group of torsion-free rank 1 is endoprimal provided none of its nonzero p-components splits off and it has a non-zero p-component for every prime p such that the torsion-free factor of the group is p-divisible. This fact is based on the next lemma.

LEMMA 3.4. *Let A be a group of torsion-free rank* 1 *and let T_p be reduced. Assume that there exist $\xi \in J_p$ and $\phi \in \operatorname{End} A$ such that $0 \neq \phi(A) \subseteq T_p$ and $\phi(x) = \xi x$ for every $x \in T_p$. Then $T_p \sqsubset A$.*

PROOF. Lemma 3.2 reduces the proof of the present lemma to the case $T = T_p$. If A/T is not p-divisible then $T = T_p \sqsubset A$ follows by Lemma 3.1. Assume that A/T is p-divisible and consider first the case $\xi = 0$. Clearly then $T \subseteq \operatorname{Ker} \phi$ and we have a homomorphism $\overline{\phi} : A/T \to T$. Hence $\overline{\phi}(A/T) = \phi(A)$ is a divisible subgroup of T. Since A is reduced, $\phi(A) = 0$, a contradiction.

If $\xi \neq 0$ then ξ can be written in the form $\xi = p^k \eta$ where η is an invertible p-adic integer and $k \geqslant 0$. The function $x \mapsto \eta^{-1} x$ is a well-defined endomorphism of T_p, hence we have $\psi = \eta^{-1} \phi \in \operatorname{End} A$ and obviously $\psi(x) = p^k x$ for every $x \in T$. Moreover, $\phi(A) \subseteq T$ implies $\psi(x) = p^k x$ iff $x \in T$. Now, if $\chi \in \operatorname{End} A$ is defined by $\chi(x) = \psi(x) - p^k x$ then $\operatorname{Ker} \chi = T$, hence $\chi(A) \cong A/T$. Consequently, $\chi(A)$ is a torsion-free subgroup of A, that is, $\chi(A) \cap T = 0$.

We show that $\chi(A) + T = A$, which proves $T \sqsubset A$. Let $\overline{A} = A/T$ and $\overline{\chi}$ be the endomorphism of $\overline{A}$ induced by χ modulo T. Then $\overline{\chi}(\overline{A}) = p^k \overline{A} = \overline{A}$ because $\overline{A}$ is p-divisible. Hence, $\overline{\chi}$ is surjective, which is equivalent to the statement that $\chi(A)$ intersects all cosets of T in A. The latter is equivalent to the equality $\chi(A) + T = A$. ■

We need one more lemma which helps to carry out the induction in the proof of the main theorem. Note that actually the same idea was used several times in [5].

LEMMA 3.5. *Let $A = B \oplus C$. Assume that every $c \in C$ is contained in some subgroup $\phi(B)$ where $\phi \in \operatorname{End} A$. Then, if the restriction of every unary endofunction of A to C is a term function, the same holds for endofunctions of A of arbitrary arity.*

PROOF. Suppose that the restrictions of all endofunctions of A of arity less than n are term functions. Let f be an n-ary endofunction of A and $c_1, \dots, c_n \in C$. We take $\phi \in \operatorname{End} A$ such that $\phi(C) = 0$ and $c_1 = \phi(b)$ where $b \in B$. Also, let $\psi \in \operatorname{End} A$ be the projection of A onto C composed with the natural embedding of C into A. Then

$$
\begin{aligned}
f(c_1, \dots, c_n) &= f((\phi + \psi)(b), (\phi + \psi)(c_2), \dots, (\phi + \psi)(c_n)) \\
&= (\phi + \psi)\, f(b, c_2, \dots, c_n) \\
&= \phi(f(b, c_2, \dots, c_n)) + \psi(f(b, c_2, \dots, c_n)) \\
&= f(\phi(b), \phi(c_2), \dots, \phi(c_n)) + f(\psi(b), \psi(c_2), \dots, \psi(c_n)) \\
&= f(c_1, 0, \dots, 0) + f(0, c_2, \dots, c_n).
\end{aligned}
$$

