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Temperate Holomorphic Solutions and Regularity of Holonomic D-modules on Curves.

ANA RITA MARTINS (*)

ABSTRACT - In [8], Kashiwara and Schapira introduced the notion of regularity for ind-sheaves and conjectured that a holonomic $\mathcal{D}$-module on a complex manifold is regular if and only if its complex of temperate holomorphic solutions is regular. Our aim is to prove this conjecture in the one-dimensional case.

Introduction.

In [8], the authors introduced the notions of microsupport and regularity for ind-sheaves. Let X be a complex manifold, $\mathcal{M}$ a coherent $\mathcal{D}_X$-module and consider its complex of temperate holomorphic solutions

$$Sol^t(\mathcal{M}) := R\mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X\mathcal{M}, \mathcal{O}^t_X).$$

It is proved in [8] that the microsupport of $Sol^t(\mathcal{M})$ coincides with the characteristic variety of $\mathcal{M}$. Moreover, if $\mathcal{M}$ is regular holonomic, then $Sol^t(\mathcal{M})$ is regular. In fact, Kashiwara and Schapira made the following conjecture:

(*K-S*)-CONJECTURE. *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then $\mathcal{M}$ is regular holonomic if and only if $R\mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X\mathcal{M}, \mathcal{O}^t_X)$ is regular.*

In this paper, we prove, in dimension one, that the regularity of $Sol^t(\mathcal{M})$ implies the regularity of the holonomic $\mathcal{D}_X$-module $\mathcal{M}$. More precisely, we show that $Sol^t(\mathcal{M})$ is irregular when the holonomic $\mathcal{D}_X$-module $\mathcal{M}$ has an irregular singularity. This proof relies in several steps. First we reduce to the case $\mathcal{M} = \mathcal{D}^m_X/\mathcal{D}^m_X P$, where P is a matrix of differential operators of the

(*) Indirizzo dell'A.: Centro de Álgebra da Universidade de Lisboa, Complexo 2, Avenida Prof. Gama Pinto,1699 Lisboa Portugal.
E-mail: arita@mat.fc.ul.pt

form $z^N \partial_z I_m + A(z)$, with $m, N \in \mathbb{N}$, I_m the identity matrix of order m and A a $m \times m$ matrix of holomorphic functions on a neighborhood of the origin. Then we show that it is enough to prove the irregularity of $\mathcal{S}^t := H^0(Sol^t(\mathcal{M}))$ and we give a characterization of $\mathcal{S}^t$ in a sector. From this characterization we easily conclude a contradiction by assuming the regularity of $\mathcal{S}^t$ at $(0;0)$, which completes the desired proof.

The contents of this paper are two sections as follows.

In Section 1, we start with a quick review on sheaves, ind-sheaves, microsupport and regularity for ind-sheaves and we recall the results on the microsupport and regularity of $Sol^t(\mathcal{M})$, proved in [8].

Section 2 is dedicated to the proof of the irregularity of $Sol^t(\mathcal{M})$, when $\mathcal{M}$ is an irregular holonomic $\mathcal{D}_X$-module on an open neighborhood X of 0 in $\mathbb{C}$.

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1. Notations and review.

We will follow the notations in [8].

SHEAVES. Let X be a real analytic n-dimensional manifold. We denote by $\pi : T^*X \to X$ the cotangent bundle to X. We identify X with the zero section of T^*X and we denote by $\dot{T}^*X$ the set $T^*X \backslash X$.

Let $\mathbf{k}$ be a field. We denote by $\mathrm{Mod}(\boldsymbol{k}_X)$ the abelian category of sheaves of $\mathbf{k}$-vector spaces on X and by $D^b(\mathbf{k}_X)$ its bounded derived category. For $a, b \in \mathbb{R}$, $a < b$ (resp. $k \in \mathbb{Z}$), we denote by $D^{[a,b]}(\mathbf{k}_X)$ (resp. $D^{\geq k}(\mathbf{k}_X)$) the full additive subcategory of $D^b(\mathbf{k}_X)$ consisting of objects F satisfying $H^j(F) = 0$, for any $j \notin [a, b]$ (resp. $j < k$).

