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Compact Topologically Torsion Elements of Topological Abelian Groups.

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ABSTRACT - In this note, we prove that in a Hausdorff topological abelian group, the closed subgroup generated by all compact elements is equal to the closed subgroup generated by all compact elements which are topologically p-torsion for some prime p. In particular, this yields a new, short solution to a question raised by Armacost [A]. Using Pontrjagin duality, we obtain new descriptions of the identity component of a locally compact abelian group.

1. Introduction.

All considered groups in this paper are Hausdorff topological abelian groups and will be written additively. Let us establish notation and terminology. The set of all natural numbers is denoted by $\boldsymbol{N}$, and $\boldsymbol{P}$ is the set of all primes. The additive topological group of real numbers is denoted by $\boldsymbol{R}$. By $\mathbf{Z}$ and $\boldsymbol{Q}$ we mean the additive, discrete group of integers and rationals, respectively, and $\boldsymbol{T}$ is the quotient $\boldsymbol{R}/\boldsymbol{Z}$. The n-element cyclic group is written $\boldsymbol{Z}(n)$. For a prime p, let $\boldsymbol{Z}(p^\infty)$ denote the quasicyclic group and $\boldsymbol{J}_p$ the compact group of p-adic integers. The groups isomorphic to a subgroup of $\boldsymbol{Z}(p^\infty)$ for some prime p are precisely the *cocyclic* groups (cf. [F]). By G_d we mean the group G with the discrete topology. We shall write $H < G$ to indicate that H is a subgroup of G, and by $\overline{H}$ we mean the closure of H in G. The subgroup of G generated by a collection $H_i < G$ $(i \in I)$ is denoted by $\sum\{H_i : i \in I\}$. An element $x \in G$ is called *compact* if $\overline{\langle x \rangle}$ is a compact subgroup of G. The compact elements of G form a subgroup of G which is denoted by bG. The *dual group* of G is the topological group $\hat{G} = \mathrm{Hom}(G, \boldsymbol{T})$ endowed with the compact-open topology. The annihilator of $H \subseteq G$ in $\hat{G}$ is written $(\hat{G}, H)$.

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If p is a prime, the *p-component* of a topological group G is defined to be $G_p = \{x \in G : \lim_{n\to\infty} p^n x = 0\}$. This set is a subgroup of G and its elements are called *topologically p-torsion elements*. If $G_p = G$, the group G is called a *topological p-group*. For example, $\boldsymbol{Z}(p^\infty)$ and $\boldsymbol{J}_p$ are topological p-groups. The p-component of $\boldsymbol{T}$ is equal to the p-torsion part of $\boldsymbol{T}$ (see [A] Lemma 2.6). Following Robertson [R], we call an element $x \in G$ *topologically torsion* if $\lim_{n\to\infty} n!x = 0$ and denote by $G!$ the set of all topologically torsion elements of G. If $G! = G$, then G is called *topologically torsion*. Notice that $G!$ is a subgroup of G and that $G!$ contains G_p for every prime p. The group $G!$ is a subgroup of bG provided that G is locally compact (cf. [HR] Theorem 9.1). If G is not locally compact, the inclusion $G! \subseteq bG$ may strongly fail. For instance, every countable subgroup H of $\boldsymbol{J}_p$ satisfies $H! = H$ and $bH = 0$. It is clear that if G is a discrete group, $G!$ [G_p] coincides with the torsion part [p-torsion part] of G. Topologically torsion groups have been studied by Armacost [A], Braconnier [B], Dikranjan [D], Dikranjan, Prodanov and Stoyanov [DPS], Robertson [R] and Vilenkin [V]. Additional references may be found in [D] and [DPS].

