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Abelian Groups that cannot be Factored Without Periodic Factor.

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ABSTRACT - The list of the finite abelian groups that cannot be factored into two of its subsets without one factor being periodic is the same as the list of the finite abelian groups that cannot be factored into any number of its subsets without one factor being periodic.

1. Introduction.

Throughout this paper we will use multiplicative notation for abelian groups. Let G be a finite abelian group. We denote the identity element of G by e. Let $B, A_1, \ldots, A_n$ be subsets of G. We define the product $A_1 \cdots A_n$ to be

$$\{a_1 \cdots a_n : a_1 \in A_1, \ldots, a_n \in A_n\}.$$

Suppose $B = A_1 \cdots A_n$. We say that the product $A_1 \cdots A_n$ is *direct* if each b in B is uniquely expressible in the form

$$b = a_1 \cdots a_n, \qquad a_1 \in A_1, \ldots, a_n \in A_n.$$

If B is a direct product of $A_1, \ldots, A_n$, then the equation $B = A_1 \cdots A_n$ is said to be a *factorization* of B.

A subset A of a finite abelian group G will be called *normalized* if $e \in A$. The factorization $G = A_1 \cdots A_n$ is called normalized if each factor is normalized. We say that A is *periodic* if there is an element $a \in G$ such that $a \neq e$ and $aA = A$. The element a is called a *period* of A. Note that if a and b are periods of A, then ab is a period of A unless $ab = e$. It follows that the

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periods of A augmented with the identity element form a subgroup H of G. Moreover there is a subset B of G such that $A = BH$ is a factorization of A. If the group G is a direct product of cyclic groups of orders $t_1, \dots, t_s$ respectively, then we express this fact shortly saying that G is of *type* $(t_1, \dots, t_s)$. If from the factorization $G = A_1 \cdots A_n$ it follows that one of the factors is periodic for any possible choice of the factors $A_1, \dots, A_n$, then we say that G has the *H_n-property*. (H_n is an abbreviation for the Hajós property with n factors.) Here we do not allow factors with only one element and do not consider groups with only one element.

In 1949, G. Hajós [4] called for determining all finite abelian groups with the H_2-property. The complete list of these groups first appeared in 1962 A. D. Sands [8].

$$
\begin{array}{llll}
(p^a, q), & (p^2, q^2), & (p^2, q, r), & (p, q, r, s) \\
(p^3, 2, 2), & (p^2, 2, 2, 2), & (p, 2^2, 2), & (p, 2, 2, 2, 2), \\
(p, q, 2, 2), & (p, 3, 3), & (3^2, 3), & (2^a, 2), \\
 & (2^2, 2^2), & (p, p). &
\end{array} \tag{1}
$$

Here p, q, r, s are distinct primes the $p = 2$ and $p = 3$ cases are not excluded and $a \geq 1$ is an integer. Groups whose type is on list (1) and their subgroups have the H_2-property and other groups do not have the H_2-property.

This result has applications in various fields. In geometry [11], combinatorics [5], coding theory [3], number theory [13], Fourier analysis [6].

If a group has the H_n-property for each possible choice of n, then we say that G has the *H-property*. It is plain that groups with the H-property have the H_2-property. We will show that groups with the H_2-property are in fact the same as groups with the H-property. We express this fact more formally as a theorem.

THEOREM 1. *Let G be a finite abelian group. If G has the H_2-property, then G has the H-property.*

2. Replacing factors.

We say that in the factorization $G = AB$ the factor B can be *replaced* by B' if $G = AB'$ is also a factorization of G. If c is an element of G, then multiplying both sides of the factorization by c we get the factorization $G = Gc = A(Bc)$. In other words B can be replaced by Bc for each $c \in G$.

Note that B is periodic if and only if Bc is periodic. Let $G = A_1 \cdots A_n$ be a factorization. Let $a_1 \in A_1, \ldots, a_n \in A_n$ multiplying the factorization by $a_1^{-1} \cdots a_n^{-1}$ we get the normalized factorization $G = (a_1^{-1}A_1) \cdots (a_n^{-1}A_n)$. Thus when we study factorizations with periodic factors we may restrict our attention to normalized factorizations.