We denote by $\mathbb{R}\mathrm{-C}(\boldsymbol{k}_X)$ the abelian category of $\mathbb{R}$-constructible sheaves of $\boldsymbol{k}$-vector spaces on X and by $D^b_{\mathbb{R}-\mathrm{C}}(\boldsymbol{k}_X)$ the full subcategory of $D^b(\boldsymbol{k}_X)$ consisting of objects with $\mathbb{R}$-constructible cohomology.

For an object $F \in D^b(\mathbf{k}_X)$, we denote by $SS(F)$ the *microsupport of* F, a closed $\mathbb{R}^+$-conic involutive subset of T^*X. We refer to [9] for details.

IND-SHEAVES ON REAL MANIFOLDS. Let X be a real analytic manifold. We denote by $\mathrm{I}(\boldsymbol{k}_X)$ the abelian category of ind-sheaves on X, that is, the category of ind-objects of the category $\mathrm{Mod}^c(\boldsymbol{k}_X)$ of sheaves with compact support on X (see [7]).

Recall the natural faithful exact functor

$$\iota_X : \mathrm{Mod}(\boldsymbol{k}_X) \to \mathrm{I}(\boldsymbol{k}_X); F \mapsto \text{“}\varinjlim_{\substack{U \subset\subset X \\ U \text{ open}}}\text{”} F_U.$$

We usually don't write this functor and identify $\mathrm{Mod}(\boldsymbol{k}_X)$ with a full abelian subcategory of $\mathrm{I}(\boldsymbol{k}_X)$ and $D^b(\boldsymbol{k}_X)$ with a full triangulated subcategory of $D^b(\mathrm{I}(\boldsymbol{k}_X))$.

The category $\mathrm{I}(\boldsymbol{k}_X)$ admits an internal hom denoted by $\mathcal{I}hom$ and this functor admits a left adjoint, denoted by $\otimes$. If $F \simeq \text{“}\varinjlim_i\text{”} F_i$ and $G \simeq \text{“}\varinjlim_j\text{”} G_j$, then:

$$\mathcal{I}hom(G, F) \simeq \varprojlim_j \text{“}\varinjlim_i\text{”} \mathcal{H}om(G_j, F_i),$$

$$G \otimes F \simeq \text{“}\varinjlim_j\text{”}\text{“}\varinjlim_i\text{”}(G_j \otimes F_i).$$

The functor ι_X admits a left adjoint

$$\alpha_X : \mathrm{I}(\boldsymbol{k}_X) \to \mathrm{Mod}(\boldsymbol{k}_X); F = \text{“}\varinjlim_i\text{”} F_i \mapsto \varinjlim_i F_i.$$

This last functor also admits a left adjoint $\beta_X : \mathrm{Mod}(\boldsymbol{k}_X) \to \mathrm{I}(\boldsymbol{k}_X)$. Both functors α_X and β_X are exact. We refer to [7] for the description of β_X.

Let X be a real analytic manifold. We denote by $\mathbb{R}\text{−}\mathrm{C}^c(\boldsymbol{k}_X)$ the full abelian subcategory of $\mathbb{R}\text{−}\mathrm{C}(\boldsymbol{k}_X)$ consisting of $\mathbb{R}$-constructible sheaves with compact support. We denote by $\mathrm{I}\mathbb{R}\text{−}\mathrm{c}(\boldsymbol{k}_X)$ the category $\mathrm{Ind}(\mathbb{R}\text{−}\mathrm{C}^c(\boldsymbol{k}_X))$ and by $D^b_{\mathrm{I}\mathbb{R}-\mathrm{c}}(\mathrm{I}(\boldsymbol{k}_X))$ the full subcategory of $D^b(\mathrm{I}(\boldsymbol{k}_X))$ consisting of objects with cohomology in $\mathrm{I}\mathbb{R}\text{−}\mathrm{c}(\boldsymbol{k}_X)$.