In this paper, we are concerned with compact topologically (p-)torsion elements of topological abelian groups. In [A] (3.23), Armacost asked whether the subgroup of all compact elements of a locally compact abelian (LCA) group G contains $G!$ as a dense subgroup. The answer to this question is affirmative and follows from the fact that in an LCA group which consists entirely of compact elements, the compact topologically p-torsion elements generate a dense subgroup (cf. [A] (4.27)). A result in a stronger form can be found in [DPS] 4.5.12. We give a new short solution to the question in [A] (3.23), reducing it to the case of a monothetic compact group which can be easily checked. Our approach is based essentially on Pontrjagin duality and allows for different proofs with the key role of the following as an example: For any topological abelian group G, the p-component of bG is generated by all subgroups H of G such that $H \cong \boldsymbol{Z}(p^n)$ for some positive integer n or $H \cong \boldsymbol{J}_p$ (see Lemma 2.1). It follows that if G is any topological abelian group G, then $\overline{bG}$ contains a dense subgroup generated by all compact topologically p-torsion elements of G (cf. Theorem 2.5). Consequently, $G!$ is a dense subgroup of bG whenever G is locally compact. In this case, Pontrjagin duality can be used to obtain new descriptions of the identity component of G: The identity component of an LCA group G is the intersection of all open [closed] subgroups H of G such that G/H is a cocyclic group [topological p-group] (Theorem 2.8).

2. Topologically torsion elements and duality.

The following lemma on p-components will be useful:

LEMMA 2.1. *Let G be a topological abelian group and let p be a prime. Then we have:*

(i) *$(bG)_p$ is the subgroup of G generated by all subgroups H of G such that $H \cong \boldsymbol{Z}(p^n)$ for some positive integer n or $H \cong \boldsymbol{J}_p$.*

(ii) *If G is locally compact, then $(\hat{G}, G_p)$ is the intersection of all open subgroups X of $\hat{G}$ such that $\hat{G}/X$ is a cocyclic p-group.*

(iii) *If G is locally compact, then $(\hat{G}, G_p)$ is the intersection of all closed subgroups X of $\hat{G}$ such that $\hat{G}/X$ is a topological p-group.*

PROOF. The first assertion follows from [A] Lemma 2.11. If G is locally compact, then every topologically p-torsion element of G is compact. Therefore, (i) yields

$$(\hat{G}, G_p) = \bigcap\{(\hat{G}, H) : H < G \text{ and } H \cong \boldsymbol{Z}(p^n) \text{ for some } n \in \boldsymbol{N} \text{ or } H \cong \boldsymbol{J}_p\}.$$

Using Pontrjagin duality (cf. [HR] Theorem 24.8), we may identify the dual group of $\hat{G}$ with G. By [HR] Theorems 23.17, 23.25 and 24.10, a subgroup X of $\hat{G}$ is open in $\hat{G}$ if and only if X is the annihilator of (G, X) in $\hat{G}$ such that (G, X) is compact. Since finite groups are self-dual and $(\boldsymbol{J}_p)\hat{} \cong \boldsymbol{Z}(p^\infty)$ (see [HR] 23.27(c) and 25.2), we obtain

$$(\hat{G}, G_p) = \bigcap\{X : X \text{ open} < \hat{G} \text{ and } \hat{G}/H \text{ is a cocyclic } p\text{-group}\},$$

as desired. The third statement follows from the first paragraph and the fact that an LCA group is a topological p-group exactly if its dual group is a topological p-group (cf. [A] Corollary 2.13). □

COROLLARY 2.2. *Let G be a compact abelian group and let p be a prime. Then:*

(i) *G_p is the subgroup of G generated by all subgroups H of G such that $H \cong \boldsymbol{Z}(p^n)$ for some positive integer n or $H \cong \boldsymbol{J}_p$.*

(ii) *$(\hat{G}, G_p)$ is the intersection of all subgroups X of $\hat{G}$ such that $\hat{G}/X$ is a cocyclic p-group.*

(iii) *$(\hat{G}, G_p)$ is the intersection of all subgroups X of $\hat{G}$ such that $\hat{G}/X$ is a p-group.*

EXAMPLE. For every prime p, $(\hat{\boldsymbol{Q}})_p$ is a dense subgroup of the compact group $\hat{\boldsymbol{Q}}$ (cf. [DPS] Example 4.1.3(c)). To see this, we consider the discrete group $\boldsymbol{Q}_{(p)}$ of rational numbers whose denominators are prime to p. Clearly, $\bigcap_{k\in\boldsymbol{N}} p^k\boldsymbol{Q}_{(p)} = 0$ and $\boldsymbol{Q}/p^k\boldsymbol{Q}_{(p)} \cong \boldsymbol{Z}(p^\infty)$ for every $k \in \boldsymbol{N}$. By Lemma 2.1(ii), the annihilator of $(\hat{\boldsymbol{Q}})_p$ in the dual group of $\hat{\boldsymbol{Q}}$ is trivial and the assertion follows.