Now our claim follows directly from the induction hypothesis. ■

PROOF OF THEOREM 1.2. *Neccessity.* By [5], Corollary 5.2 the torsion part must be bounded. If $T_p = 0$ for $p \in P$, then $x \mapsto \frac{1}{p} x$ is an endofunction which is not a term function, hence A is not endoprimal. Finally, assume that there exists $p \in P$ such that T_p is a nonzero, reduced direct summand of A and let $A = B \oplus T_p$. Clearly $\operatorname{Hom}(T_p, B) = 0$. Let $\phi \in \operatorname{Hom}(B, T_p)$. Obviously, T_p^* is the torsion part of B, $T_p^* \subseteq \operatorname{Ker} \phi$, and $A/T \cong B/T_p^*$. Hence ϕ induces a homomorphism $\psi : A/T \to T_p$. Since A/T is p-divisible, $\psi(A/T)$ is a divisible subgroup of T_p. Since T_p is reduced, we have $\psi = 0$ but then also $\phi = 0$. Thus, $\operatorname{Hom}(B, T_p) = 0$, and using [5], Corollary 2.6, we conclude that A is not endoprimal.

Proof of Theorem 1.2. *Sufficiency.* We first prove that A is 1-endoprimal. Let f be a unary endofunction of A. By Lemma 2.5 f is an endomorphism of A. Hence f preserves T and induces an endomorphism $\bar{f}$ of A/T defined by $\bar{f}(a+T) = f(a) + T$. Thus there is a rational number α such that $f(a) + T = \alpha(a + T)$ for any $a \in A$. Consider first the case when $\alpha \in \mathbb{Z}$. Define a new function $f_1(x) = f(x) - \alpha x$. Clearly, f_1 is both an endomorphism and an endofunction of A, and $f_1(A) \subseteq T$. We are going to show that $f_1 = 0$, which implies that f is a term function.

Given a prime number p, let π_p be the natural projection of T onto T_p composed with the embedding of T_p into T. Then π_p is an idempotent endomorphism of T with range T_p and also an endofunction of T. Since $f_1(A) \subseteq T$ and $f_1|_T \in \operatorname{End} T$, the composition $f_p = \pi_p f_1$ is an endofunction of A. We shall show, by checking three cases separately, that $f_p = 0$ for any prime p. This will prove that $f_1 = 0$.

Case 1: $p \notin P$. By Lemma 3.3 we have $A = B \oplus T_p$ for a suitable subgroup B of A. Since $f_p(A) \subseteq T_p$, Lemma 2.2 implies $f_p|_B = 0$. Obviously $B/T(B) \cong A/T$, therefore $B/T(B)$ is not p-divisible. Hence, given any $t \in T_p$, there exists $\phi \in \operatorname{End} A$ such that $\phi(b) = t$, for a suitable element $b \in B$. Since f_p is an endofunction of A, we have $0 = \phi(f_p(b)) = f_p(\phi(b)) = f_p(t)$. Thus, $f_p|_{T_p} = 0$. Now Lemma 2.2 gives $f_p = 0$.

Case 2: $p \in P$, $T_p \sqsubset A$ and T_p is not reduced. Now T_p has a nonzero divisible part D and we have direct decompositions $A = B \oplus T_p$, $T_p = T_p' \oplus D$. As in Case 1, we get $f_p|_B = 0$. Due to $p \in P$ the group $B/T(B)$ is p-divisible, hence, given any $t \in D$, there exist $\phi \in \operatorname{End} A$ and $b \in B$ such that $\phi(b) = t$. Now again as in Case 1 we conclude $f_p(t) = 0$, hence $f_p|_D = 0$. It remains to show that $f_p|_{T_p'} = 0$. First, any cyclic direct summand C of T_p' can be embedded by an endomorphism of A into D. Since f_p permutes with that endomorphism, we get $f_p|_C = 0$. Now there are two possibilities: 1) T_p' is bounded; 2) T_p' is not bounded. In the first case we take a cyclic direct summand C of T_p' of the highest order. Then C can be mapped by an endomorphism of A onto any cyclic subgroup of T_p'. Hence $f_p|_C = 0$ implies $f_p|_{T_p'} = 0$. In the second case we know that T_p' has cyclic direct summands C_i, $i = 1, 2, \dots$, of arbitrarily high orders and obviously the collection of subgroups $\phi(C_i)$, $\phi \in \operatorname{End} A$, $i = 1, 2, \dots$, includes all cyclic subgroups of T_p'. As above, we conclude that $f_p|_{T_p'} = 0$.

