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On the Chow Rings of Classifying Spaces for Classical Groups.

LUIS ALBERTO MOLINA ROJAS (*) - ANGELO VISTOLI (**)

ABSTRACT - We show how the stratification method, introduced by Vezzosi in his study of PGL_3, provides a unified approach to the known computations of the Chow rings of the classifying spaces of GL_n, SL_n, Sp_n, O_n and SO_n.

1. Introduction.

To algebraic topologists, the cohomology of classifying spaces of linear algebraic groups (or, equivalently, of compact Lie groups) has been an important object of study for a long time. Recently, B. Totaro ([Tot99]) has introduced an algebraic analogue of this cohomology, the Chow ring of the classifying space of a linear algebraic group G, denoted by A_G^*. If H_G^* denotes the integral cohomology of the classifying space of G, there is a natural ring homomorphism $A_G^* \to H_G^*$, which is, in general, neither surjective not injective.

Remarkably, the Chow ring A_G^* seems to be smaller and easier to control than H_G^*, while still containg a lot of information: for example, if G is a finite abelian group A_G^* is the symmetric algebra on the group of characters, while H_G^* is much larger (unless G is cyclic). This is truly surprising to someone who is familiar with the theory of Chow rings of smooth projective algebraic varieties, because these tends to be infinitely more complicated and less computable than their cohomology.

Rationally, the situation is very well understood. If G is a connected algebraic group, then the homomorphism $A_G^* \otimes \mathbb{Q} \to H_G^* \otimes \mathbb{Q}$ is an isomorphism, and both rings coincide with the ring of invariants under the

(*) Indirizzo dell'A.: Dipartimento di Matematica, Università di Roma Tre, Largo San Leonardo Murialdo 1, I-00146 Roma, Italy; e-mail: molina@mat.uniroma3.it

(**) Indirizzo dell'A.: Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato 5, I-40126 Bologna, Italy; e-mail: vistoli@dm.unibo.it

Weyl group in the symmetric algebra of the ring of characters of a maximal torus; this is classical, due to Leray and Borel, in the case of cohomology, and to Edidin and Graham ([EG97]) for the Chow ring. Furthermore, this ring of invariants is always a polynomial ring, as was shown by Chevalley. With integral coefficients, the situation is much more subtle.

The Chow ring A^*_G has been computed for the classical groups GL_n, SL_n, Sp_n, O_n or SO_n, but not for the PGL_n series. The results are as follows. Each of the groups above comes with a tautological representation, of dimension n (or $2n$, in the case of Sp_n). Every representation V of an algebraic group G has Chern classes $\mathrm{c}_i(V) \in \mathrm{A}^i_G$. When G is a classical group, we denote the Chern classes of the tautological representation simply by c_i.

Burt Totaro ([Tot99]) and R. Pandharipande ([Pan98]) described A^*_G when $G = \mathrm{GL}_n$, SL_n, Sp_n, O_n and SO_n when n is odd. We will use the following notation: if R is a ring, $t_1, \dots, t_n$ are elements of R, $f_1, \dots, f_r$ are polynomials in $\mathbb{Z}[x_1, \dots, x_n]$, we write

$$R = \mathbb{Z}[t_1, \dots, t_n]/\bigl(f_1(t_1, \dots, t_n), \dots, f_r(t_1, \dots, t_n)\bigr)$$

to indicate the the ring R is generated by $t_1, \dots, t_n$, and the kernel of the evaluation map $\mathbb{Z}[x_1, \dots, x_n] \to R$ sending x_i to t_i is generated by $f_1, \dots, f_r$. When there are no f_i this means that R is a polynomial ring in the t_i.

First the case of the special groups.

THEOREM [B. Totaro].

(1) $\mathrm{A}^*_{\mathrm{GL}_n} = \mathbb{Z}[c_1, \dots, c_n]$.
(2) $\mathrm{A}^*_{\mathrm{SL}_n} = \mathbb{Z}[c_2, \dots, c_n]$.
(3) $\mathrm{A}^*_{\mathrm{Sp}_n} = \mathbb{Z}[c_2, c_4, \dots, c_{2n}]$.

The first two cases follow very easily from the well known description via generators and relations of the Chow ring of a Grassmannian.

In all three cases, the Chow ring is isomorphic to the cohomology ring.

THEOREM [R. Pandharipande, B. Totaro]

(1) $\mathrm{A}^*_{\mathrm{O}_n} = \mathbb{Z}[c_1, \dots, c_n]/(2c_{\mathrm{odd}})$.
(2) *If n is odd, then* $\mathrm{A}^*_{\mathrm{SO}_n} = \mathbb{Z}[c_2, \dots, c_n]/(2c_{\mathrm{odd}})$.

The notation $2c_{\mathrm{odd}}$ means "all the elements $2c_i$ for i odd".

In these cases the Chow ring is not isomorphic to the cohomology ring: the ring $\mathrm{H}^*_{\mathrm{O}_n}$ was computed independently by Brown (see [Bro82]) and Feshbach ([Fes83]); the result is considerably more involved. The fact that these formulae are so simple is another manifestation of the tamer nature of A^*_G, as opposed to H^*_G.

When n is odd, then $\mathrm{O}_n \simeq \mathrm{SO}_n \times \boldsymbol{\mu}_2$, and this allows to obtain the result for SO_n from that for O_n. When n is even this fails, and the situation is more complicated. Even rationally, the Chern classes of the tautological representation do not generate the Chow ring, or the cohomology. It is well known that when $n = 2m$ the tautological representation has an Euler class $\varepsilon_m \in \mathrm{H}^{2m}_{\mathrm{SO}_n}$, whose square is $(-1)^m c_n$: this class, together with the even Chern classes $c_2, c_4, \dots, c_{n-2}$ generate $\mathrm{A}^*_{\mathrm{SO}_n} \otimes \mathbb{Q} = \mathrm{H}^*_{\mathrm{SO}_n} \otimes \mathbb{Q}$. Totaro noticed that when $n = 4$ the class ε_2 is not in the image of $\mathrm{A}^*_{\mathrm{SO}_n}$; shortly afterwards, Edidin and Graham ([EG95]) constructed a class $y_m \in \mathrm{A}^m_{\mathrm{SO}_n}$, whose image in $\mathrm{H}^*_{\mathrm{SO}_n}$ is, rationally, $2^{m-1}\varepsilon_m$.

Subsequently, Pandharipande computed $\mathrm{A}^*_{\mathrm{SO}_4}$: he showed that it is generated by c_2, c_3, c_4 and y_2, and gave the relations (his description of the class y_2 is different, but equivalent to that of Edidin and Graham). Finally, in her Ph.D. thesis Rebecca Field obtained the general result ([Fie04]), which is as follows.

THEOREM [R. Field]. *When $n = 2m$ is even, then*

$$\mathrm{A}^*_{\mathrm{SO}_n} = \mathbb{Z}[c_2, \dots, c_n, y_m] / \big(y_m^2 - (-1)^m 2^{n-2} c_n,\ 2c_{\mathrm{odd}},\ y_m c_{\mathrm{odd}}\big).$$

Furthermore there are many results, due to Totaro himself ([Tot99]), to P. Guillot ([Gui04a] and [Gui04c]), and to N. Yagita ([Yag02]), for finite groups. Chow rings of classifiying spaces of exceptional groups have been studied by Yagita (see [Yag04], [Yag05]) and Guillot ([Gui04b]).

As we already mentioned, the PGL_n series is much harder (this is an example of a universal phenomenon, that of all the classical groups, these are the ones giving rise to the deepest problems). For $n = 2$ we have that $\mathrm{PGL}_2 = \mathrm{SO}_3$, and for this group everything is well understood. The cohomology with $\mathbb{Z}/3\mathbb{Z}$ coefficients of the classifying space of PGL_3 has been computed in [KMS75], and that of PGL_n with $\mathbb{Z}/2\mathbb{Z}$ coefficients when $n \equiv 2 \pmod 4$ in [KM75] and [Tod87]; furthermore, several results on the cohomology of PGL_p with $\mathbb{Z}/p\mathbb{Z}$ coefficients were proved in [VV03]. To our knowledge, not much else was known about the cohomology of the classifying space of PGL_n with $\mathbb{Z}/p\mathbb{Z}$ coefficients, when p divides n.

Concerning the Chow ring, for $n = 3$ there is a difficult paper of G. Vezzosi ([Vez00]), where he describes $\mathrm{A}^*_{\mathrm{PGL}_3}$ almost completely. Here is his basic idea. The fundamental tool is the equivariant intersection theory that Edidin and Graham ([EG98]) have forged starting from Totaro's idea. Vezzosi stratifies the adjoint representation $\mathfrak{sl}_3$ of PGL_3 by type of Jordan canonical form, compute the Chow ring of each stratum, and then get generators for $\mathrm{A}^*_{\mathrm{PGL}_3}$ using the localization sequence for equivariant Chow groups. To get relations he restricts to appropriate subgroups of PGL_3. His technique has been refined and improved by the second author in [Vis05], where he studies the Chow ring and the cohomology of the classifying space of PGL_p, where p is an odd prime.

The purpose of this article is to show how this stratification method provides a unified approach to all the known results on the Chow ring of classical groups. Consider a classical group G with its tautological representation V. Then one stratifies V in strata in which the stabilizers are, up to an extension by a unipotent group, smaller classical group. Using the localization sequence for equivariant Chow groups this gives generators for the Chow rings, with relations that come out naturally. To show that the relations suffice, one restricts to appropriate subgroups of G: a maximal torus first, to show that the relations suffice up to torsion, then to some finite subgroup to handle torsion. This turns out to be reasonably straightforward for all the classical groups, except for PGL_n. From our point of view, this is due to the fact that the natural representation to use for PGL_n, which is the adjoint representation, has a much more complicated orbit structure then in the cases of the other groups.

For the cases of Sp_n and O_n, Totaro's proofs, based on his very interesting and important [Tot99, Proposition 14.2], are much simpler. In the case of SO_n for even n, Totaro's method, as implemented by Field, does not seem easier than the stratification method. In the case of PGL_n, the stratification method provides the best known results; but there is also a very recent preprint of Kameko and Yagita where they also compute the additive structure of $\mathrm{A}^*_{\mathrm{PGL}_n}$ and $\mathrm{H}^*_{\mathrm{PGL}_n}$, with completely different methods, using the Adams spectral sequence for Brown–Peterson cohomology ([KN05]).

A few words about the future ([1]). Despite its elementary nature, the stratification methods is powerful; also, as was pointed out to the second author by N. Yagita, it might yield interesting results when applied to more general cohomology theories. However, it seems clear that to proceed much further one will eventually need to introduce substantial amounts of homological machinery, as provided by the theory of motivic cohomology of Voevodsky and Morel. Thus, the way is indicated by the work of N. Yagita (see for example [Yag03] and [Yag05]).

