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6-BFC Groups.

CLIFF DAVID (*) - JAMES WIEGOLD (**)

To Professor Guido Zappa on his 90th birthday

1. Introduction and preliminaries.

A group is said to be BFC if its conjugacy classes (of elements) are boundedly finite, and n-BFC if the largest conjugacy classes have order n. B.H. Neumann proved in [4] that a group is BFC if and only if its derived group is finite; in [8], the second author showed that the derived group of an n-BFC group is of order bounded in terms of n. He formulated there the following conjecture, in which $\lambda(n)$ stands for the number of prime factors of n, multiplicities included.

CONJECTURE. *For every n-BFC group G, the order of G' is at most $n^{\frac{1}{2}(1+\lambda(n))}$.*

There are nilpotent groups of class 2 and arbitrarily large n where this bound is achieved. Further, it is proved in [8] that the conjecture is true when n is prime and when $n = 4$, in which case G' is of order 4 or 8. There is a wide literature attacking this problem; the best bound achieved so far is that of Segal and Shalev [5], namely $n^{\frac{1}{2}(13+\log_2(n))}$. Vaughan-Lee [6] established the conjecture for nilpotent groups. The smallest value of n for which the conjecture is not known to be true is 6, and the aim of this note is to rectify this by proving the following result.

THEOREM. *Let G be a 6-BFC group. Then G' is either C_6 or Q_8.*

Throughout, notation is standard unless otherwise stated. For example, we write $n = \beta(G)$ for an n-BFC-group G. It is easy to see [8] that the proof

(*) Indirizzo dell'A.: 68, Ty'n-y-Twr, Baglan, West Glamorgan, Wales, UK.
(**) Indirizzo dell'A.: School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF24 4AG Wales, UK.

of the theorem reduces to one for finite groups, and henceforward we shall consider finite groups only.

We state some obvious facts here.

1.1. *For any groups A and B,* $\beta(A \times B) = \beta(A)\beta(B)$.

1.2. *If N is normal subgroup of a group G and g an element of G, then the number of conjugates of gN in G/N divides the number of conjugates of g in G.*

A useful tool in the proof of the Theorem is this. Let a be an element of a group G with exactly t conjugates $a = a_1, a_2, \ldots, a_t$. Then the map $\mu(a)$ of G into the symmetric group S_t defined by

$$g\mu(a) = \begin{pmatrix} a_1 & a_2 & \ldots & a_t \\ a_1^g & a_2^g & \ldots & a_t^g \end{pmatrix}$$

is a homomorphism of G onto a transitive subgroup of S_t.

1.3. *The kernel of* $\mu(a)$ *is* $\mathrm{Core}_G\, C(a)$.

Note that both alternatives for G' mentioned in the Theorem occur. There are in fact four 6-BFC group of order 24, the smallest possible order. Three of them have derived group cyclic of order 6, collected together as Examples 1.4; see Coxeter and Moser [1].

EXAMPLE 1.4. *The dihedral group of order* 24 *is* 6-BFC *and its derived group is* C_6. *The same is true of the dicyclic group of order* 24 *and the group with the following presentation:*

$$\langle a, b : a^4 = b^6 = (ab)^2 = (a^{-1}b)^2 = 1 \rangle.$$

EXAMPLE 1.5. *The binary tetrahedral group, with presentation*

$$\langle a, b, c : a^2 = b^2 = [a, b], c^3 = 1, a^c = b, b^c = a^{-1}b^{-1} \rangle,$$

is 6-BFC *and has derived group* Q_8.

Finally in this section, we show that the problem reduces to one in $\{2, 3\}$-groups. Specifically, we have the following result.

LEMMA 1.6. (1) *Every* 6-BFC *group is soluble.*

(2) *If G is a* 6-BFC *group, then* $G = A \times B$, *where A is a* $\{2, 3\}$*-group and B is an abelian* $\{2, 3\}'$*-group.*

PROOF. The proof of (1) is almost immediate. If there is an insoluble 6-BFC group, some composition factor H of it is a nonabelian simple group with $\beta(H) \leq 6$. By the remarks preceding 1.3, there is an isomorphism of H onto a simple subgroup of S_6; these are A_5 and A_6, and they have conjugacy classes of order more than 6.