Note that $G = AB$ is a factorization if and only if $|G| = |A||B|$ and $AA^{-1} \cap BB^{-1} = \{e\}$. From this it is plain that B can be replaced by B^{-1}. Let $G = AB$ be a factorization of G. Then each $g \in G$ is uniquely expressible in the form $g = ab$, $a \in A$, $b \in B$. We call a the *A-coordinate* of g and we denote it by $\alpha(g)$. Similarly, we call b the *B-coordinate* of g and we denote it by $\beta(g)$. The coordinates of g make sense only relative to the factorization $G = AB$. If A is a subset of a finite abelian group G and q is an integer, then A^q will denote the set $\{a^q : a \in A\}$.

LEMMA 1. *Let $G = AB$ be a factorization of G and let $A = \{a_1, \ldots, a_n\}$. For each $g \in G$ the elements $\alpha(ga_1), \ldots, \alpha(ga_n)$ form a permutation of $a_1, \ldots, a_n$.*

PROOF. Clearly, $\alpha(ga_i) \in A$. So we will show that $\alpha(ga_i) = \alpha(ga_j)$ implies $a_i = a_j$. From the equations

$$ga_i = \alpha(ga_i)\beta(ga_i), \quad ga_j = \alpha(ga_j)\beta(ga_j)$$

we get the equations

$$g = \alpha(ga_i)\beta(ga_i)a_i^{-1}, \quad g = \alpha(ga_j)\beta(ga_j)a_j^{-1}.$$

Then $\beta(ga_i)a_i^{-1} = \beta(ga_j)a_j^{-1}$ and $a_j\beta(ga_i) = a_i\beta(ga_j)$. Now as $a_i, a_j \in A$, $\beta(ga_i), \beta(ga_j) \in B$, from the factorization $G = AB$ it follows that $a_i = a_j$.

This completes the proof.

LEMMA 2. *Let $G = AB$ be a factorization of G and let q be a prime such that $q \nmid |A|$. Then $G = A^qB$ is a factorization of G.*

PROOF. Choose an $a \in A, g \in G$ and define T to be the set of all q tuples

$$(x_1, x_2, \ldots, x_q), \quad x_1, x_2, \ldots, x_q \in A$$

for which $\alpha(gx_1x_2 \cdots x_q) = a$. First note that $|T| = |A|^{q-1}$. Indeed, choose $x_1, x_2, \ldots, x_{q-1} \in A$ arbitrarily, then by Lemma 1, $\alpha[(gx_1x_2 \cdots x_{q-1})x_q] = a$ has a unique solution for x_q. Next note that if $(x_1, x_2, \ldots, x_q) \in T$, then $(x_2, \ldots, x_q, x_1) \in T$. We define a graph Γ. The vertices of Γ are the elements of T and we draw an arrow from the node $(x_1, x_2, \ldots, x_q)$ to the node

$(x_2, \dots, x_q, x_1)$. The graph Γ is a union of disjoint cycles. The cycles are of length 1 or of length q. When $x_1 = x_2 = \dots = x_q$, then the node $(x_1, x_2, \dots, x_q)$ is on a cycle of length 1. When $x_1, x_2, \dots, x_q$ are not all equal, then the node $(x_1, x_2, \dots, x_q)$ is on a cycle of length q. As $q \not| |A|$ there must be a cycle of length 1 in Γ. In other words there is an $x_1 \in A$ such that $\alpha(gx_1^q) = a$.

Consider the factorization $G = AB$. From $gx_1^q = \alpha(gx_1^q)\beta(gx_1^q)$ using $\alpha(gx_1^q) = a$ we get $gx_1^q = a\beta(gx_1^q)$, then $g = ax_1^{-q}\beta(gx_1^q)$. Here $ax_1^{-q} \in aA^{-q}$, $\beta(gx_1^q) \in B$ and the equation holds for each $g \in G$. It follows that $G = (aA^{-q})B$. Then $G = A^{-q}B$. Note that $|G| = |A||B|$, $|A^{-q}| \leq |A|$ imply $|G| = |A^{-q}||B|$ and so $G = A^{-q}B$ is a factorization. Now A^{-q} can be replaced by A^q and so it follows that $G = A^qB$ is a factorization.