([1]) The authors are well aware of the risks involved in making predictions, as people always play Chesterton's game "Cheat the Prophet".

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2. Preliminaries and notation.

In this section we recall some definitions and notations, and state some tecnical results that will be used throughout this paper.

All schemes and algebraic spaces are assumed to be of finite type over a fixed field k. All algebraic groups will be linear algebraic group schemes over k, and all representations will be k-representations. If X is a scheme over k and k' is an extension of k, we set $X_{k'} \overset{\text{def}}{=} X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$.

The notation for algebraic groups will be standard: thus $\mathbb{G}_{\mathrm{m}}$ will be the multiplicative group over k, and $\boldsymbol{\mu}_n$ the group scheme of n^{th} roots of 1.

Let G a linear algebraic group over k, and X a smooth scheme over k with a G-action.

Edidin and Graham ([EG98]), expanding on the idea of Totaro, have defined the G-equivariant Chow ring of X, denoted $\mathrm{A}^*_G(X)$, as follows. For each $i \geq 0$, choose a representation V of G with an open susbscheme $U \subset V$ on which G acts freely (in which case we call (V, U) a *good pair for* G), and such that the codimension of $V \setminus U$ is greater than i. The action of G on $X \times U$ is also free, and the quotient $(X \times U)/G$ exists as a smooth algebraic space; then Edidin and Graham define

$$\mathrm{A}^i_G(X) \overset{\text{def}}{=} \mathrm{A}^i\big((X \times U)/G\big),$$

where the right hand term is the usual Chow group of rational equivalence classes of cycles of codimension i. This is easily seen to be independent of the good pair (V, U) chosen. Then one sets

$$\mathrm{A}^*_G(X) \overset{\text{def}}{=} \bigoplus_{i \geq 0} \mathrm{A}^i_G(X).$$

If G acts freely on X, then there is a quotient X/G as an algebraic space of finite type over k, and the projection $X \to X/G$ makes X into a G-torsor over X/G; and in this case the ring $\mathrm{A}^*_G(X)$ is canonically isomorphic to $\mathrm{A}^*(X/G)$.

Totaro's definition of the Chow ring of a classifying space is a particular case of this, as

$$\mathrm{A}^*_G = \mathrm{A}^*_G(\operatorname{Spec} k).$$

The formal properties of ordinary Chow rings extend to equivariant Chow rings. We recall briefly the properties that we need, which will be used without comments in the paper, referring to [EG98] for the details.

If $f\colon X \to Y$ is an equivariant morphism of smooth G-schemes there is an induced ring homomorphism $f^*\colon \mathrm{A}^*_G(Y) \to \mathrm{A}^*_G(X)$, making A^*_G into a contravariant functor from smooth G-schemes to graded commutative rings. Furthermore, if f is proper there is an induced homomorphism of groups $f_*\colon \mathrm{A}^*_G(X) \to \mathrm{A}^*_G(Y)$; the projection formula holds. This means that if $\eta \in \mathrm{A}^*_G(Y)$ and $\xi \in \mathrm{A}^*_G(X)$, then

$$f_*(\xi \cdot f^*\eta) = f_*\xi \cdot \eta\,;$$

in other words, f_* is a homomorphism of $\mathrm{A}^*_G(Y)$-modules.

There is also a functoriality in the group: if $H \to G$ is a homomorphism of algebraic groups, the action of G on X induces an action of H on X, and there is homomorphism of graded rings $\mathrm{A}^*_G(X) \to \mathrm{A}^*_H(X)$. When H is a subroup of G we will refer to this as a *restriction homomorphism*.

If H is a subgroup of G, then there is an H-equivariant embedding X into $X \times G/H$, defined in set-theoretic terms by sending x into $(x, 1)$. Then the composite of the restriction homomorphism $\mathrm{A}^*_G(X \times G/H) \to$ $\to \mathrm{A}^*_H(X \times G/H)$ with the pullback $\mathrm{A}^*_H(X \times G/H) \to \mathrm{A}^*_H(X)$ is an isomorphism.

Of paramount importance is the localization sequence; if Y is a closed G-invariant subscheme of X, and we denote by $i\colon Y \hookrightarrow X$ and $j\colon X \setminus Y \hookrightarrow X$ the inclusions, then the sequence

$$\mathrm{A}^*_G(Y) \xrightarrow{i_*} \mathrm{A}^*_G(X) \xrightarrow{j^*} \mathrm{A}^*_G(X \setminus Y) \to 0$$

is exact.

Furthermore, if E is a G-equivariant vector bundle on X, there are Chern classes $\mathrm{c}_i(E) \in \mathrm{A}^i_G(X)$, enjoying the usual properties. Also, the pullback $\mathrm{A}^*_G(X) \to \mathrm{A}^*_G(E)$ is an isomorphism.

In particular, since the equivariant vector bundles over $\operatorname{Spec} k$ are the representations of G, we get Chern classes $\mathrm{c}_i(V) \in \mathrm{A}^i_G$ for every representation of G; and the pullback $\mathrm{A}^*_G \to \mathrm{A}^*_G(V)$ is an isomorphism.

We also need other easy properties of equivariant Chow rings, for which we do not have a suitable reference.

LEMMA 2.1. *Let G a linear algebraic group, X a smooth G-scheme, H a normal algebraic subgroup G. Suppose that the action of H on X is free with quotient X/H. Then there is canonical isomorphism of graded rings*

$$\mathrm{A}^*_G(X) \simeq \mathrm{A}^*_{G/H}(X/H).$$

PROOF. Let (V, U) be a good pair for G, such that the codimension of $V \setminus U$ is greater then i. Then

$$\begin{aligned}\mathrm{A}^i_G(X) &= \mathrm{A}^i\big((X \times U)/G\big)\\ &= \mathrm{A}^i\big(((X \times U)/H)/(G/H)\big)\\ &= \mathrm{A}^i_{G/H}\big((X \times U)/H\big).\end{aligned}$$

Now, the quotient $(X \times V)/H$ is a G/H-equivariant vector bundle over X/H, $(X \times U)/H$ is an open subscheme of $(X \times V)/H$ whose complement has codimension larger than i. This yields isomorphisms

$$\begin{aligned}\mathrm{A}^i_{G/H}\big((X \times U)/H\big) &\simeq \mathrm{A}^i_{G/H}\big((X \times V)/H\big)\\ &\simeq \mathrm{A}^i_{G/H}(X/H).\end{aligned}$$

The resulting isomorphisms $\mathrm{A}^i_G(X) \simeq \mathrm{A}^i_{G/H}(X/H)$ yield the desired ring isomorphism $\mathrm{A}^*_G(X) \simeq \mathrm{A}^*_{G/H}(X/H)$. ∎

LEMMA 2.2. *Let G be an affine linear group acting on a smooth scheme X, $E \to X$ an equivariant vector bundle of rank r. Call $E_0 \subseteq E$ the complement of the zero section of E. Then the pullback homomorphism $\mathrm{A}^*_G(X) \to \mathrm{A}^*_G(E_0)$ is surjective, and its kernel is generated by the top Chern class $\mathrm{c}_r(E) \in \mathrm{A}^r_G(X)$.*

PROOF. Call $s: X \to E$ the zero-section. Then the statement follows immediately from the exactness of the localization sequence

$$\mathrm{A}^*_G(X) \xrightarrow{s_*} \mathrm{A}^*_G(E) \to \mathrm{A}^*_G(E_0) \to 0,$$

from the fact that the pullback $s^*: \mathrm{A}^*_G(E) \to \mathrm{A}^*_G(X)$ is an isomorphism, and from the self-intersection formula, which implies that the composite $\mathrm{A}^*_G(X) \xrightarrow{s_*} \mathrm{A}^*_G(E) \xrightarrow{s_*} \mathrm{A}^*_G(X)$ is multiplication by $\mathrm{c}_r(E)$. ∎

LEMMA 2.3. *Let H a linear algebraic group with an isomorphism $\phi: H \simeq \mathbb{A}^n_k$ of varieties, such that for any field extension $k \subseteq k'$ and any $h \in H(k')$, the automorphism of $\mathbb{A}^n_{k'}$ that corresponds under ϕ to the action of h on $H_{k'}$ by left multiplication is affine. Furthermore, let G be a linear* *algebraic group acting on H via group automorphisms, corresponding to linear automorphisms of $\mathbb{A}^n_k$ under ϕ.*

If G acts on a smooth scheme X, form the semidirect product $G \ltimes H$, and let $G \ltimes H$ act on X via the projection $G \ltimes H \to G$. Then the homomorphism

$$\mathrm{A}^*_G(X) \to \mathrm{A}^*_{G \ltimes H}(X)$$

induced by the projection $G \ltimes H \to G$ is an isomorphism.

PROOF. Let (V, U) (resp. (V', U')) be a good pair for $G \ltimes H$ (resp. G). Then $G \ltimes H$ acts on U' via the projection $G \ltimes H \to G$: it follows that $G \ltimes H$ acts on $X \times H \times U \times U'$, and since the action of $G \ltimes H$ on H is transitive, and the stabilizer of the origin is G, there is an isomorphism

$$\begin{aligned}(X \times H \times U \times U')/(G \ltimes H) &= (X \times (G \ltimes H)/H \times U \times U')/(G \ltimes H)\\ &\simeq (X \times U \times U')/G.\end{aligned}$$

Look at the following commutative diagram:

$$\begin{array}{ccc}(X \times H \times U \times U')/(G \ltimes H) & \longrightarrow & (X \times U \times U')/G \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ (X \times U \times U')/(G \ltimes H) & \xrightarrow{f} & (X \times U')/G.\end{array}$$

Note that π_1 is an affine bundle: in fact, it is a fiber bundle with fiber isomorphic to $\mathbb{A}^n$, and structure group $G \ltimes H$ that acts on $\mathbb{A}^n$ by affine transformations, since the action of G on H is affine and the action of H on itself is affine. It follows from [Gro58a, p. 35] that π_1^* is an isomorphism. On the other hand, since $U \times U'$ is an open set of $V \times V'$ on which G acts freely, π_2^* is the identity on the equivariant Chow ring $\mathrm{A}^*_G(X)$, up to a degree that can be made arbitrarily large: so we have a commutative triangle

$$\begin{array}{ccccc} & & \mathrm{A}^*_{G \ltimes H}(X \times H) & & \\ & {\scriptstyle \pi_1^*}\nearrow {\scriptstyle \simeq} & & \nwarrow {\scriptstyle \simeq} & \\ \mathrm{A}^*_{G \ltimes H}(X) & & \xleftarrow{\quad f^* \quad} & & \mathrm{A}^*_G(X)\end{array}$$

where the horizontal arrow is exactly the map induced by the projection $G \ltimes H \to G$. ■