Now let G be any 6-BFC group. All Sylow p-subgroups of G with $p > 5$ are central. Otherwise, there is an element g of p-power order not commuting with some element a of G, and the conjugates a^{g^i} of a with $0 \leq i < p$ are all distinct, too many for a 6-BFC group.

Less obvious but still easy is that the Sylow 5-subgroup is central. We shall show first that every element x of 5-power order in G commutes with all elements having 1, 2, 3, 4, or 6 conjugates. The first four are proved in a manner like that in the previous paragraph. Now suppose that y has 6 conjugates. Then, by 1.3 above, $G/\mathrm{Core}\, C(y)$ is a transitive group of degree 6. If y fails to commute with x, then the element $x\mathrm{Core}\, C(y)$ is of order 5, and thus $G/\mathrm{Core}\, C(y)$ is doubly transitive and therefore primitive; but it is soluble and thus imprimitive because 6 is not a prime-power. Thus x does indeed commute with all elements having 1, 2, 3, 4 or 6 conjugates. If x is not central, then G must be generated by elements having exactly 5 conjugates, namely the elements outside the subgroup generated by elements having 1, 2, 3, 4 or 6 conjugates. Finally, suppose that u is an element of 3-power order not commuting with an element v with 5 conjugates. Then $G/\mathrm{Core}\, C(v)$ is a transitive group of degree 5 having the non-trivial element $u\mathrm{Core}\, G(v)$ of order 3; thus it is either S_5 or A_5 and that is impossible because $\beta(G) = 6$. So every element u of 3-power order does commute with every element with 5 conjugates; since these generate G, we have that the Sylow 3-subgroup P is central. This is the final contradiction, as then $G = P \times R$ with R a $3'$-group and so by 1.1, G cannot have an element with 6 conjugates. Thus the elements of 5-power order are central, which is all that is needed to complete the proof of part (2) of the lemma. All Sylow p-subgroups with $p > 3$ are central and the direct decomposition stated in (2) is obvious.

Thus $G' = A'$ and $\beta(G) = \beta(A)$, so we have reduced the problem to one in $\{2, 3\}$-groups.

2. Proof of the Theorem.

The proof goes by induction on the group order. The theorem holds for the smallest 6-BFC groups, namely those in Examples 1.4 and 1.5. So we assume that G is of order more than 24 and that the theorem holds for smaller groups. We divide the proof into several sections. Recall that we may assume that G is a $\{2,3\}$-group.

2.1. Suppose that G is nilpotent. Then $G = A \times B$, where A is a 2-group and B a 3-group. Then A must be 2-BFC and B must be 3-BFC by 1.1, so that A' is of order 2 and B' of order 3, and $G' = A' \times B'$ is cyclic of order 6, as required.

2.2. Suppose that G has A_4 as a homomorphic image, and let N be such that $G/N \cong A_4$. We shall show that $N = Z$, the centre of G, and that $G' \cong Q_8$. The conjugacy classes of A_4 are: the identity class, the three elements of order 2 in A', and two classes consisting of four elements of order 3 outside A'. The possible conjugacy class sizes of G are 1, 2, 3, 4, 6; by 1.2, elements with 1, 2, 3 or 6 conjugates map to A_4' mod N. The set of elements of G mapping to A_4' is $G'N$, and the elements outside $G'N$ have four conjugates each and map to elements with 4 conjugates each outside A_4'. Take a in $G - G'N$. Then $|G : C(a)| = 4$ and $|G/N : C(aN)| = 4$, which means that $C(a)$ contains N. But G is generated by elements like a, so that N is central. As G/N is A_4, this means that $N = Z$, as claimed, and $G/Z \cong A_4$. Next, $G'/(G' \cap Z) \cong A_4' \cong C_2 \times C_2$. Further, $G' \cap Z$ is isomorphic to a subgroup of the Schur multiplier of A_4, that is, of C_2. Indeed it must be of order 2, else G' is of order 4, too small for a 6-BFC group. Thus G' is of order 8. Let H be a stem-group [3] in the isoclinism class of G. Then $Z(H) \subseteq H'$, $H' \cong G'$, $G/Z(G) \cong H/Z(H)$ by definition, and it is very easy to see that $\beta(G) = \beta(H)$. As H is not nilpotent, this means that Z is of order 2 and H of order 24. It is now an easy matter to check that the only group with the required properties is that in Example 1.5, and so G' is Q_8, as claimed.