Case 3: $p \in P$ and $T_p \not\sqsubset A$. Let D be the divisible part of T_p, $A = B \oplus D$ and $T_p' = T_p \cap B$. Then $T_p' \not\sqsubset B$, hence T_p' is unbounded and obviously $T_p' = T_p(B)$ is reduced. Assume that $f_p \neq 0$. Then by Lemma 2.2 the restriction of f_p either to B or D must be nonzero. We show that $f_p|_B \neq 0$.

Indeed, by Lemma 2.4 there exists $\xi \in J_p$ such that $f_p(x) = \xi x$ for every $x \in T_p$. Since T_p' is unbounded, $f_p\,|_{T_p'} = 0$ would imply $\xi = 0$ but then also $f_p\,|_D = 0$. Now Lemma 3.4 applies with B and $f_p\,|_B$ in the roles of A and ϕ, respectively, to claim that $T_p' \sqsubset B$, a contradiction.

We have finished the proof of the claim that f is a term function of A if $\alpha \in \mathbb{Z}$. In general, α can be written as an irreducible fraction

$$\alpha = \frac{m}{p_1^{k_1} \dots p_s^{k_s}}$$

where $m, k_1, \dots, k_s \in \mathbb{Z}$, $k_1, \dots, k_s \geqslant 1$, and the p_i are prime numbers. Since $x \mapsto \alpha x$ is an endomorphism of A/T, all the p_i must belong to P. If p_i is a prime factor of the denominator of α, then $T_{p_i} \neq 0$ by the assumptions of the theorem. Let $t \in T_{p_i}$ be an element of order p_i, then

$$p_1^{k_1} \dots p_s^{k_s} f(t) = f(p_1^{k_1} \dots p_s^{k_s} t) = f(0) = 0$$

but, on the other hand,

$$p_1^{k_1} \dots p_s^{k_s} f(t) = mt \neq 0$$

for m is prime to p_i. Hence $\alpha \notin \mathbb{Z}$ cannot take place, and this completes the proof of the claim that A is 1-endoprimal.

We continue by induction on the arity of the endofunction f. We assume that all endofunctions of A whose arity is less than n are term functions of A. Let f be an n-ary endofunction of A, $n \geqslant 2$.

We start with showing that $f|_{T_p}$ is a term function of T_p for every prime p. Again we have to handle several cases separately. Obviously, we may assume that $T_p \neq 0$.

Case 1. $p \notin P$. Then Lemma 3.3 implies that T_p splits off, say, $A = B \oplus \oplus T_p$. Since $B/T_p^* \cong A/T$ is not p-divisible, for every $c \in T_p$ there exist $b \in B$ and $\phi \in \operatorname{End} A$ such that $\phi(b) = c$. Hence our claim follows by Lemma 3.5.

Case 2. $p \in P$, $T_p \not\sqsubset A$. Now T_p contains cyclic direct summands of A of arbitrarily high orders. This implies that, for any positive integer k, the group A has an endomorphic image S_k isomorphic to $\mathbb{Z}_{p^k}^n$. Being an endofunction, f preserves every endomorphic image of A, moreover, the restriction of f to that image is an endofunction of the latter. Since all groups $\mathbb{Z}_{p^k}^n$ are endoprimal, it follows that the restrictions of f to all subgroups S_k are term functions. Moreover, since for every k, l with $k < l$ there exists an endomorphism of A that embeds S_k into S_l, there exist p-

adic integers $\xi_1, \dots, \xi_n$ such that

$$f(x_1, \dots, x_n) = \xi_1 x_1 + \dots + \xi_n x_n \tag{2}$$

for every k and all $x_1, \dots, x_n \in S_k$, $k = 1, 2, \dots$. Finally, if $x_1, \dots, x_n \in T_p$ are arbitrary then there are an integer k and an endomorphism ϕ of A such that $x_1, \dots, x_n \in \phi(S_k)$. From this we conclude that the formula (2) holds for all $x_1, \dots, x_n \in T_p$. Since $g_1(x) = f(x, 0, \dots, 0)$ is a term function by the induction hypothesis and $g_1(x) = \xi_1 x$ for $x \in T_p$, we conclude that $\xi_1 \in \mathbb{Z}$. Similarly, all other coefficients ξ_i are integers.