Here is another auxiliary result: it is well known (see for instance [Vez00]) that $\mathrm{A}^*_{\boldsymbol{\mu}_n} \simeq \mathbb{Z}[\xi]/(n\xi)$, where ξ is the first Chern class of the character given by the inclusion $\boldsymbol{\mu}_n \hookrightarrow \mathbb{G}_{\mathrm{m}}$. If G is an algebraic group, we will denote by $\xi \in \mathrm{A}^*_{G\times\boldsymbol{\mu}_n}$ the image of ξ under the map $\mathrm{A}^*_{\boldsymbol{\mu}_n} \to \mathrm{A}^*_{G\times\boldsymbol{\mu}_n}$ induced by the projection $G \times \boldsymbol{\mu}_n \to \boldsymbol{\mu}_n$. Using the projection $G \times \boldsymbol{\mu}_n \to G$, we can consider $\mathrm{A}^*_{G\times\boldsymbol{\mu}_n}$ as an A^*_G-algebra. Then $\mathrm{A}^*_{G\times\boldsymbol{\mu}_n}$ admits the following description:

LEMMA 2.4. *As an A^*_G-algebra, $\mathrm{A}^*_{G\times\boldsymbol{\mu}_n}$ is generated by the element ξ, and the kernel of the evaluation map $\mathrm{A}^*_G[x] \to \mathrm{A}^*_{G\times\boldsymbol{\mu}_n}$ sending x into ξ is the ideal (nx). In other words,*

$$\mathrm{A}^*_{G\times\boldsymbol{\mu}_n} = \mathrm{A}^*_G[\xi]/(n\xi).$$

PROOF. See [Vis05, Lemma 4.3]. ■

3. The special groups: GL_n, SL_n and Sp_n.

Let us fix a field k: we write GL_n, SL_n and Sp_n for the corresponding algebraic groups over k.

These groups are always much easier to study: they are special, in the sense that every étale principal bundle is Zariski locally trivial (this terminology is due to Grothendieck, see [Gro58b]). For GL_n and Sp_n the idea works in a very similar way: let us work out Sp_n, which is marginally harder. We proceed by induction on n, the case $n = 0$ being trivial.

Consider $V = \mathbb{A}^{2n}$, the tautological representation of Sp_n, with its symplectic form $h\colon V \times V \to k$ given in coordinates by

$$h(x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}) = x_1 y_{n+1} + \ldots + x_n y_{2n} - x_{n+1} y_1 - \ldots - x_{2n} y_n.$$

Denote by $e_1, \ldots, e_{2n}$ the canonical basis of V.

The orbit structure of V is very simple: there are two orbits, the origin and its complement $U \stackrel{\text{def}}{=} V \setminus \{0\}$. Consider the subspace

$$V' = \langle e_1, \ldots, e_{n-1}, e_{n+1}, \ldots, e_{2n-1} \rangle;$$

the restriction of h to V' is a non-degenerate symplectic form, and $V = V' \oplus \langle e_n, e_{2n} \rangle$. This induces an embedding $\mathrm{Sp}_{n-1} \hookrightarrow \mathrm{Sp}_n$, identifying Sp_{n-1} with the stabilizer of the pair (e_n, e_{2n}).

Let G the stabilizer of the element e_n: then we have that $\mathrm{Sp}_{n-1} \subseteq G \subseteq \mathrm{Sp}_n$. The first inclusion admits a splitting: if $A \in G$, then A stabilizes the orthogonal complement $\langle e_n \rangle^\perp$. It follows that A induces a linear endomorphism on the quotient $\langle e_n \rangle^\perp / \langle e_n \rangle \simeq V'$, and this endomorphism is easily seen to preserve the symplectic form $h|_{V'}$, so it is an element of Sp_{n-1}. Thus we have a projection $G \to \mathrm{Sp}_{n-1}$: let H its kernel, so that $G = \mathrm{Sp}_{n-1} \ltimes H$.

The structure of H is as follows; the matrices in H are exactly those for which there are scalars $a_1, \dots, a_{2n-1}$ such that

$$Ae_i = \begin{cases} e_i - a_{i+n}e_n & \text{if } i = 1, \dots, n-1 \\ e_n & \text{if } i = n \\ e_i + a_{i-n}e_n & \text{if } i = n+1, \dots, 2n-1 \\ a_1e_2 + \dots + a_{2n-1}e_{2n-1} + e_{2n} & \text{if } i = 2n. \end{cases}$$

This yields an isomorphism of varieties $H \simeq \mathbb{A}^{2n-1}$. It is not hard to see that the conditions of Lemma 2.3 are satisfied for the action of Sp_{n-1} on H; hence the embedding $\mathrm{Sp}_{n-1} \subseteq G$ induces an isomorphism of rings $\mathrm{A}^*_G \simeq \mathrm{A}^*_{\mathrm{Sp}_{n-1}}$, so the composite

$$\mathrm{A}^*_{\mathrm{Sp}_n}(U) \to \mathrm{A}^*_{\mathrm{Sp}_{n-1}}(U) \to \mathrm{A}^*_{\mathrm{Sp}_{n-1}}(e_n) = \mathrm{A}^*_{\mathrm{Sp}_{n-1}}$$

is an isomorphism. The restriction of the representation V to Sp_{n-1} is the direct sum of V' and of a trivial 2-dimensional representation: hence the Chern classes $c_i = \mathrm{c}_i(V)$ restrict to the $\mathrm{c}_i(V')$. From the induction hypothesis, we conclude that $\mathrm{A}^*_{\mathrm{Sp}_n}(U)$ is generated by the images of $c_2, \dots, c_{2n-2}$.

From Lemma 2.2 we conclude that every class in $\mathrm{A}^*_{\mathrm{Sp}_n}$ can be written as a polynomial in $c_2, \dots, c_{2n-2}$, plus a multiple of c_{2n}. By induction on the degree we conclude that $c_2, \dots, c_{2n}$ generate $\mathrm{A}^*_{\mathrm{Sp}_n}$.

To prove their algebraic independence, let us restrict to $\mathrm{A}^*_{T_n}$, where $T_n \simeq \mathbb{G}_{\mathrm{m}}^n$ is the standard maximal torus in Sp_n, consisting of diagonal matrices with entries $(t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1})$. Then $\mathrm{A}^*_{T_n}$ is the polynomial ring $\mathbb{Z}[\xi_1, \dots, \xi_n]$, where ξ_i is the first Chern class of the 1-dimensional representation given by the i^{th} projection $T_n \to \mathbb{G}_{\mathrm{m}}$. Then the total Chern class of the restriction of V_n to T_n is

$$(1+\xi_1)\dots(1+\xi_n)(1-\xi_1)\dots(1-\xi_n) = (1-\xi_1^2)\dots(1-\xi_n^2);$$

hence the restriction of c_{2i} is the i^{th} elementary symmetric function of $-\xi_1^2, \dots, -\xi_n^2$. This proves the independence of the c_{2i}.

As we mentioned, the argument for GL_n is very similar. For SL_n, one can proceed similarly, but it is easier to use the fact that, if GL_n acts freely on an algebraic variety U, the induced morphism $U/\mathrm{SL}_n \to$ $\to U/\mathrm{GL}_n$ makes U/SL_n into a principal $\mathbb{G}_{\mathrm{m}}$-bundle on U/GL_n, associated with the determinant homomorphism $\det\colon \mathrm{GL}_n \to \mathbb{G}_{\mathrm{m}}$. Hence, by

Lemma 2.2, we have an isomorphism $\mathrm{A}^*_{\mathrm{SL}_n} \simeq \mathrm{A}^*_{\mathrm{GL}_n}/(c_1)$, which gives us what we want.

REMARK 3.1. All these arguments work with cohomology, when $k = \mathbb{C}$. The localization sequence in cohomology does not quite work in the same way, as the restriction homomorphism from the cohomology of the total space to that of an open subset is not necessarily surjective. However, if Y is a smooth closed subvariety of a smooth complex algebraic variety X, of pure codimension d, then there is an exact sequence

$$\cdots \to \mathrm{H}^{i-2d}_G(Y) \to \mathrm{H}^i_G(X) \to \mathrm{H}^i_G(X \setminus Y) \to \mathrm{H}^{i-2d+1}_G(Y) \to \cdots.$$

Hence if we know that either the pullback $\mathrm{H}^*_G(X) \to \mathrm{H}^*_G(X \setminus Y)$ is surjective, or the pushforward $\mathrm{H}^*_G(Y) \to \mathrm{H}^*_G(X)$ is injective, we can conclude that we have an exact sequence

$$0 \to \mathrm{H}^*_G(Y) \to \mathrm{H}^*_G(X) \to \mathrm{H}^*_G(X \setminus Y) \to 0;$$

and this is sufficient to mimic the arguments above and give the result for cohomology.

REMARK 3.2. These results can also be proved very simply from a result of Edidin and Graham (see [EG97]): if G is a special algebraic group, T a maximal torus and W the Weyl group, the natural restriction homomorphism $\mathrm{A}^*_G \to (\mathrm{A}^*_T)^W$ is an isomorphism.

4. The Chow ring of the classifying space of O_n.

Let us fix a field k of characteristic different from 2. If $V = k^n$ is an n-dimensional vector space, we define a quadratic form $q\colon V \to k$ in the standard form

$$q(x_1, \ldots, x_n) = x_1 x_{m+1} + \ldots + x_m x_{2m}$$

when $n = 2m$, and

$$q(x_1, \ldots, x_n) = x_1 x_{m+1} + \ldots + x_m x_{2m} + x_{2m+1}^2$$

when $n = 2m + 1$. We will denote by O_n the algebraic group of linear transformations preserving this quadratic form.

THEOREM 4.1. [R. Pandharipande, B. Totaro].

$$\mathrm{A}^*_{\mathrm{O}_n} = \mathbb{Z}[c_1, \ldots, c_n]/(2c_{\mathrm{odd}}).$$

REMARK 4.2. Let V' be another n-dimensional vector space over k, with a non-degenerate quadratic form $q' \colon V' \to k$. We can associate with this another algebraic group $\mathrm{O}(q')$, which will not be isomorphic to $\mathrm{O}_n = \mathrm{O}(q)$, in general, unless k is algebraically closed.