Thus we may now assume from now on:

2.3. *G is not nilpotent and does not have A_4 as a homomorphic image.*

LEMMA 2.3.1. *G is not generated by the set of all elements with* 1, 2, *or* 4 *conjugates.*

PROOF. Suppose the contrary, namely that G is so generated. The elements with 1 conjugate are central. If a has 2 conjugates, then $G' \subseteq C(a)$ as $|G : C(a)| = 2$. If a has 4 conjugates, then $G/\mathrm{Core}\, C(a)$ is a transitive group on 4 symbols that is at most 6-BFC. Since A_4 is not a candidate, the only possibilities are D_8, C_4, $C_2 \times C_2$. Thus $G/\mathrm{Core}\, C(a)$ is nilpotent of class at most 2 and so $\gamma_3(G) \subseteq C(a)$ in all cases considered. But then $\gamma_3(G)$ is central and G is nilpotent, which is not allowed. This proves the lemma.

Thus G is generated by the set of all elements with 3 or 6 conjugates, and we prove:

LEMMA 2.3.2. *The Sylow subgroups of G/Z are elementary abelian, and the derived group is a 3-group.*

PROOF. As before, we may assusme that $Z \subseteq G'$; that is, we replace G if necessary by the stem-group of its isoclinism class. Let S be the set of all elements with 3 or 6 conjugates. Then $Z = \bigcap_{g\in S} C(g) = \bigcap_{g\in S} \operatorname{Core} C(g)$, so that G/Z is a subgroup of the direct product $\operatorname{Dr}_{g\in G} G/\operatorname{Core} C(g)$; so all we have to do is to establish that each $G/\operatorname{Core} C(g)$ has elementary abelian Sylow subgroups and that the derived group is a 3-group.

When g has 3 conjugates, the result is clear since the choices for $G/\operatorname{Core} C(g)$ are C_3 and S_3. When g has 6 conjugates, $G/\operatorname{Core} C(g)$ is a soluble transitive group of degree 6 which is at most 6-BFC, and the only such groups are (abstractly) A_4, S_3, $S_3 \times C_2$, $C_2 \operatorname{wr} C_3$, $C_3 \operatorname{wr} C_2$, C_6. But $C_2 \operatorname{wr} C_3$ maps to A_4, so in our case $G/\operatorname{Core} C(g)$ must be one of C_6, S_3, $S_3 \times C_2$, $C_3 \operatorname{wr} C_2$ and so it has Sylow subgroups of the required type.

To sum up: G is a $\{2,3\}$-group, not nilpotent, does not map to A_4, the Sylow subgroups are elementary abelian, and $(G/Z)'$ is a 3-group.

CASE 1. $Z \neq 1$. Then G/Z is smaller than G and is not nilpotent. Further, it not 2-BFC nor 4-BFC since the derived groups of such groups are 2-groups [8]. Further, it is not 6-BFC either, as if it were the induction hypothesis would give that $(G/Z)'$ is C_6 or Q_8, neither of which are 3-groups. Thus G/Z is 3-BFC and by [8] again, $(G/Z)'$ is of order 3. In particular, G' is abelian.