This completes the proof.

LEMMA 3. *Let $G = AB$ be a factorization such that $e \in A$, $|A| = p$ is a prime. Then $G = A'B$ is a factorization of G, where $A' = \{e, a, a^2, \dots, a^{p-1}\}$, $a \in A \setminus \{e\}$.*

PROOF. Note that $G = A^tB$ is a factorization of G whenever $t \geq 2$, $p \not| t$. Indeed, as t is a product of primes we can apply Lemma 2 several times starting with the factorization $G = AB$. Let $A = \{e, a_1, a_2, \dots, a_{p-1}\}$. The fact that $G = A^tB$ is a factorization is equivalent to that the sets

$$eB, a_1^tB, a_2^tB, \dots, a_{p-1}^tB$$

form a partition of G. Set $A' = \{e, a_k, a_k^2, \dots, a_k^{p-1}\}$. The fact that $G = A'B$ is a factorization is equivalent to that the sets

$$eB, a_kB, a_k^2B, \dots, a_k^{p-1}B$$

form a partition of G. Since G is finite it is enough to show that $a_k^iB \cap a_k^jB = \emptyset$ for each i, j, $0 \leq i < j \leq p-1$. Assume the contrary that $a_k^iB \cap a_k^jB \neq \emptyset$. Multiplying by a_k^{-i} we get $eB \cap a_k^{j-i}B \neq \emptyset$. Set $t = j - i$. Clearly, $1 \leq t \leq p-1$ and so t is prime to p. Now $eB \cap a_k^tB \neq \emptyset$ contradicts the fact that $G = A^tB$ is a factorization of G.

This completes the proof.

3. Periodicity forcing factorization types.

If $G = A_1 \cdots A_n$ is a factorization of G and $|A_1| = q_1, \dots, |A_n| = q_n$, then we call the n tuple $(q_1, \dots, q_n)$ the *type* of the factorization. If from each $G = A_1 \cdots A_n$ factorization of type $(q_1, \dots, q_n)$ it follows that at least one of the factors is always periodic we call $(q_1, \dots, q_n)$ a *periodicity forcing* factorization type for G. In 1965 L. Rédei [7] proved that a factorization type $(q_1, \dots, q_n)$ is always periodicity forcing if $q_1, \dots, q_n$ are primes.

LEMMA 4. *Let $q_1 = p_1 \cdots p_s$, where $p_1, \dots, p_s$ are primes. If $(q_1, \dots, q_n)$ is a periodicity forcing factorization type for an abelian group, then so is $(p_1, \dots, p_s, q_2, \dots, q_n)$.*

PROOF. Suppose that $(q_1, \dots, q_n)$ is a periodicity forcing factorization type for the finite abelian group G and consider a normalized factorization

$$G = B_1 \cdots B_s A_2 \cdots A_n, \tag{2}$$

where

$$|B_1| = p_1, \dots, |B_s| = p_s, |A_2| = q_2, \dots, |A_n| = q_n.$$

We would like to show that at least one of the factors $B_1, \dots, B_s, A_2, \dots, A_n$ is periodic. If one of the factors is periodic then there is nothing to prove so we assume that none of the factors is periodic.

If the order of each element in B_i is a power of p_i, then we define C_i to be B_i. Assume that there are elements in B_i whose order is not a power of p_i. In factorization (2), by Lemma 2, B_i can be replaced by a normalized subset C_i such that the orders of each element in C_i is a product of at most two distinct primes and C_i is not a subgroup of G. Note that as $|C_i|$ is a prime and $e \in C_i$, C_i is periodic if and only if C_i is a subgroup of G. Setting $C = C_1 \cdots C_s$ we get a factorization $G = CA_2 \cdots A_n$. The type of this factorization is $(q_1, q_2, \dots, q_n)$. So by our assumption one of the factors $C, A_2, \dots, A_n$ is periodic. If one of $A_2 \dots, A_n$ is periodic, then we are done so we assume that C is periodic. Now a periodic subset C of G is factored into subsets $C_1, \dots, C_s$. In addition, there is a subset A of G such that $G = CA$ is a factorization. Simply set $A = A_2 \cdots A_n$. The hypotheses of Theorem 2 of [2] are satisfied therefore this theorem gives that at least one of the factors $C_1, \dots, C_s$ must be periodic.