However, one can show that there is an isomorphism of Chow rings $\mathrm{A}^*_{\mathrm{O}_n} \simeq \mathrm{A}^*_{\mathrm{O}(q')}$, such that the classes $\mathrm{c}_i(V)$ in the left hand side correspond to the classes $\mathrm{c}_i(V')$ in the right hand side. The principle that allows to prove this has been known for a long time ([Gir71, Remarque 1.6.7]): it is the existence of a bitorsor $I \to \operatorname{Spec} k$. This is the scheme representing the functor of isomorphisms of (V, q) with (V', q'). On I there is a left action of $\mathrm{O}(q')$ and right action of O_n, by composition. These two actions commute, and make I into a torsor under both groups (because (V, q) and (V', q') become isomorphic after a base extension).

In general, assume that G and G' are algebraic groups over a field k (in fact, any algebraic space will do as a base), and $I \to \operatorname{Spec} k$ is a (G', G)-bitorsor: that is, on I there is a right action of G and left action of G', and this makes I into a torsor under both groups. If X is a k-algebraic space on which G' acts on the left, then we can produce a k-algebraic space $I \times^G X$ on which G acts on the left, by dividing the product $I \times_{\operatorname{Spec} k} X$ by the right action of G, defined by the usual formula $(i, x)g = (ig, g^{-1}x)$. The left action of G' is by multiplication on the first component: the quotients $G \backslash X$ and $G' \backslash (I \times^G X)$ are canonically isomorphic.

This operation gives an equivalence of the category of G-algebraic spaces with the category of G'-algebraic spaces. When applied to representations, it yields representations, and gives an equivalence of the category of representations of G and of G'. Furthermore, given a representation V of G, with an open subset $U \subseteq V$ on which G acts freely, we get a representation $V' = I \times^G V$ with an open subset $U' = I \times^G U$ on which G' acts freely, so that the quotients $G \backslash U$ and $G' \backslash U'$ are isomorphic. In Totaro's construction this gives an isomorphism of A^*_G with $\mathrm{A}^*_{G'}$.

So, in particular, the result that we have stated for O_n also holds for $\mathrm{O}(q')$ for any other non-degenerate n-dimensional quadratic form q', and we have

$$\mathrm{A}^*_{\mathrm{O}(q')} = \mathbb{Z}[c_1, \ldots, c_n]/(2c_{\mathrm{odd}}).$$

The proof of the Theorem will be split into two parts: first we show that the c_i generate $\mathrm{A}^*_{\mathrm{O}_n}$, then that ideal of relations is generated by the given ones.

For the first part we proceed by induction on n.

For $n = 1$, $q(x) = x_1^2$, and $\mathrm{O}_1 = \boldsymbol{\mu}_2$, so

$$\mathrm{A}^*_{\mathrm{O}_1} = \mathrm{A}^*_{\boldsymbol{\mu}_2} \simeq \mathbb{Z}[c_1]/(2c_1).$$

For $n > 1$, let $B = \{v \in \mathbb{A}^n \mid q(v) \neq 0\}$, and set $Q = q^{-1}(1)$. Then $q\colon B \to \mathbb{G}_{\mathrm{m}}$ is a fibration, with fibers isomorphic to Q. This fibration is not trivial, but it becomes trivial after an étale base change. Set

$$\widetilde{B} = \{(t, v) \in \mathbb{G}_{\mathrm{m}} \times B \mid t^2 = q(v)\},$$

and consider the cartesian diagram

$$\begin{array}{ccc} \widetilde{B} & \longrightarrow & B \\ \downarrow & & \downarrow q \\ \mathbb{G}_{\mathrm{m}} & \xrightarrow{(-)^2} & \mathbb{G}_{\mathrm{m}} \end{array}$$

where the first column is projection onto the first factor, and the top row is defined by the formula $(t, v) \mapsto tv$.

There are obvious commuting actions of $\boldsymbol{\mu}_2$ and O_n on $\widetilde{B}$, the first defined by $\varepsilon \cdot (t, v) = (\varepsilon t, v)$, and the second by $M \cdot (t, v) = (t, Mv)$. The quotient $\widetilde{B}/\boldsymbol{\mu}_2$ is isomorphic to B, and the induced action of O_n on the quotient coincides with the given action on B. From Lemma 2.1, we obtain an isomorphism

$$\mathrm{A}^*_{\mathrm{O}_n}(B) \simeq A_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(\widetilde{B}).$$

Then there is an isomorphism of $\mathbb{G}_{\mathrm{m}}$-schemes $\widetilde{B} \simeq \mathbb{G}_{\mathrm{m}} \times Q$ defined by the formula $(t, v) \mapsto (t, v/t)$. The given actions of $\boldsymbol{\mu}_2$ and of O_n on $\widetilde{B}$ induce commuting actions on $\mathbb{G}_{\mathrm{m}} \times Q$ given by $\varepsilon \cdot (t, v) = (\varepsilon t, \varepsilon v)$ for $\varepsilon \in \boldsymbol{\mu}_2$ and $M(t, v) = (t, Mv)$ for $M \in \mathrm{O}_n$. These define an action of $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ on $\mathbb{G}_{\mathrm{m}} \times Q$, and $\mathrm{A}^*_{\mathrm{O}_n}(B)$ is isomorphic to $\mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(\mathbb{G}_{\mathrm{m}} \times Q)$.

This action of $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ on $\mathbb{G}_{\mathrm{m}} \times Q$ extends uniquely to an action of $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ on $\mathbb{A}^1 \times Q$, defined by the same formulae. This action is defined by two separate action on $\mathbb{A}^1$ and Q, and the action on $\mathbb{A}^1$ is linear, defined by the non-trivial character of $\boldsymbol{\mu}_2$ through the projection $\boldsymbol{\mu}_2 \times \mathrm{O}_n \to \boldsymbol{\mu}_2$. Call ξ the first Chern class of this representation. From Lemma 2.2, we have an isomorphism

$$\mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(\mathbb{G}_{\mathrm{m}} \times Q) \simeq \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(Q)/(\xi). \tag{4.1}$$

To investigate $\mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(Q)$ we will also use an orthogonal basis $e'_1, \ldots, e'_n$ of V, in which q has the form

$$q(x_1 e'_1 + \ldots + x_n e'_n) = x_1^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_n^2$$

when $n = 2m$, and

$$q(x_1 e'_1 + \ldots + x_n e'_n) = x_1^2 + \ldots + x_{m+1}^2 - x_{m+2}^2 - \ldots - x_n^2$$

when $n = 2m + 1$.

Now, the action of $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ on Q is transitive; let H the stabilizer of the point $e'_1 \in Q$. The structure of H is as follows. Set $V' \overset{\text{def}}{=} \langle e'_2, \ldots, e'_n \rangle$, so that V is the orthogonal sum $\langle e'_1 \rangle \oplus V'$, and call q' the restriction of q to V'. Then the group $\mathrm{O}_{q'}$ of linear automorphisms of V' preserving q' is naturally embedded into O_n, as the stabilizer of e'_1. Notice that in an appropriate basis q' has the standard form

$$q'(x_1, \ldots, x_{n-1}) = x_1 x_{m+1} + \ldots + x_m x_{2m}$$

when $n = 2m + 1$, and the opposite of the standard form

$$q'(x_1, \ldots, x_{n-1}) = -(x_1 x_m + \ldots + x_{m-1} x_{2m-2} + x_{2m-1}^2)$$

when $n = 2m$; in both cases the orthogonal group $\mathrm{O}(q')$ is isomorphic to O_{n-1}, and we identify it with O_{n-1}.

The stabilizer of e'_1 in $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ is the group $\boldsymbol{\mu}_2 \times \mathrm{O}_{n-1}$, embedded into $\boldsymbol{\mu}_2 \times \mathrm{O}_n$ with the injective homomorphism

$$(\varepsilon, M) \mapsto (\varepsilon, \varepsilon M).$$

It follows that

$$\begin{aligned} \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(Q) &\simeq \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}\big((\boldsymbol{\mu}_2 \times \mathrm{O}_n)/(\boldsymbol{\mu}_2 \times \mathrm{O}_{n-1})\big) \\ &\simeq \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_{n-1}}. \end{aligned}$$

We obtain a chain of isomorphisms

$$\begin{aligned} \mathrm{A}^*_{\mathrm{O}_n}(B) &\simeq \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_n}(Q)/(\xi) \\ &\mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_{n-1}}/(\xi). \end{aligned}$$

Finally, from Lemma 2.1 we get an isomorphism

$$\begin{aligned} \mathrm{A}^*_{\boldsymbol{\mu}_2 \times \mathrm{O}_{n-1}}/(\xi) &\simeq \mathrm{A}^*_{\mathrm{O}_{n-1}}[\xi]/(\xi) \\ &\simeq \mathrm{A}^*_{\mathrm{O}_{n-1}}. \end{aligned}$$

The composite $\mathrm{A}^*_{\mathrm{O}_n} \to \mathrm{A}^*_{\mathrm{O}_n}(U) \to \mathrm{A}^*_{\mathrm{O}_{n-1}}$ is the pullback induced by the embedding $\mathrm{O}_{n-1} \subseteq \mathrm{O}_n$.

The restriction of V to O_{n-1} is the direct sum of V' and a trivial 1-dimensional representation, hence the restriction $\mathrm{A}^*_{\mathrm{O}_n} \to \mathrm{A}^*_{\mathrm{O}_{n-1}}$ carries c_i into $c_i(V')$. Therefore, by induction hypothesis, the images of $c_1, \ldots, c_{n-1}$ generate $\mathrm{A}^*_{\mathrm{O}_n}(B)$.

Next, we claim that the restriction homomorphism $A^*_{O_n}(\mathbb{A}^n \setminus \{0\}) \to$ $\to A^*_{O_n}(B)$ is an isomorphism. To see this, set

$$C = \{v \in \mathbb{A}^n \setminus \{0\} \mid q(v) = 0\}$$

with its reduced scheme structure, and consider the fundamental exact sequence

$$A^*_{O_n}(C) \xrightarrow{i_*} A^*_{O_n}(\mathbb{A}^n \setminus \{0\}) \to A^*_{O_n}(B) \to 0.$$

We need to show that i_* is the zero map. In fact, $q\colon \mathbb{A}^n \setminus \{0\} \to \mathbb{A}^1$ is smooth, since the characteristic of the base field is not 2, so C is the scheme-theoretic inverse image of $\{0\}$. The map $q\colon \mathbb{A}^n \setminus \{0\} \to \mathbb{A}^1$ is O_n-equivariant, if we let O_n act trivially on $\mathbb{A}^1$; and the fundamental class $[0] \in A^*_{O_n}(\mathbb{A}^1)$ equals zero. Since the inverse image of $[0]$ in $A^*_{O_n}(\mathbb{A}^n \setminus \{0\})$ is $[C]$, we can conclude that

$$[C] = 0 \in A^*_{O_n}(\mathbb{A}^n \setminus \{0\}).$$

Next we show that the pullback $i^*\colon A^*_{O_n}(\mathbb{A}^n \setminus \{0\}) \to A^*_{O_n}(C)$ is surjective: in this case, for every $\alpha \in A^*_{O_n}(C \setminus \{0\})$, we have $\alpha = i^*\beta$ for some $\beta \in A^*_{O_n}(\mathbb{A}^n \setminus \{0\})$, so

$$i_*(\alpha) = i_* i^*(\beta) = [C] \cdot \beta = 0$$

by the projection formula, and i_* is the zero map, as claimed.