Suppose first that Z contains an element x of order 2. If $Z = \langle x\rangle$, we have $G' \cong C_6$ as $G'/Z \cong C_3$, and all is well. If $\langle x\rangle \neq Z$, then $G'/\langle x\rangle$ is of order more than 3 and so $G/\langle x\rangle$ is 6-BFC; it cannot be 3-BFC since its derived group is of order more than 3, and it cannot be 2-BFC nor 4-BFC as $G'/\langle x\rangle$ is not a 2-group. The induction hypothesis now applies to give that $G'\langle x\rangle$ is C_6, since it cannot be Q_8 as G is metabelian. Thus G' is of order at most 12. If G' as order less that 12, it must have order 6 and so it is C_6 since S_3 is not a derived group. Thus we may assume that G' is of order 12, and we can write $G' = A \times Z$, where A is of order 3 and the centre Z of G is of order 4. Then $(G/A)'$ is of order 4, so G/A is 4-BFC; it is also nilpotent of class 2 since its derived group is ZA/A. The nilpotent residual of G is contained in A, and indeed it must be A since A is of order 3 and G is not nilpotent. By [2], G splits over A, say $G = AU$, where $A \cap U = 1$. Note that $U \cong G/A$, so U is 4-BFC and U is nilpotent of class 2. The Sylow 3-subgroup X of U is abelian and central in U, being a direct factor; since A is normal and of order 3, X centralizes A. Thus X is in the centre of G and is therefore trivial since the centre is of order 4. So U is a 2-group. By Lemma 2.3 of [7], U is generated by elements with four conjugates; since A does not centralize U, it must fail to centralize an element u in U with four U-conjugates. Let a be a generator of A. Since $[a, u]$ is not 1, it must be a since $a^u = a^{-1}$. Thus u does not commute with any element of G of the form av, with v in U; that is, the G-centralizer of u is in U and thus it is the U-centralizer of u. This is a contradiction: $|G : C_G(u)| = |G : C_U(u)| = = 3|U : C_U| = 12$, which gives u 12 conjugates, impossible as G is 6-BFC.

So still in Case 1, we may assume that Z is a 3 -group and thus that G' is a 3-group since G'/Z is of order 3. We shall show that Z must be of order 3. If not, there is an element y of order 3 such that $\langle y\rangle \neq Z$; as above, the factor-group $G/\langle y\rangle$ must be 6-BFC and by induction this is a contradiction since G' is a 3-group. Thus Z is of order 3 and G' of order 9.

If G' is cyclic, say $G' = \langle t\rangle$, then $Z = \langle t^3\rangle$. For every g in G, $t^G = t^m$ for some integer m and so $t^3 = (t^3)^g = t^{3m}$, which means that $9|3(m-1)$ and $3|(m-1)$, say $m = 3r+1$ and then $[t, g] = t^{-1}t^g = t^{3m} \in Z$. Thus $[G', G]$ is central, a contradiction as G is not nilpotent. Thus G' is not cyclic. There are two possibilities for the nilpotent residual of G. It is either G' or a non-central subgroup of order 3 in G'. We deal with the two cases separately.

Suppose first that G' is the nilpotent residual of G. By [2] again, G splits over G', say $G = G'U$ where $G' \cap U = 1$ and thus U is abelian. We shall show that G has order at most 54 in these circumstances. The 6 non-central elements in G' can split into G-conjugacy classes only of the following sizes: 6; 3, 3; 4, 2; 2, 2, 2. Suppose that G' contains an element x with just two conjugates. Then $|G : C(x)| = 2$ and thus $|UC(x) : C(x)|$ is at most 2, that 2, that is, $|U : U \cap C(x)|$ is at most 2. But $G' = \langle Z, x\rangle$ so $C(G') = C(x)$ and thus $U \cap C(G')$ has index at most 2 in U. But U is abelian and G is generated by U and G', so that $U \cap C(G')$ is central. But $U \cap Z = 1$, so U has order at most 2 and G has order at most 18. Such a group cannot be 6-BFC since it has centre of order 3: all centralizers are bigger than the centre. If there is a conjugacy class of size 3, a similar argument shows that G has order 27, impossible as groups of that order cannot be 6-BFC. Thus we may assume that the 6 non-central elements in

G' form a conjugacy class, and an argument like the one above shows that U has order at most 6, and the only non-trivial case is where it has order 6 and so G has order $9.6 = 54$. A rather fussy argument now completes the proof. If u is an element of U of order 2, then its centralizer has index a 3-power and therefore index 3; so u centralizes a subgroup X of order 9. If the Sylow 3-subgroup P is abelian, this means that X is central, impossible as Z has order 3. Otherwise P is one of the two non-abelian groups of order 27, and it is readily proved that it does not have an automorphism of order 2 such that the splitting extension of P by it produces a group with the required properties.