REMARK. The example above provides an alternative proof of the property that $(\hat{\boldsymbol{Q}})_p$ is dense in $\hat{\boldsymbol{Q}}$. The proof in [DPS] requires the knowledge of the structure of the group $\hat{\boldsymbol{Q}}$, whereas our proof is based on Lemma 2.1(ii).

Recall that a topological group is said to be *monothetic* if it contains a dense cyclic subgroup.

PROPOSITION 2.3. *If G is a compact monothetic group, then the p-components of G ($p \in \boldsymbol{P}$) generate a dense subgroup of G.*

PROOF. Any homomorphism $f : \boldsymbol{Z} \to G$ with $\overline{f(\boldsymbol{Z})} = G$ induces a monomorphism $f^* : \hat{G} \to \hat{\boldsymbol{Z}} \cong \boldsymbol{T}$ (cf. [HR] Theorem 24.38). Since the dual group of a compact group is discrete, $\hat{G}$ is isomorphic to a subgroup of $\boldsymbol{T}_d \cong \bigoplus_{p\in\boldsymbol{P}} \boldsymbol{Z}(p^\infty) \oplus \bigoplus_{2^{\aleph_0}} \boldsymbol{Q}$. This gives rise to a continuous epimorphism

$$\phi : (\boldsymbol{T}_d)\hat{} \cong \prod_{p\in\boldsymbol{P}} \boldsymbol{J}_p \oplus \prod_{2^{\aleph_0}} \hat{\boldsymbol{Q}} \longrightarrow G.$$

Since ϕ maps topological p-groups onto topological p-groups and every p-component of $\hat{\boldsymbol{Q}}$ is a dense subgroup (see the example above), the groups $\overline{G_p}$ ($p \in \boldsymbol{P}$) generate a dense subgroup of G and the proof is complete. $\square$

Let G be a topological abelian group. Then we denote by $A(G)$ the subgroup of G generated by all subgroups of G which are topologically isomorphic to $\boldsymbol{Z}(p^n)$ or $\boldsymbol{J}_p$ ($p \in \boldsymbol{P}, n \in \boldsymbol{N}$). Further, let $B(G)$ denote the subgroup of G generated by all compact totally disconnected subgroups of G. Recall that an element $x \in G$ is said to be *quasi-p-torsion* if either the cyclic subgroup $\langle x\rangle$ of G is a finite p-group, or $\langle x\rangle$ is topologically isomorphic to $\boldsymbol{Z}$ endowed with the p-adic topology. The subgroup $wtd(G)$ of G generated by all quasi-p-torsion elements ($p \in \boldsymbol{P}$) was defined and studied by Stoyanov [S] (see also [D], [DPS] and the references there). An element x of G is called *quasi-torsion* if either $\langle x\rangle$ is finite, or $\langle x\rangle$ is equipped with a non-discrete topology generated by open subgroups of $\langle x\rangle$. For any prime p, quasi-p-torsion elements are both topologically p-torsion and quasi-torsion. The quasi-torsion elements of G form a subgroup of G! which is denoted by $td(G)$. For a compact abelian group G, this subgroup was introduced by Dikranjan and Stoyanov [DS] as the subgroup of G generated by all closed totally disconnected subgroups of G. In [S], Stoyanov considered $td(G)$ for an arbitrary topological group G (see also [D] and [DPS]).

PROPOSITION 2.4. *Let G be a topological abelian group. Then we have:*

(i) $A(G) = \sum\{(bG)_p : p \in \boldsymbol{P}\} = wtd(bG)$.

(ii) $B(G) = td(bG)$.