Case 3. $p \in P$, $T_p \sqsubset A$. Now it follows from our assumptions that T_p cannot be reduced. Let $D \neq 0$ be the divisible part of T_p. Hence we have direct decompositions $A = B \oplus T_p$ and $T_p = C \oplus D$. We first consider the subgroup $B \oplus D$. Since $B/T_p^* \cong A/T_p$ is p-divisible, the assumption of Lemma 3.5 is satisfied, that is, for every $d \in D$ there exist $b \in B$ and $\phi \in \in \operatorname{End} A$ such that $d = \phi(b)$. Therefore we can conclude that the restriction of f to D is a term function; let

$$f(x_1, \dots, x_n) = \alpha_1 x_1 + \dots + \alpha_n x_n \tag{3}$$

for all $x_1, \dots, x_n \in D$ and for fixed $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$.

Now, if $x_1, \dots, x_n \in T_p$ and ϕ is any endomorphism of A such that $\phi(T_p) \subseteq D$, easy calculations show that

$$f(x_1, \dots, x_n) - \alpha_1 x_1 - \dots - \alpha_n x_n \in \operatorname{Ker} \phi .$$

Thus the equality (3) will be verified for all $x_1, \dots, x_n \in T_p$ if we prove that the intersection of the kernels of all such endomorphisms and the subgroup T_p is zero. But this follows easily from the injectivity of D. Indeed, every cyclic subgroup of T_p can be embedded into D. Since D is injective, this embedding can be extended to an endomorphism of T_p and since $T_p \sqsubset A$, it can be further extended to an endomorphism of A.

Thus we have shown that the restrictions of f to all p-components of T are term functions. It is important to observe that there is a common term function t which coincides with f on all p-components. Indeed, this follows from the fact that, by the induction hypothesis, all unary functions $f(0, \dots, 0, x, 0, \dots, 0)$ are term functions.

Our next step is to show that the restriction of f to T is a term function. Since every element of T belongs to a sum of finitely many p-components, we can use again an induction argument. Let

$$x_1, \dots, x_n \in T_{p_1} \oplus \dots \oplus T_{p_m}$$

and assume that the formula (3) holds if $x_1, \dots, x_n \in T_{p_1}$ or $x_1, \dots, x_n \in T_{p_2} \oplus \dots \oplus T_{p_m}$. Using the Chinese Remainder Theorem, we find integers k and l such that

$$(k+l)x_i = x_i, \quad kx_i \in T_{p_1}, \quad lx_i \in T_{p_2} \oplus \dots \oplus T_{p_m}, \quad i = 1, \dots, n .$$

It remains to calculate:

$$\begin{aligned} f(x_1, \dots, x_n) &= f((k+l)\,x_1, \dots, (k+l)\,x_n) \\ &= (k+l)f(x_1, \dots, x_n) \\ &= kf(x_1, \dots, x_n) + lf(x_1, \dots, x_n) \\ &= f(kx_1, \dots, kx_n) + f(lx_1, \dots, lx_n) \\ &= \alpha_1(kx_1) + \dots + \alpha_n(kx_n) + \alpha_1(lx_1) + \dots + \alpha_n(lx_n) \\ &= \alpha_1(k+l)x_1 + \dots + \alpha_n(k+l)x_n \\ &= \alpha_1 x_1 + \dots + \alpha_n x_n . \end{aligned}$$

For the rest we assume, without loss of generality, that the restriction of f to T is zero. The next step is to prove the equality

$$f(a_1, \dots, a_{n-1}, a_n + t) = f(a_1, \dots, a_{n-1}, a_n) \tag{4}$$

for all $a_1, \dots, a_n \in A$ and $t \in T$. Obviously, it is enough to consider the case $t \in T_p$, for an arbitrary prime p. We treat two cases separately.