To show surjectivity, notice that the action of O_n on C is transitive. Let us investigate the stabilizer G of $e_1 \in C$. Set $n = 2m$ or $n = 2m + 1$, as usual. If we define

$$V' = \langle e_2, \ldots, e_m, e_{m+2}, \ldots, e_n \rangle$$

then the restriction of q to V' has the standard form, and V is the orthogonal sum $V' \oplus \langle e_1, e_{m+1} \rangle$. This gives an embedding $O_{n-2} \subseteq O_n$, identifying O_{n-2} with the stabilizer of the pair (e_1, e_{m+1}).

An analysis very similar to that we have carried out for the stabilizer of a vector under Sp_n leads to the conclusion that the stabilizer G of e_1 is a semidirect product $O_{n-2} \ltimes H$, where H is isomorphic to $\mathbb{A}^{n-1}$ as a variety, the action of an element of H on itself is given by an affine map, and the action of O_{n-2} on H is linear: by Lemma 2.3, the embedding $O_{n-2} \subseteq G$ induces an isomorphism of rings $A^*_G \simeq A^*_{O_{n-2}}$, so the composite

$$A^*_{O_n}(C) \to A^*_{O_{n-2}}(C) \to A^*_{O_{n-2}}(e_1) = A^*_{O_{n-2}}$$

is an isomorphism. But the c_i restrict in $A^*_{O_{n-2}}$ to the Chern classes of V': hence, by inductions hypothesis, they generate $A^*_{O_{n-2}}$. Hence the pullback

$\mathrm{A}^*_{\mathrm{O}_n} \to \mathrm{A}^*_{\mathrm{O}_n}(C)$ is surjective, as claimed. This ends the proof that the c_i generate $\mathrm{A}^*_{\mathrm{O}_n}$. Let us investigate the relations.

The quadratic form q induces an isomorphism $V \simeq V^\vee$ of representations of O_n, hence for each i we have $\mathrm{c}_i(V) = (-1)^i \mathrm{c}_i(V)$. This shows that $2c_i = 0$ when i is odd.

To show that these generate the ideal of relations among the c_i, let $J \subseteq \mathbb{Z}[x_1, \dots, x_n]$ be the ideal generated by $2x_1$, $2x_3, \dots$. Let $P \in \mathbb{Z}[x_1, \dots, x_n]$ be a homogeneous polynomial such that $P(c_1, \dots, c_n) = 0 \in \mathrm{A}^*_{\mathrm{O}_n}$: we need to check that P is in J. By modifying P by an element of J, we may assume that P is of the form $Q + R$, where Q is a polynomial in the even x_i, while R is a polynomial in which every monomial contains some x_i with i odd, and all of whose coefficients are either 0 or 1.

Let $T_m \simeq \mathbb{G}_{\mathrm{m}}^m$ be the standard torus in O_n: the embedding $T_m \subseteq \mathrm{O}_n$ sends $(t_1, \dots, t_m)$ into the diagonal matrix with entries $(t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1})$ if $n = 2m$, and $(t_1, \dots, t_m, t_1^{-1}, \dots, t_m^{-1}, 1)$ if $n = 2m + 1$. Then $\mathrm{A}^*_{T_m} = \mathbb{Z}[\xi_1, \dots, \xi_m]$, where ξ_i is the first Chern class of the i^{th} projection $\chi_i \colon T_n \to \mathbb{G}_{\mathrm{m}}$. The restriction of V to T_n splits as $\rho \overset{\text{def}}{=} \chi_1 + \dots + \chi_m + \chi_1^{-1} + \dots + \chi_m^{-1}$ when m is even, and $\rho + 1$ when n is odd. Hence the total Chern class of the restriction of V to T_n is

$$(1 + \xi_1) \dots (1 + \xi_m)(1 - \xi_1) \dots (1 - \xi_m) = (1 - \xi_1^2) \dots (1 - \xi_m^2);$$

and this means that the restrictions of the c_i is 0 when i is odd, while c_{2j} restricts to the j^{th} symmetric function of $-\xi_1^2, \dots, -\xi_m^2$. Hence the restrictions of even Chern classes are algebraically independent. In the decomposition $0 = P(c_1, \dots, c_n) = Q(c_2, \dots, c_{2m}) + R(c_1, \dots, c_n)$ the summand $R(c_1, \dots, c_n)$ restricts to 0, so $Q(c_2, \dots, c_{2m})$ also restricts to 0. This implies that $Q = 0$. So we have that P has coefficients that are either 0 or 1.

Now take a basis $e'_1, \dots, e'_n$ of V in which q has a diagonal form. Consider the subgroup $\boldsymbol{\mu}_n^n \subseteq \mathrm{O}_n$ consisting of linear transformations that take each e'_i into e'_i or $-e'_i$. If we call η_i the first Chern class of the character obtained composing the i^{th} projection $\boldsymbol{\mu}_n^n \to \boldsymbol{\mu}_n$ with the embedding $\boldsymbol{\mu}_n \hookrightarrow \mathbb{G}_{\mathrm{m}}$, then by Lemma 2.4 we have

$$\mathrm{A}^*_{\boldsymbol{\mu}_n^n} = \mathbb{Z}[\eta_1, \dots, \eta_n]/(2\eta_1, \dots, 2\eta_n).$$

There is a natural ring homomorphism from $\mathrm{A}^*_{\boldsymbol{\mu}_n^n}$ into the polymomial ring $\mathbb{F}_2[y_1, \dots, y_n]$ that sends each η_i into y_i. The restriction of V to $\boldsymbol{\mu}_n^n$ has total Chern class $(1 + \eta_1) \dots (1 + \eta_n)$; hence the image of c_i in $\mathbb{F}_2[y_1, \dots, y_n]$ is the i^{th} elementary symmetric polynomial s_i in the y_i. The s_i are algebraically independent in $\mathbb{F}_2[y_1, \dots, y_n]$, the image of $0 = P(c_1, \dots, c_n)$ is $P(s_1, \dots, s_n)$, and P has coefficients that either 0 or 1. This implies that $P = 0$, and completes the proof of the theorem.

5. The Chow ring of the classifying space of SO_n.

Let k be a field of characteristic different from 2, set $V = k^n$, and let $q\colon V \to k$ be the same quadratic form as in the previous section. Consider the subgroup $\mathrm{SO}_n \subseteq \mathrm{O}_n$ of orthogonal linear transformations of determinant 1.

If n is odd, $\mathrm{A}^*_{\mathrm{SO}_n}$ can be easily computed from $\mathrm{A}^*_{\mathrm{O}_n}$, as was noticed in [Pan98] and [Tot99].

THEOREM 5.1. [R. Pandharipande, B. Totaro]. *If n is odd, then*

$$\mathrm{A}^*_{\mathrm{SO}_n} = \mathbb{Z}[c_2, \dots, c_n]/(2c_{\mathrm{odd}}).$$

PROOF. When is n odd there is an isomorphism $\mathrm{O}_n \simeq \boldsymbol{\mu}_n \times \mathrm{SO}_n$; the determinant character $\det\colon \mathrm{O}_n \to \boldsymbol{\mu}_n$ (whose first Chern class in $\mathrm{A}^*_{\mathrm{O}_n}$ is c_1) corresponds to the projection $\boldsymbol{\mu}_n \times \mathrm{SO}_n \to \boldsymbol{\mu}_n$. Then from Lemma 2.4 we get that

$$\mathrm{A}^*_{\mathrm{SO}_n} \simeq \mathrm{A}^*_{\mathrm{O}_n}/(c_1)$$

and the conclusion follows. ■

5.1 – *The Edidin–Graham construction*

From now on we shall assume that n is even, and write $n = 2m$.

In this case, $\mathrm{A}^*_{\mathrm{SO}_n}$ is not generated by the Chern classes of the standard representation, not even rationally. This can be seen easily for $n = 2$. We have that SO_2 consists of matrices of the form

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

and so is isomorphic to $\mathbb{G}_{\mathrm{m}}$. Then

$$\mathrm{A}^*_{\mathrm{SO}_2} = \mathrm{A}^*_{\mathbb{G}_{\mathrm{m}}} = \mathbb{Z}[\xi],$$

where ξ is the first Chern class of the tautological representation $L = \mathbb{A}^1$, on which $\mathbb{G}_{\mathrm{m}}$ acts via multiplication. Hence $V = L \oplus L^\vee$, so $\mathrm{c}_2(V) = -\xi^2$.

For general n, the vector space V will still split as the direct sum of two totally isotropic subspaces, one dual to the other: however, when $n > 2$ this splitting is not unique, and the totally isotropic subspaces are not invariant under the action of SO_n, so V is not a direct sum of two nontrivial representations (and V is in fact irreducible). Still, in topology V has an

Euler class $\varepsilon_m \in \mathrm{H}^{2m}_{\mathrm{SO}_n}$, whose square is $(-1)^m c_m$. Let us recall Edidin and Graham's construction of an algebraic multiple of ε_m (see [EG95]).

In what follows we will use the classical conventions for projectivizations and Grassmannians; those seem a little more natural in intersection theory than Grothendieck's. So, if W is a vector space, we denote by $\mathbb{P}(W)$ the vector space of lines in W, and by $\mathbb{G}(r, W)$ the Grassmannian of subspaces of dimension r; and similarly for vector bundles.

Denote by $\mathbb{I}(m, V)$ the smooth subvariety of $\mathbb{G}(m, V)$ consisting of maximal totally isotropic subspaces of V. It is well known that O_n acts transitively on $\mathbb{I}(m, V)$, and that $\mathbb{I}(m, V)$ has two connected components, each of which is an orbit under the action of SO_n. Let us choose one of the orbits, for example, the one containing the subspace $\langle e_1, \dots, e_m \rangle$. Every totally isotropic subspace of dimension $m-1$ of V is contained in exactly two maximal totally isotropic subspaces, one in each connected component.

