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On the Exponent of the Product of two Groups.

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To Guido Zappa on his 90th birthday

Let the group $G = AB$ be the product of two subgroups A and B, and assume that these subgroups have finite exponents e and f. If A and B are abelian, then G is metabelian, by a famous result of N.Ito [AFG, 2.1.1], and in that case R.W.Howlett has shown that the exponent of G satisfies $exp(G) \mid ef$ [AFG, 3.3.1]. Apart from that, it seems that very little is known about the exponent of G in general. If G is soluble, it has a finite exponent [AFG, 3.2.11]. From this we deduce

THEOREM 1. *Let the soluble group G of derived length d be the product of two subgroups A and B of finite exponents e and f. Then the exponent of G is bounded by some function of d, e, and f.*

PROOF. If no such function exists, then we can find soluble groups $G_n = A_nB_n$, of derived length d, such that A_n and B_n have exponents e and f, and the exponents of the groups G_n are unbounded. Then the direct product of all the G_n's has derived length d, is the product of subgroups of exponents e and f, but has infinite exponent, a contradiction.

COROLLARY 2. *Let $G = AB$, where A and B have exponents e and f, respectively. Assume that A and $[A, B]$ are soluble of length d. Then the exponent of G is bounded by a function of e, f, and d.*

PROOF. We have $A^G = A[A, B]$. Therefore A^G is soluble of length $2d$. But also $A^G = AC$, where $C = A^G \cap B$ has exponent f. By the previous theorem, $exp(A^G)$ is bounded, while $exp(G/A^G) = exp(B/C) \mid f$.

Using Howlett's result, we can give explicit bounds for two cases: when G is metabelian, or nilpotent-by-abelian, and when one of the factors is abelian.

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THEOREM 3. *Let the metabelian group G be a product of two subgroups A and B, of exponents e and f. Then $exp(G) \mid (ef)^3$. If either A or B is abelian, then $exp(G) \mid (ef)^2$.*

This follows immediately by substituting $A_1 = A'$ and $B_1 = B'$ in the following more elaborate result, in which no solubility assumption is made.

PROPOSITION 4. *Let G be a product AB of two subgroups, of exponents e and f. Assume that there exist two subgroups, A_1 and B_1, such that $A' \leq A_1 \leq A$, $B' \leq B_1 \leq B$, and $[A,B]$ normalizes both A_1 and B_1. Then $exp(G) \mid (ef)^3$. If either A or B is abelian, then $exp(G) \mid (ef)^2$.*

PROOF. As above, write $A^G = A[A,B] = AC$. Then $A_1 \triangleleft A^G$, and we have a normal series $A_1 \triangleleft A^G \triangleleft G$. Here $A^G = AC$, for $C = A^G \cap B$, $G/A^G \cong B/C$, and $A^G/A_1 = A/A_1 \cdot CA_1/A_1$. Assume first that B is abelian. Then $exp(A^G/A_1) \mid ef$, by Howlett, implying $exp(G) \mid e^2f^2$. If B is not abelian, then the factorization $A^G/A_1 = A/A_1 \cdot CA_1/A_1$ satisfies the assumptions of the proposition, with B_1 replaced by $C_1 = B_1 \cap A^G$, and with A/A_1 abelian. By what was just proved, $exp(A^G/A_1) \mid e^2f^2$, and $exp(G) \mid e^3f^3$.

THEOREM 5. *Let $G = AB$, where $exp(A) = e$ and $exp(B) = f$. Assume that A is abelian, and that G is soluble of length $d \geq 2$. Then $exp(G) \mid (ef)^{2d-2}$.*

PROOF. If $d = 2$, this follows from Theorem 3. Otherwise, let $N = G^{(d-1)}$. Then N is abelian, and AN is metabelian, and can be factored as $AN = AC$, with $C = AN \cap B$. Theorem 3 shows that $exp(AN) \mid e^2f^2$, and in particular $exp(N) \mid e^2f^2$. Apply induction to G/N.

A similar argument establishes the following variation:

PROPOSITION 6. *Let $G = AB$, where $exp(A) = e$ and $exp(B) = f$. Assume that A is abelian, and that $[A,B]$ is soluble of length d. Then $exp(G) \mid e^{2d}f^{2d+1}$.*

PROOF. Let $N = [A,B]^{(d-1)}$. Then N is abelian, and AN is metabelian, and can be factored as $AN = AC$, with $C = AN \cap B$. Theorem 3 shows that $exp(AN) \mid e^2f^2$. If $d = 1$, then $N = [A,B]$, and then $AN = A^G \triangleleft G$, with $G/A^G \cong B/C$, and the result follows. In the general case we have $exp(N) \mid e^2f^2$, and we apply induction to G/N.

PROPOSITION 7. *Let $G = AB$ be nilpotent-by-abelian, with $cl(G') = c$, and let A and B have exponents e and f. Then $exp(G) \mid (ef)^{2c+1}$.*

PROOF. For $c = 1$ this follows from Theorem 3. In the general case, write $Z = Z(G')$, and $AZ = AC$, with $C = AZ \cap B$. Then $A' \triangleleft AZ$, and AZ/A' is metabelian, with A/A' abelian, and so Theorem 3 implies that $exp(AZ) \mid e^2 f^2$, therefore $exp(Z) \mid e^2 f^2$, and we can apply induction to G/Z.

NOTE. Theorem 5 yields a better bound than Proposition 7 whenever both are applicable.

REFERENCES

[AFG] B. AMBERG - S. FRANCIOSI - F. DE GIOVANNI, *Products of Groups*, Oxford Univ. Press, Oxford 1992.

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