(iii) *If H is a subgroup of G, then $A(H) \subseteq H \cap A(G)$ and $B(H) \subseteq \subseteq H \cap B(G)$.*

(iv) *If H is a closed subgroup of G, then $A(H) = H \cap A(G)$ and $B(H) = H \cap B(G)$.*

(v) *If H is a topological abelian group and $f : G \to H$ is a continuous homomorphism, then $f(A(G)) \subseteq A(H)$ and $f(B(G)) \subseteq B(H)$.*

(vi) *If $\{G_i : i \in I\}$ is a family of topological abelian groups, then $A(\prod_{i\in I} G_i) \subseteq \prod_{i\in I} A(G_i)$ and $B(\prod_{i\in I} G_i) = \prod_{i\in I} B(G_i)$.*

PROOF. The first equality in (i) follows from Lemma 2.1(i). Since $A(G)$ is the subgroup of G generated by all elements x such that $\overline{\langle x\rangle}$ is topologically isomorphic to $\boldsymbol{Z}(p^n)$ or $\boldsymbol{J}_p$ ($p \in \boldsymbol{P}, n \in \boldsymbol{N}$), we have $A(G) = wtd(bG)$. Obviously, $B(G)$ coincides with $td(bG)$. Properties (iii) - (vi) follow immediately from the corresponding statements on $wtd(G)$ and $td(G)$ in [DPS] Proposition 4.1.2 and from properties of the group bG (see [DPS] p. 91). $\square$

THEOREM 2.5. *Let G be a topological abelian group, Then we have:*

(i) $A(G) \subseteq B(G) \subseteq (bG)! \subseteq bG$.

(ii) $A(G) \neq B(G) \neq (bG)! \neq bG$ in general.

(iii) $\overline{A(G)} = \overline{B(G)} = \overline{(bG)!} = \overline{bG}$.

PROOF. (i) follows from the fact that a compact totally disconnected abelian group is topologically torsion (cf. [A] Corollary 3.6). To show (ii), we consider the compact group $G = \boldsymbol{T}^{\aleph_0}$ (This example was suggested by the referee). By Proposition 2.4 or [DPS] Proposition 4.1.2 we have $A(G) = \sum\{\boldsymbol{Z}(p^\infty)^{\aleph_0} : p \in \boldsymbol{P}\}$ and $B(G) = (\boldsymbol{Q}/\boldsymbol{Z})^{\aleph_0}$. Since the group $\boldsymbol{T}!$ is neither torsion nor equal to $\boldsymbol{T}$ (see [A] Lemma 3.4), we conclude that $A(G) \neq B(G) \neq G! \neq G$. To prove (iii), let x be a compact element of G. Then Proposition 2.3 shows that $\overline{\langle x\rangle}$ contains a dense subgroup generated by all its p-components. Therefore, x is an element of $\overline{\sum\{(bG)_p : p \in \boldsymbol{P}\}}$ and by Proposition 2.4(i), the assertion follows. $\square$

In [A] (3.23), Armacost asked whether $G!$ is a dense subgroup of bG if G is an LCA group. Since in a locally compact abelian group G, the compact elements form a closed subgroup (cf. [HR] Theorem 9.10) containing all topologically torsion elements of G, we obtained a new, short solution of this problem.

Corollary 2.6. *If G is an LCA group, then $\overline{A(G)} = bG$.*

If G is a locally compact abelian group, the annihilator of $G!$ in $\hat{G}$ can be described as follows:

Corollary 2.7. *Let G be an LCA group. Then:*

(i) $(\hat{G}, G!)$ *is the intersection of all open subgroups X of $\hat{G}$ such that $\hat{G}/X$ is cocyclic.*

(ii) $(\hat{G}, G!)$ *is the intersection of all closed subgroups X of $\hat{G}$ such that $\hat{G}/X$ is a topological p-group ($p \in \boldsymbol{P}$).*

Proof. The assertions follow from Lemma 2.1 and Theorem 2.5. $\square$

The identity component of an LCA group G is the intersection of all open subgroups of G (see [HR] Theorem 7.8). Since the annihilator of bG in $\hat{G}$ coincides with the identity component of $\hat{G}$ (cf. [HR] Theorem 24.17), we can use Pontrjagin duality to obtain the following new descriptions of the identity component of an LCA group:

Theorem 2.8. *Let G be an LCA group and let G_0 be the identity component of G. Then we have:*

(i) G_0 *is the intersection of all open subgroups H of G such that G/H is cocyclic.*

(ii) G_0 *is the intersection of all closed subgroups H of G such that G/H is a topological p-group ($p \in \boldsymbol{P}$).*

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