This contradiction completes the proof.

If A is a subset and χ is a character of G, then we will use the notation $\chi(A)$ to denote the sum

$$\sum_{a \in A} \chi(a).$$

In this paper character of G always means irreducible character of G. The set of all characters χ of G with $\chi(A) = 0$ is called the *annihilator* set of A and will be denoted by $\mathrm{Ann}(A)$.

Lemma 5. *Let G be a finite abelian group with the H_2-property and let $q_1 = p$ be a prime. In the $p \geq 3$ case we assume that the p-component of G is cyclic. (In the $p = 2$ case nothing is assumed about the p-component of G.) Then (q_1, q_2, q_3) is a periodicity forcing factorization type for G.*

Proof. Let $G = A_1A_2A_3$ be a normalized factorization of G such that $|A_1| = q_1$, $|A_2| = q_2$, $|A_3| = q_3$. We would like to show that one of the factors A_1, A_2, A_3 is periodic. If one of A_1, A_2, A_3 is periodic, then there is nothing to prove. So we assume that none of the factors is periodic. Assume that $p \geq 3$. In this case the p-component of G is cyclic. By Lemma 3, A_1 can be replaced by $A_1' = \{e, a, a^2, \dots, a^{p-1}\}$ for each $a \in A_1 \setminus \{e\}$. If A_1' is a subgroup of G for each $a \in A_1 \setminus \{e\}$, then as the p-component of G is cyclic, A_1 is equal to the unique subgroup of G that has p elements. But A_1 is not a subgroup. This contradiction gives that for some $a \in A_1 \setminus \{e\}$ the subset A_1' is not a subgroup. This is equivalent to $a^p \neq e$. For the remaining part of the proof we choose an A_1' which is not a subgroup. In the $p = 2$ case we set $A_1' = A_1 = \{e, a\}$. Here $a^2 \neq e$ as A_1 is not a subgroup. Thus $a^p \neq e$ holds in both of the cases $p = 2$ or $p \geq 3$.

From the factorization $G = (A_1'A_2)A_3$, by the H_2-property of G, it follows that $A_1'A_2$ or A_3 is periodic. Since A_3 is not periodic, $A_1'A_2$ is periodic, say with period g, that is, $A_1'A_2g = A_1'A_2$. We claim that $\chi(A_2) = 0$ holds for all characters χ of G with $\chi(g) \neq 1$ and $\chi(a^p) \neq 1$. In order to prove the claim assume that $\chi(g) \neq 1$ and $\chi(a^p) \neq 1$. As $\chi(g) \neq 1$ from $\chi(A_1'A_2)\chi(g) = \chi(A_1'A_2)$ we get $0 = \chi(A_1'A_2) = \chi(A_1')\chi(A_2)$. From $\chi(a^p) \neq 1$ it follows that $\chi(A_1') \neq 0$. Therefore $\chi(A_2) = 0$. The fact that $\chi(a^p) \neq 1$ and $\chi(g) \neq 1$ imply $\chi(A_2) = 0$ is equivalent to

$$\operatorname{Ann}\big(\langle a^p\rangle\big) \cap \operatorname{Ann}\big(\langle g\rangle\big) \subset \operatorname{Ann}(A_2).$$

By Theorem 2 of [10], there are subsets X, Y of G such that

$$A_2 = X\langle a^p\rangle \cup Y\langle g\rangle,$$

where the products are direct and the union is disjoint. Similarly, from the factorization $G = A_2(A_1'A_3)$ it follows that $A_1'A_3$ is periodic, say with period h. Then there are subsets U, V of G such that

$$A_3 = U\langle a^p\rangle \cup V\langle h\rangle,$$

where the products are direct and the union is disjoint.