There is a well known equivalence of categories between O_n-torsors and vector bundles of rank n with a non-degenerate quadratic form. If E is a vector bundle on a scheme X with a non-degenerate quadratic form, this corresponds to a O_n-torsor $\pi: P \to X$, the torsor of isometries between E and $V \times X$; with this torsor we can associate a μ_2-torsor (that is, an étale double cover) $P/\mathrm{SO}_n \to X$ via the determinant homomorphism $\det: \mathrm{O}_n \to \mu_2$. This cover can be described geometrically as follows.

Consider the subscheme $\mathbb{I}(m, E)$ of totally isotropic subbundles in the relative Grassmannian $\mathbb{G}(m, E) \to X$; the projection $\mathbb{I}(m, E) \to X$ is proper and smooth, and each of its geometric fibers has two connected components. Let $\mathbb{I}(m, E) \to \widetilde{\mathbb{I}}(m, E) \to X$ be the Stein factorization; then $\widetilde{\mathbb{I}}(m, E) \to X$ is an étale double cover, and is precisely the double cover $P/\mathrm{SO}_n \to X$. This can be seen as follows.

On P we have, by definition, an isometry of $\pi^* E$ with $V \times P$. In $V \times P$ we have a maximal totally isotropic subbundle $\langle e_1, \dots, e_m \rangle \times P$, so we get a maximal totally isotropic subbundle of $\pi^* E$. This defines a morphism $P \to \mathbb{I}(m, E)$ over X; the composite $P \to \mathbb{I}(m, E) \to \widetilde{\mathbb{I}}(m, E)$ induces the desired isomorphism $P/\mathrm{SO}_n \simeq \widetilde{\mathbb{I}}(m, E)$.

Hence, giving a reduction of structure group of $P \to X$ to SO_n is equivalent to assigning a section $X \to \widetilde{\mathbb{I}}(m, E)$. This yields an equivalence of the groupoid of SO_n-torsors on X with the groupoid of vector bundles $E \to X$ of rank n with a non-degenerate quadratic form, and a section $X \to \widetilde{\mathbb{I}}(m, E)$. We shall refer to such a structure as an SO_n*-structure on* E.

Furthermore, given an SO_n-structure on E, if $f: T \to X$ is a morphism of algebraic varieties, and L is a totally isotropic subbundle of $f^* E$ of rank m, we say that L is *admissible* if the image of T under the morphism

$T \to \mathbb{I}(m, X)$ corresponding to L is contained in the inverse image of the given embedding $X \subseteq \widetilde{\mathbb{I}}(m, E)$.

Here is the construction of Edidin and Graham ([EG95, Section 6]). We will follow their notation. Let E be a vector bundle of rank n with an SO_n-structure on a smooth algebraic variety X. For each $i = 1, \dots, m$ consider the flag variety $f_i\colon Q_i \to X$ of totally isotropic flags $L_1 \subseteq L_2 \subseteq \dots \subseteq L_{m-i} \subseteq E$, with each L_s of rank s. For each i, denote by $L_1 \subseteq L_2 \subseteq \dots \subseteq L_{m-i} \subseteq f_i^* E$ the universal flag on Q_i. The restriction of the quadratic form to $L_{m-i}^{\perp}$ is degenerate, with radical equal to L_{m-i}; hence on Q_i there lives a vector bundle $E_i \overset{\mathrm{def}}{=} L_{m-i}^{\perp}/L_{m-i}$ of rank $2i$ with a non-degenerate quadratic form. For each $i = 1, \dots, m-1$ we have a projection $\pi_i\colon Q_{i-1} \to Q_i$, obtained by dropping the last totally isotropic subbundle in the chain; and Q_{i-1} is canonically isomorphic, as a scheme over Q_i, to the smooth quadric bundle in $\mathbb{P}(E_i)$ defined by the quadratic form on E_i. This means that Q_{i-1} is a family of quadrics of dimension $2(i-1)$ over Q_i. Let us denote by $h_i \in \mathrm{A}^1(Q_{i-1})$ the restriction to Q_{i-1} of the class $\mathrm{c}_1\big(\mathcal{O}_{\mathbb{P}(E_i)}(1)\big) \in \mathrm{A}^1\big(\mathbb{P}(E_i)\big)$.

Each bundle E_i has a canonical SO_{n-2i}-structure. Call $\pi_i\colon L_{m-i}^{\perp} \to E_i$ the projection. From each totally isotropic vector subbundle $L \subseteq E_i$ of rank $m-i$, we get a totally isotropic vector subbundle $\pi_i^* L \subseteq L_{m-i}^{\perp} \subseteq f_i^* E$ of rank m; then L is admissible if and only if $\pi_i^* L$ is admissible.

The universal flag $L_1 \subseteq L_2 \subseteq \dots \subseteq L_{m-1} \subseteq f_1^* E$ on Q_1 can be completed in a unique way to a maximal totally isotropic flag $L_1 \subseteq \dots \subseteq L_{m-1} \subseteq$ $\subseteq L_m \subseteq f_1^* E$ in such a way that L_m is admissible. Then Edidin and Graham define

$$y_m(E) = f_{1*}\big(s \cdot \mathrm{c}_m(L_m)\big) \in \mathrm{A}^m(X)$$

where we have set

$$s = h_2^2 h_3^4 \dots h_m^{2m-2} \in \mathrm{A}^*(Q_1).$$

Remark 5.2. In this formula each of the classes h_i should be pulled back to Q_1. Here, and in what follows, we use the following convention: when $f\colon Y \to X$ is a morphism of smooth varieties, and $\xi \in \mathrm{A}^*(X)$, we will also write ξ for $f^*\xi \in \mathrm{A}^*(Y)$. Similarly, if $E \to X$ is a vector bundle, we will also write E for f^*E. This has the advantage of considerably simplifying notation, and should not lead to confusion. With this notation, when f is proper the projection formula reads: if $\xi \in \mathrm{A}^*(X)$ and $\eta \in \mathrm{A}^*(Y)$, then

$$f_*(\xi\eta) = \xi f_*\eta.$$

There is also an inductive definition of $y_m(E)$. If $m = 1$ then there is precisely one totally admissible isotropic line subbundle of E, and we have $y_1(E) = \mathrm{c}_1(L)$, by definition.

For $m > 1$ we have a vector bundle E_{m-1} on Q_{m-1} with an SO_{n-2}-structure.

LEMMA 5.3. *The formula*

$$y_m(E) = -f_{m-1*}\big(h_m^{2m-1} y_{m-1}(E_{m-1})\big)$$

holds.

PROOF. To prove this, call $g\colon Q_1 \to Q_{m-1}$ the projection: on Q_1 we have a flag

$$L_2/L_1 \subseteq L_3/L_1 \subseteq \ldots \subseteq L_{m-1}/L_1 \subseteq g^* E_{m-1}$$

that makes Q_1 into the variety of totally isotropic flags of length $m-2$ in E_{m-1}; we complete this to a maximal totally isotropic flag by adding L_m/L_1. So we get

$$y_{m-1}(E_{m-1}) = g_*\big(h_2^2 h_3^4 \ldots h_{m-1}^{2m-4} \mathrm{c}_{m-1}(L_m/L_1)\big).$$

On the other hand, on $Q_{m-1} \subseteq \mathbb{P}(E)$, the line bundle $L_1 \subseteq f_{m-1}^* E$ is the pullback of the tautological bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$, so $\mathrm{c}_1(L_1) = -h_m$. Hence we have

$$\mathrm{c}_m(L_m) = -h_m \mathrm{c}_{m-1}(L_m/L_{m-1})$$

and

$$\begin{aligned}
-f_{m-1*}\big(h_m^{2m-1} y_{m-1}(E_{m-1})\big) &= -f_{m-1*}\big(h_m^{2m-1} g_*\big(h_2^2 h_3^4 \ldots h_{m-1}^{2m-4} \mathrm{c}_{m-1}(L_m/L_1)\big)\big)\\
&= -f_{1*}\big(h_2^2 h_3^4 \ldots h_{m-1}^{2m-4} h_m^{2m-1} \mathrm{c}_{m-1}(L_m/L_1)\big)\\
&= f_{1*}\big(h_2^2 h_3^4 \ldots h_{m-1}^{2m-4} h_m^{2m-2} \mathrm{c}_m(L_m)\big)\\
&= y_m(E)
\end{aligned}$$

as claimed. ∎

The Edidin–Graham class $y_m \in \mathrm{A}^m_{\mathrm{SO}_n}$ is defined as follows. Take a representation W of SO_n with an open subset U on which SO_n acts freely, and whose complement has codimension larger than m. Call E the vector bundle with an SO_n-structure associated with the SO_n-torsor $U \to U/\mathrm{SO}_n$. Then we set

$$y_m = y_m(E) \in \mathrm{A}^m(U/\mathrm{SO}_n) = \mathrm{A}^m_{\mathrm{SO}_n}.$$

It is easy to verify that this is independent of the W and U chosen.

5.2 – *The main result*

THEOREM [R. Field]. *If $n = 2m$, then*

$$\mathrm{A}^*_{\mathrm{SO}_n} = \mathbb{Z}[c_2, \dots, c_n, y_m]/\big(y_m^2 - (-1)^m 2^{n-2} c_n,\ 2c_{\mathrm{odd}},\ y_m c_{\mathrm{odd}}\big).$$

REMARK 5.4. Once again, this result can be extended to other quadratic forms (compare with Remark 4.2). Let V' be another n-dimensional vector space over k, with a non-degenerate quadratic form $q'\colon V' \to k$. This induces a non-degenerate quadratic form on the exterior powers $\bigwedge^i V'$. Let us assume that there is an isometry $\bigwedge^n V \simeq \bigwedge^n V'$.

This is equivalent to the following more concrete condition. We will write $\det q' \in k^*/k^{*2}$ for the class in k^*/k^{*2} of the determinant of a matrix representing q' in some basis. Then two n-dimensional quadratic forms have isomorphic top exterior powers if and only if they have the same determinant. Hence the condition above is equivalent to the equality

$$\det q' = (-1)^m \in k^*/k^{*2}.$$

Fix an isometry $\bigwedge^n V \simeq \bigwedge^n V'$. We can construct an $\big(\mathrm{SO}(q'), \mathrm{SO}_n\big)$-bitorsor $I \to \operatorname{Spec} k$, as the scheme representing the functor of isometries $V \simeq V'$ inducing the fixed isometry $\bigwedge^n V \simeq \bigwedge^n V'$. So we deduce the following result: if the condition above is satisfied, there exists a class $y_m \in \mathrm{A}^m_{\mathrm{SO}(q')}$, such that

$$\mathrm{A}^*_{\mathrm{SO}(q')} = \mathbb{Z}[c_2, \dots, c_n, y_m]/\big(y_m^2 - (-1)^m 2^{n-2} c_n,\ 2c_{\mathrm{odd}},\ y_m c_{\mathrm{odd}}\big).$$

The proof of the theorem will be split into three parts: first we verify that the classes c_i and y_m generate $\mathrm{A}^*_{\mathrm{SO}_n}$, next that the relations holds, and finally that they generate the ideal of relations.