Thus we may assume that the nilpotent residual V has order 3, and thus $G' = Z \times V$. Again by [2], $G = VU$ for some subgroup U with $V \cap U = 1$. Then $G' = V'U'[V, U] = VU'$ and U' is of order 3, so U is 3-BFC. Some element u of U with 3 conjugates in U must fail to commute with a generator v of V. It follows as above that $C_G(u) = C_U(u)$ and so $|G : C_G(u)| = 9$, false as G is 6-BFC. This completes Case 1.

Case 2. $Z = 1$.

By Lemma 2.3.2, G has elementary abelian Sylow subgroups and G' is a 3-group. Let P be the Sylow 3-subgroup. Then $P = G' \times L$ for some subgroup L. By Maschke's theorem, L can be chosen to be normal in G. But then $L = 1$ since L is a normal subgroup missing G' and therefore central, and we have $G' = P$. Next, $C(G')$ is nilpotent and therefore of the form $G' \times X$, where X is a 2-group characteristic in G' and therefore normal in G. Since it misses G', it too is trivial and so $C(G') = G'$.

We claim that G' contains a normal subgroup of order 3. Let M be a maximal subgroup of G containing G'. Then M is normal and thus of index 2. It is smaller than G and therefore, by the induction hypothesis, it is not 6-BFC since its derived group is a 3-group. It is not 4-BFC nor 2-BFC, because such groups have 2-group derived groups. If it is 1-BFC, that is, abelian, then M is G' because its 2-part is normal and therefore trivial. So when M is abelian, G is $G'\langle a\rangle$, where a is of order 2. The centralizer of a has 3-power index and so is of index 3; as G' is evidently of order more than 3 (at least 6 since G is 6-BFC), this means that a centralizes a non-trivial subgroup Y of G'. But Y is central since it centralizes a and G', and this is a contradiction. Thus M must be 3-BFC, its derived group M' is the normal subgroup of G of order 3 that we claimed exists. Note that G/M' is smaller than G, and the by now familiar argument shows that it is 3-BFC, which means that G'/M' is of order 3 and G' of order 9.

Further, M' is a direct factor of G', and so by Maschke again, there is a G-normal subgroup A such that $G' = M' \times A$. Further, $G/C(G')$ is an elementary 2-group; as a subgroup of $\mathrm{Aut}(C_3 \times C_3)$, it has order at most 4 and G has order at 18 or 36 since $C(G') = G'$. When G has order 18, we have $G = G'\langle a \rangle$ for some element a of order 2; as in the previous case, G has non-trivial centre and this is a contradiction. Suppose finally that G' has order 36. We have that G' is the direct product of two G-normal subgroups $\langle a \rangle, \langle b \rangle$, of order 3. A Sylow 2-subgroup of G is a four-group $\langle c, d \rangle$. As above, c has centralizer of index 3 and must centralize a subgroup of order 3, generating the centre of $\langle a, b, c \rangle$ and therefore normal in G. Without loss, we may assume that c centralizes a; since a is not central, (conjugation by) d must invert a. Since c is not central, it must invert b. If d inverts b, then d inverts everything in G' and so has centralizer of order 4, meaning that it has 9 conjugates, which is impossible. Thus d centralizes b and inverts a. But then cd inverts a and b and so has too many conjugates. This completes the Case 2 and the theorem is proved.

The next lowest value of n for which the conjecture is not known is $n = 8$. To confirm it in this case would be a much longer undertaking than that in this short note.

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