If $X = \emptyset$, then A_2 is periodic with period g. If $U = \emptyset$, then A_3 is periodic with period h. So we may assume that $X \neq \emptyset$ and $U \neq \emptyset$. Choose $x \in X$,

$u \in U$. Multiply the factorization $G = A_1A_2A_3$ by $g = x^{-1}u^{-1}$ to get the factorization

$$G = Gg = A_1(A_2x^{-1})(A_3u^{-1}).$$

Now $\langle a^p \rangle \subset A_2x^{-1}$, $\langle a^p \rangle \subset A_3u^{-1}$ contradict the definition of the factorization.

This completes the proof.

4. Proof of Theorem 1.

We consider a finite abelian group G with the H_2-property. Thus G is a subgroup of a group whose type is on list (1) and we try to establish that G has the H_n-property for each possible choice of n. The $n = 1$ case is trivial so we assume that $n \geq 2$.

Let G be a subgroup of a group of type (p^a, q) and let $G = A_1 \cdots A_n$ be a factorization of G. If G is a p-group, then $|A_1|, \dots, |A_n|$ are powers of p. If G is not a p-group, then q divides one of $|A_1|, \dots, |A_n|$ and q can divide only one of $|A_1|, \dots, |A_n|$. Say $q \big| |A_1|$ and plainly each of $|A_2|, \dots, |A_n|$ must be a power of p. Now the p-component of G is cyclic and each of $|A_2|, \dots, |A_n|$ is a power of p. The conditions of Theorem 2 of [9] are met. By this theorem, one of the factors $A_1, \dots, A_n$ is periodic and so G has the H_n-property. Since $n \geq 2$ was arbitrary G has the H-property.

By Theorem 1 of [12] (or by Theorem 10 of [1]), a group of type $(2^a, 2)$ has the H-property.

We inspect the remaining groups G in a case by case manner and sort out the arising cases using Rédei's theorem or Lemma 4 or Lemma 5.

(1) Suppose the type of G is one of the following

$$(3) \qquad (p^3, 2, 2), \quad (p^2, 2, 2, 2), \quad (p, 2, 2, 2, 2).$$

Note that $|G|$ is a product of 5 primes. This is why we treat these cases together. Let $G = A_1 \cdots A_n$ be a factorization of G. If the type of the factorization is $(q_1, q_2, q_3, q_4, q_5)$, then each q_i is a prime and Rédei's theorem gives that one of the factors is periodic. If the factorization type is (q_1, q_2, q_3, q_4), then only one q_i can be composite, say q_4. The H_2-property of G gives that $(q_1q_2q_3, q_4)$ is a periodicity forcing factorization type for G. Now by Lemma 4, (q_1, q_2, q_3, q_4) is a periodicity forcing factorization type for G. If the factorization type is (q_1, q_2, q_3), then at least one of the q_i's is a prime, say q_1 is a prime, and Lemma 5 applies. If G is a proper subgroup of a group whose type is on list (3), then we can use a similar but to some extent simpler argument.

(2) Let us assume that the type of G is one of the next

$$(4)\qquad \begin{array}{lll} (p^2,q^2) & (p^2,q,r) & (p,q,r,s) \\ (p,4,2) & (p,q,2,2) & (2^2,2^2). \end{array}$$

Note that $|G|$ is a product of 4 primes. Let $G = A_1 \cdots A_n$ be a factorization of G. If the factorization type is (q_1, q_2, q_3, q_4), then Rédei's theorem implies that this factorization type is periodicity forcing. If the factorization type is (q_1, q_2, q_3), then Lemma 4 is applicable. If G is a proper subgroup of a group whose type is on list (4), then an analogous but slightly simpler reasoning can be used.

(3) The case when G is of type $(p,3,3)$, $(3^2,3)$, (p,p), that is, when $|G|$ is a product of at most 3 primes is rather trivial.

This inspection completes the proof.

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