Step 1: The generators. We proceed by induction on m. In the case $m = 1$ the statement says that

$$\mathrm{A}^*_{\mathrm{SO}_1} = \mathbb{Z}[c_2, y_1]/(y_1^2 + c_2) = \mathbb{Z}[y_1]$$

we have seen that $\mathrm{SO}_1 = \mathbb{G}_{\mathrm{m}}$, that y_1 is the first Chern class of the identity character on $\mathbb{G}_{\mathrm{m}}$, and that $c_2 = -y_1^2$.

Suppose $m > 1$. Set $B = \{x \in \mathbb{A}^n \mid q(x) \neq 0\}$ and $C = \{x \in \mathbb{A}^n \setminus \{0\} \mid q(x) - 1 = 0\}$. Proceeding precisely as for O_n, one establishes the following results.

(1) Let $e'_1, \dots, e'_n$ be an orthogonal basis of V in which q has the form

$$q(x_1 e'_1 + \dots + x_n e'_n) = x_1^2 + \dots + x_m^2 - x_{m+1}^2 - \dots - x_n^2.$$

Then the stabilizer of $e_1' \in B$ in SO_n is isomorphic to SO_{n-1}, and the composite

$$\mathrm{A}^*_{\mathrm{SO}_n}(B) \to \mathrm{A}^*_{\mathrm{SO}_{n-1}}(B) \to \mathrm{A}^*_{\mathrm{SO}_{n-1}}(e_1') = \mathrm{A}^*_{\mathrm{SO}_{n-1}}$$

is an isomorphism.

(2) The stabilizer of the pair (e_1, e_{m+1}) is isomorphic to SO_{n-2}. The composite

$$\mathrm{A}^*_{\mathrm{SO}_n}(C) \to \mathrm{A}^*_{\mathrm{SO}_{n-2}}(C) \to \mathrm{A}^*_{\mathrm{SO}_{n-2}}(e_1) = \mathrm{A}^*_{\mathrm{SO}_{n-2}}$$

is an isomorphism.

Call $i\colon C \subseteq \mathbb{A}^n \setminus \{0\}$ and $j\colon B \subseteq \mathbb{A}^n \setminus \{0\}$ the inclusions. Then we have an exact sequence

$$\mathrm{A}^*_{\mathrm{SO}_n}(C) \xrightarrow{i_*} \mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n \setminus \{0\}) \xrightarrow{j^*} \mathrm{A}^*_{\mathrm{SO}_n}(B) \to 0.$$

By induction hypothesis, we have that $\mathrm{A}^*_{\mathrm{SO}_n}(C) \simeq \mathrm{A}^*_{\mathrm{SO}_{n-2}}$ is generated as a ring by $c_2, \dots, c_{n-2}$ and y_{m-1}. From this, and from the relation $y_{m-1}^2 - (-1)^{m-1} 2^{n-4} c_{n-2}$, we see that $\mathrm{A}^*_{\mathrm{SO}_n}(C)$ is generated as a module over $\mathrm{A}^*_{\mathrm{SO}_n}$ by 1 and y_{m-1}; hence, since i_* is a homomorphism of $\mathrm{A}^*_{\mathrm{SO}_n}$-modules, by the projection formula, we see that the kernel of the pullback $\mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n \setminus \{0\}) \to \mathrm{A}^*_{\mathrm{SO}_n}(B)$ is generated as an ideal by $i_*1 = [C]$ and $i_* y_{m-1}$.

As in the case of O_n, we see that the fundamental class $[C] \in \mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n \setminus \{0\})$ is 0, because C is the scheme-theoretic zero-locus of the invariant function q. Furthermore, the images of $c_2, \dots c_{n-1}$ generate $\mathrm{A}^*_{\mathrm{SO}_n}(U) \simeq \mathrm{A}^*_{\mathrm{SO}_{n-1}}$: and this implies that $c_2, \dots, c_{n-1}$, together with $i_* y_{m-1}$, generate $\mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n \setminus \{0\}) = \mathrm{A}^*_{\mathrm{SO}_n}/(c_n)$. Hence $c_2, \dots, c_n$, $i_* y_{m-1}$ generate $\mathrm{A}^*_{\mathrm{SO}_n}$. Next, we have a Lemma.

LEMMA 5.5.

$$i_* y_{m-1} = -y_m \in \mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n \setminus \{0\}).$$

PROOF. Let W be a representation of SO_n, and U an open set of W on which the action of SO_n is free, and such that the codimension of $W \setminus U$ in W is larger than m. The vector bundle associated with the SO_n-torsor $U \to U/\mathrm{SO}_n$ is $E \overset{\mathrm{def}}{=} (\mathbb{A}^n \times U)/\mathrm{SO}_n$. We set $X \overset{\mathrm{def}}{=} \big((\mathbb{A}^n \setminus \{0\}) \times U\big)/\mathrm{SO}_n$, so that $X \subseteq E$ is the complement of the zero section, while $Y \overset{\mathrm{def}}{=} (C \times U)/\mathrm{SO}_n \subseteq X$ is the closed subscheme consisting of non-zero isotropic vectors, and $Z \overset{\mathrm{def}}{=} X \setminus Y$. By a slight abuse of notation, we will denote $i\colon Y \hookrightarrow X$ and $j\colon Z \hookrightarrow X$ the inclusions. Note that there is a tautological section $s\colon X \to E$ defined set-theoretically by $[u, x] \mapsto [u, x, x]$.

Let us first prove that $j^* y_m = 0 \in \mathrm{A}^*_{\mathrm{SO}_n}(B)$. In fact, the tautological section restricted to Z has the property that $q(s(x)) \neq 0$ for all x, and so $j^* y_m(E) = y_m(j^* E) = 0$, due to the following result.

LEMMA 5.6. *Let $(E, q) \to X$ be a rank $n = 2m$ vector bundle with a non-degenerate quadratic form. Suppose that there exists a section $s\colon X \to E$ such that $q(s(x)) \neq 0$ for all $x \in X$. Then $y_m(E) = 0$.*

PROOF. Pulling back to the flag variety $Q_1 \to X$, it suffices to show that if $L \subset E$ is a rank m totally isotropic subbundle, then $\mathrm{c}_m(L) = 0$. The quadratic form gives a perfect pairing $L \times E/L \to \mathcal{O}_X$, so $L^\vee \simeq E/L$. On the other hand the line subbundle $\langle s \rangle$ generated by s has intersection with L equal to 0 at every point of X; hence the composite $\mathcal{O}_X \xrightarrow{w} E \to E/L$ gives a nowhere vanishing section of E/L, so that

$$\mathrm{c}_m(L) = (-1)^m \mathrm{c}_m(E/L) = 0$$

as claimed. ■

It follows that $y_m = d \cdot i_* y_{m-1}$ with $d \in \mathbb{Z}$. We will compute d by restricting to a maximal torus; but first observe that since SO_{n-2} is included in SO_n as the stabilizer of the pair (e_1, e_{m+1}), there is an isomorphism

$$\begin{aligned}
(\mathbb{A}^n \times U)/\mathrm{SO}_{n-2} &\longrightarrow \mathbb{A}^2 \times \big((\mathbb{A}^{n-2} \times U)/\mathrm{SO}_{n-2}\big) \\
[(x_1, \ldots, x_n), u] &\longmapsto \big((x_1, x_{m+1}), [(x_2, \ldots, x_m, x_{m+2}, \ldots, x_n), u]\big),
\end{aligned}$$

and that $y_{m-1} \in \mathrm{A}^*_{\mathrm{SO}_{n-2}}$ is the Edidin-Graham class of the vector bundle $\big(\mathbb{A}^{n-2} \times U\big)/\mathrm{SO}_{n-2} \to U/\mathrm{SO}_{n-2}$.

Now, let $T_m \subset \mathrm{SO}_n$ is, as before, the torus of diagonal matrices with diagonal entries $t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}$, and ξ_i is the first Chern class of the i^{th} projection $T_m \to \mathbb{G}_\mathrm{m}$.

LEMMA 5.7. *The formulae*

$$c_n = (-1)^m \xi_1^2 \ldots \xi_m^2$$

and

$$y_m = 2^{m-1} \xi_1 \ldots \xi_m$$

hold in $\mathrm{A}^*_{T_m} = \mathbb{Z}[\xi_1, \ldots, \xi_m]$.

PROOF. Reducing the structure group to T_m, the vector bundle E on U/T_m associated with the standard representation $T_m \hookrightarrow \mathrm{SO}_n \hookrightarrow \mathrm{GL}_n$ splits into a direct sum of line bundles $\Lambda_1 \oplus \ldots \oplus \Lambda_{2m}$, where the i^{th} summand is the subbundle associated with the 1-dimensional subspace $\langle e_i \rangle \subseteq V$. For each $i = 1, \ldots, m$ we have $\Lambda_{i+n} \simeq \Lambda_i^\vee$. Then E has an admissible maximal totally isotropic subbundle $\Lambda_1 \oplus \ldots \oplus \Lambda_m$, which pulls back to an admissible totally isotropic subbundle on Q_1. The first Chern class of Λ_i in $\mathrm{A}^1(U/T_m) = \mathrm{A}^1_{T_m}$ is ξ_i, for $i = 1, \ldots, m$, hence

$$\mathrm{c}_m(\Lambda_1 \oplus \ldots \oplus \Lambda_m) = \xi_1 \ldots \xi_m \ \in \mathrm{A}^m_{T_m}$$

On the other hand, the top Chern classes of any two admissible totally isotropic subbundles of Q_1 are the same, by [EG95, Theorem 1], so

$$\begin{aligned} y_m &= f_*\big(s \cdot \mathrm{c}_m(\Lambda_1 \oplus \ldots \oplus \Lambda_m)\big) \\ &= (f_* s)\xi_1 \ldots \xi_m; \end{aligned}$$

and it is easy to verify that $f_* s = 2^{m-1}$. ■

It follows that

$$(\mathbb{A}^{n-2} \times U)/T_{m-1} = \Lambda_2 \oplus \ldots \oplus \Lambda_m \oplus \Lambda_2^\vee \oplus \ldots \oplus \Lambda_m^\vee;$$

moreover, since

$$\big(U \times (\mathbb{A}^n \setminus \{0\})\big)/T_m = (\Lambda_1 \oplus \ldots \oplus \Lambda_m \oplus \Lambda_1^\vee \oplus \ldots \oplus \Lambda_m^\vee) \setminus \{0\},$$

we have

$$\begin{aligned} \mathrm{A}^*(X) &= \mathrm{A}^*_{T_m}/(c_n) \\ &= \mathbb{Z}[\xi_1, \ldots, \xi_m]/(\xi_1^2 \ldots \xi_m^2) \end{aligned}$$

and our aim is to verify that the equation

$$i_* y_{m-1} = -2^{m-1}\xi_1 \ldots \xi_m \tag{5.1}$$

holds in $\mathbb{Z}[\xi_1, \ldots, \xi_m]/(\xi_i^2 \ldots \xi_m^2)$.