Case 1. $T_p \sqsubset A$. Let $A = B \oplus T_p$ and $a_i = b_i + t_i$ where $b_i \in B$, $t_i \in T_p$, $i = 1, \dots, n$. Then, using Lemma 2.2, we have

$$\begin{aligned} f(a_1, \dots, a_{n-1}, a_n + t) &= f(b_1 + t_1, \dots, b_{n-1} + t_{n-1}, b_n + t_n + t) \\ &= f(b_1, \dots, b_n) + f(t_1, \dots, t_{n-1}, t_n + t) \\ &= f(b_1, \dots, b_n) + f(t_1, \dots, t_n) \\ &= f(b_1 + t_1, \dots, b_n + t_n) \\ &= f(a_1, \dots, a_n) . \end{aligned}$$

Case 2. $T_p \not\sqsubset A$. Now we can find in T_p a cyclic direct summand C of A whose order is not less than the order of t. Let $A = B \oplus C$, $C = \langle c \rangle$, $a_i = b_i + c_i$ where $b_i \in B$, $c_i \in C$. Also, let $t = u + v$ where $u \in B$, $v \in C$, and take the endomorphism ϕ of A such that $\phi|_B = 1_B$ and $\phi(c) = u$. Then calculate, using again Lemma 2.2:

$$\begin{aligned}
f(a_1, \dots, a_{n-1}, a_n + t) &= f(b_1 + c_1, \dots, b_{n-1} + c_{n-1}, b_n + u + c_n + v) \\
&= f(b_1, \dots, b_{n-1}, b_n + u) + f(c_1, \dots, c_{n-1}, c_n + v) \\
&= f(\phi(b_1), \dots, \phi(b_{n-1}), \phi(b_n + c)) \\
&= \phi(f(b_1, \dots, b_{n-1}, b_n + c)) \\
&= \phi(f(b_1, \dots, b_n) + f(0, \dots, 0, c)) \\
&= \phi(f(b_1, \dots, b_n)) \\
&= f(\phi(b_1), \dots, \phi(b_n)) \\
&= f(b_1, \dots, b_n) + f(c_1, \dots, c_n) \\
&= f(a_1, \dots, a_n) .
\end{aligned}$$

So we have proved (4), from which it follows that $f(a_1, \dots, a_{n-1}, t) = 0$ for arbitrary $a_1, \dots, a_n \in A$ and $t \in T$. Indeed, (4) implies

$$f(a_1, \dots, a_{n-1}, t) = f(a_1, \dots, a_{n-1}, 0) .$$

Since by the induction hypothesis $f(x_1, \dots, x_{n-1}, 0)$ is a term function and its restriction to the unbounded group T is zero, we must have $f(a_1, \dots, a_{n-1}, 0) = 0$.

It remains to show that $f(a_1, \dots, a_n) = 0$ holds also in the case when none of the a_i is a torsion element. Since $\operatorname{rank}(A/T) = 1$, there are integers k, l and elements $a \in A$, $u, v \in T$ such that $a_{n-1} = ka + u$, $a_n = la + v$. Then

$$f(a_1, \dots, a_n) = f(a_1, \dots, a_{n-2}, ka + u, la + v) = f(a_1, \dots, a_{n-2}, ka, la) .$$

Again by the induction hypothesis, $f(x_1, \dots, x_{n-2}, kx_{n-1}, lx_{n-1})$ is a term function and its restriction to T is zero, so we conclude that $f(a_1, \dots, a_{n-2}, ka, la) = 0$. This completes the proof. ■

Remark. In some cases it may be useful to consider an abelian group A as a module not over $\mathbb{Z}$ but over the largest subring N of $\mathbb{Q}$ over which it is a module – this ring N is called the *nucleus* of A. For example, the nucleus of every divisible torsion-free group is $\mathbb{Q}$. Clearly, if N is the nucleus of a group A and $q \in N \setminus \mathbb{Z}$ then the function $x \mapsto qx$ is an endofunction but not a term function of A. Thus A cannot be endoprimal as an abelian group. On the other hand, this function is a term function of the

N-module A, so A can be endoprimal over N. Therefore, when investigating endoprimality of abelian groups, it may be convenient to consider them as modules over their nuclei, that is, to regard the functions (1) with $k_i \in N$ as term functions. It is easy to understand that the nucleus of a mixed group A is the subring of $\mathbb{Q}$ generated by the inverses of all primes p such that A/T is p-divisible and $T_p = 0$.

For this case, as is easy to see, our main theorem takes the following form.

THEOREM 1.2$'$. *Let A be a group and P be the set of those primes p for which A/T is p-divisible. Assume A has torsion-free rank* 1. *Then A is endoprimal over its nucleus if and only if T is unbounded and, for every $p \in P$, at least one of the following three possibilities occurs:* 1) $T_p = 0$; 2) T_p *is not reduced*; 3) T_p *is not a direct summand of* A. ■

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