The inclusion of schemes on U/T_m

$$(\Lambda_1 \oplus \Lambda_1^\vee) \setminus \{0\} \hookrightarrow (\Lambda_1 \oplus \ldots \oplus \Lambda_m \oplus \Lambda_1^\vee \oplus \ldots \oplus \Lambda_m^\vee) \setminus \{0\}$$

induces a surjection of rings

$$\mathbb{Z}[\xi_1, \ldots, \xi_m]/(\xi_1^2 \ldots \xi_m^2) \to \mathbb{Z}[\xi_1, \ldots, \xi_m]/(\xi_1^2);$$

since $\mathbb{Z}\xi_1 \ldots \xi_m$ has trivial intersection with the kernel of this map, we can restrict to $(\Lambda_1 \oplus \Lambda_1^\vee) \setminus \{0\}$ to verify equation 5.1. There is a cartesian diagram

$$\begin{array}{ccc}
(\Lambda_1 \setminus \{0\}) \sqcup (\Lambda_1^\vee \setminus \{0\}) & \longrightarrow & (\Lambda_1 \oplus \Lambda_1^\vee) \setminus \{0\} \\
\big\downarrow & & \big\downarrow \\
Y & \xrightarrow{\quad i \quad} & X
\end{array}$$

We set

$$X' = (\Lambda_1 \oplus \Lambda_1^\vee) \setminus \{0\},$$

and

$$\begin{aligned} Y' &= Y_1' \sqcup Y_2' \\ &= (\Lambda_1 \setminus \{0\}) \sqcup (\Lambda_1^\vee \setminus \{0\}); \end{aligned}$$

call $i' \colon Y' \hookrightarrow X'$ the inclusion.

Also, form the vector bundle on Y' defined as

$$\begin{aligned} F &\overset{\text{def}}{=} \Lambda_2 \oplus \ldots \oplus \Lambda_m \oplus \Lambda_2^\vee \oplus \ldots \oplus \Lambda_m^\vee \\ &= \langle s(Y') \rangle^\perp / \langle s(Y') \rangle. \end{aligned}$$

We need to check that

$$i'_* y_{m-1}(F) = -2^{m-1} \xi_1 \ldots \xi_m \in \mathrm{A}^*(X').$$

For $l = 1$, 2, call $i_l' \colon Y_l' \hookrightarrow X'$ the inclusion, $s_l' \colon Y_l' \to i_l'^* E$ the tautological section, F_l the restriction of F to Y_l'.

Observe that the bundle $\Lambda_2 \oplus \ldots \oplus \Lambda_m$ of F is totally isotropic: however, its inverse image in E is $\Lambda_1 \oplus \ldots \Lambda_m$ on Y_1, but is $\Lambda_2 \oplus \ldots \oplus \Lambda_m \oplus \Lambda_1^\vee$ on Y_2. The first bundle is admissible, the second one is not. Hence we have

$$y_{m-1}(F_1) = 2^{m-2} \xi_2 \ldots \xi_m \in \mathrm{A}^*(Y_1')$$

and

$$y_{m-1}(F_2) = -2^{m-2} \xi_2 \ldots \xi_m \in \mathrm{A}^*(Y_1').$$

Since we also have $[Y_1] = -\xi_1$ and $[Y_2] = \xi_1$ in $\mathrm{A}^*(X')$, we get

$$\begin{aligned} i_* y_{m-1} &= i_{1*} y_{m-1}(F_1) + i_{2*} y_{m-1}(F_2) \\ &= i_{1*} i_1^* 2^{m-2} \xi_2 \ldots \xi_m - i_{2*} i_2^* 2^{m-2} \xi_2 \ldots \xi_m \\ &= \xi_1 2^{m-2} \xi_2 \ldots \xi_m + \xi_1 2^{m-2} \xi_2 \ldots \xi_m \\ &= 2^{m-1} \xi_1 \ldots \xi_m \end{aligned}$$

and Lemma 5.5 is proved. ■

This proves that $c_2, \ldots, c_n, y_m$ generate $\mathrm{A}^*_{\mathrm{SO}_n}$.

Step 2: the relations are satisfied. The fact that $2c_i = 0$ when i is odd follows immediately, as for O_n, from the fact that V is self-dual.

To prove that $y_m c_i = 0$, it is sufficient to show that $c_m(L_m)c_i = 0$ in $\mathrm{A}^*(Q_1)$, for any vector bundle E on X, with an SO_n structure, as $y_m c_i = f_{1*}(s \cdot \mathrm{c}_m(L_m)c_i)$. But on Q_1 there is an exact sequence of vector bundles

$$0 \to L_m \to f^*E \to L_m^\vee \to 0$$

so the total Chern class $\mathrm{c}({f_1}^*E)$ is $\mathrm{c}(L_m)\mathrm{c}(L_m^\vee)$ and $\mathrm{c}_i(f^*E) = 0$ when i is odd.

Finally, the normal bundle N of C in $\mathbb{A}^n \setminus \{0\}$ is trivial, since the ideal of C is generated by an invariant function on $\mathbb{A}^n - \{0\}$, so

$$\begin{aligned} y_m^2 &= i_* y_{m-1} \cdot i_* y_{m-1} \\ &= i_*(y_{m-1} \cdot i^* i_* y_{m-1}) \\ &= i_*(y_{m-1}^2 \cdot \mathrm{c}_1(N)) \\ &= 0 \end{aligned}$$

in $\mathrm{A}^*_{\mathrm{SO}_n}(\mathbb{A}^n \setminus \{0\}) = \mathrm{A}^*_{\mathrm{SO}_n}/(c_n)$, by the projection formula and the self-intersection formula. Hence there is an integer d such that $y_m^2 = dc_n$; we will compute d once again by restricting to a maximal torus. By Lemma 5.7 we have

$$\begin{aligned} y_m^2 &= 2^{2m-2}\xi_1^2 \dots \xi_m^2 \\ &= 2^{n-2}(-1)^m c_n \in \mathrm{A}^n_{T_m}; \end{aligned}$$

hence, since c_n is not a torsion element of $\mathrm{A}^*_{T_m}$, we get that $d = 2^{n-2}$, as claimed.

Step 3: the relations suffice. Consider the ideal J in the polynomial ring $\mathbb{Z}[x_2, \dots, x_n, z]$ generated by the polynomials $z^2 - (-1)^m 2^{n-2} x_n$, $2x_{\mathrm{odd}}$, zx_{odd}. Let $P \in \mathbb{Z}[x_2, \dots, x_n, z]$ a homogeneous polynomial such that

$$P(c_2, \dots, c_n, y_m) = 0;$$

we need to show that P is in J.

By modifying P by an element of J, we may assume that it is of the form $Q_1 + zQ_2 + R$, where Q_1 and Q_2 are polynomials in the even x_i, while R is a polynomial in the x_i with coefficients that are all 0 or 1, and all of whose non-zero monomial contain some x_i with i odd.

The odd c_i restrict to 0 in $\mathrm{A}^*_{T_m}$, while c_{2j} restricts to the j^{th} symmetric function s_j of $-\xi_1^2, \dots, -\xi_m^2$; also, y_m restricts to $\xi_1 \dots \xi_m$. Hence $P(c_2, \dots, c_m, y_m) = 0$ restricts to $Q_1(s_2, s_4, \dots) + \xi_1 \dots \xi_m Q_2(s_2, s_4, \dots)$; and this is easily seen to imply that $Q_1 = Q_2 = 0$.

Hence P is a polynomial in $x_2, \ldots, x_n$, all of whose coefficients are 0 or 1. Now consider the basis $e'_1, \ldots, e'_n$ of V, and the subgroup $\boldsymbol{\mu}_2^n \subseteq \mathrm{O}_n$ considered in the previous section, consisting of linear transformations that take each e'_i into e'_i or $-e'_i$. The subgroup $\Gamma_n \stackrel{\mathrm{def}}{=} \boldsymbol{\mu}_2^n \cap \mathrm{SO}_n$ consists of the elements $(\varepsilon_1, \ldots, \varepsilon_n)$ of $\boldsymbol{\mu}_2^n$ such that $\varepsilon_1 \ldots \varepsilon_n = 1$ in $\boldsymbol{\mu}_2$. The group Γ_n is isomorphic to $\boldsymbol{\mu}_2^{n-1}$; if we call $\eta_i \in \mathrm{A}^1_{\Gamma_n}$ the first Chern class of the restriction to Γ_n of the i^{th} projection $\boldsymbol{\mu}_2^n \to \boldsymbol{\mu}_2 \subseteq \mathbb{G}_{\mathrm{m}}$, then we have

$$\mathrm{A}^*_{\Gamma_n} = \mathbb{Z}[\eta_1, \ldots, \eta_n]/(\eta_1 + \ldots + \eta_n).$$

We have a natural homomorphism $\mathrm{A}^*_{\Gamma_n} \to \mathbb{F}_2[\eta_1, \ldots, \eta_n]/(\eta_1 + \ldots + \eta_n)$, which is an isomorphism in positive degree. If we denote by $r_1, \ldots, r_n$ the elementary symmetric functions of the h_i, we have that c_i restricts to the image of r_i in $\mathbb{F}_2[\eta_1, \ldots, \eta_n]/(r_1)$; hence all we need to show is that the images of $r_2, \ldots, r_n$ are algebraically independent in $\mathbb{F}_2[\eta_1, \ldots, \eta_n]/(r_1)$. But $r_1, \ldots, r_n$ are algebraically independent in $\mathbb{F}_2[\eta_1, \ldots, \eta_n]$, so $r_2, \ldots, r_n$ are algebraically independent in $\mathbb{F}_2[r_1, \ldots, r_n]/(r_1)$; and the homomorphism

$$\mathbb{F}_2[r_1, \ldots, r_n]/(r_1) \to \mathbb{F}_2[\eta_1, \ldots, \eta_n]/(r_1)$$

is injective, because the extension $\mathbb{F}_2[r_1, \ldots, r_n] \subseteq \mathbb{F}_2[\eta_1, \ldots, \eta_n]$ is faithfully flat. This shows that $P = 0$, and completes the proof of the theorem.

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