THEOREM 1.1 ([7]). *The natural functor* $D^b(\mathrm{I}\mathbb{R}\text{−}\mathrm{c}(\boldsymbol{k}_X)) \to D^b_{\mathrm{I}\mathbb{R}-\mathrm{c}}(\mathrm{I}(\boldsymbol{k}_X))$ *is an equivalence of categories.*

Recall that there is an alternative construction of $\mathrm{I}\mathbb{R}\text{−}\mathrm{c}(\boldsymbol{k}_X)$, using Grothendieck topologies. Denote by $\mathrm{Op}_{X_{sa}}$ the category of open subanalytic subsets of X. We may endow this category with a Grothendieck topology by deciding that a family $\{U_i\}_i$ in $\mathrm{Op}_{X_{sa}}$ is a covering of $U \in \mathrm{Op}_{X_{sa}}$ if for any compact subset K of X, there exists a finite subfamily which covers $U \cap K$. One denotes by X_{sa} the site defined by this topology and by $\mathrm{Mod}(\boldsymbol{k}_{X_{sa}})$ the category of sheaves on X_{sa} (see [1] and [7]). We denote by $\mathrm{Op}^c_{X_{sa}}$ the subcategory of $\mathrm{Op}_{X_{sa}}$ consisting of relatively compact open subanalytic subsets of X and for $U \in \mathrm{Op}_{X_{sa}}$ we denote by $U_{X_{sa}}$ the category $\mathrm{Op}_{X_{sa}} \cap U$ with the topology induced by X_{sa}.

Let $\rho : X \to X_{sa}$ be the natural morphism of sites. We have functors

$$\mathrm{Mod}(\boldsymbol{k}_X) \underset{\rho^{-1}}{\overset{\rho_*}{\rightleftarrows}} \mathrm{Mod}(\boldsymbol{k}_{X_{sa}}),$$

and we still denote by ρ_* the restriction of ρ_* to $\mathbb{R}-\mathrm{C}(\boldsymbol{k}_X)$ and to $\mathbb{R}-\mathrm{C}^c(\boldsymbol{k}_X)$.

We may extend the functor $\rho_* : \mathbb{R}-\mathrm{C}^c(\boldsymbol{k}_X) \to \mathrm{Mod}(\boldsymbol{k}_{X_{sa}})$ to $\mathrm{I}\mathbb{R}-\mathrm{c}(\boldsymbol{k}_X)$, by setting:

$$\begin{array}{rccc} \lambda : & \mathrm{I}\mathbb{R}-\mathrm{c}(\boldsymbol{k}_X) & \to & \mathrm{Mod}(\boldsymbol{k}_{X_{sa}}) \\ & \text{“}\varinjlim_i\text{”} F_i & \mapsto & \varinjlim_i \rho_* F_i. \end{array}$$

For $F \in \mathrm{I}\mathbb{R}-\mathrm{c}(\boldsymbol{k}_X)$, an alternative definition of $\lambda(F)$ is given by the formula

$$\lambda(F)(U) = \mathrm{Hom}_{\mathrm{I}\mathbb{R}-\mathrm{c}(\boldsymbol{k}_X)}(\boldsymbol{k}_U, F).$$

THEOREM 1.2 ([7]). *The functor λ is an equivalence of abelian categories.*

Most of the time, thanks to λ, we identify $\mathrm{I}\mathbb{R}-\mathrm{c}(\boldsymbol{k}_X)$ with $\mathrm{Mod}(\boldsymbol{k}_{X_{sa}})$.

TEMPERED DISTRIBUTIONS. Let X be a real analytic manifold. Denote by $\mathcal{D}b_X$ the sheaf of distributions on X. For each open subanalytic subset $U \subset X$, we denote by $\mathcal{D}b^t_X(U)$ the space of tempered distributions on U, defined by the exact sequence

$$0 \to \Gamma_{X \setminus U}(X; \mathcal{D}b_X) \to \Gamma(X; \mathcal{D}b_X) \to \mathcal{D}b^t_X(U) \to 0.$$

It is proved in [7] that $U \mapsto \mathcal{D}b^t_X(U)$ is a sheaf on the subanalytic site X_{sa}, hence defines an ind-sheaf. We call $\mathcal{D}b^t_X$ the ind-sheaf of tempered distributions. This ind-sheaf is well-defined in the category $\mathrm{Mod}(\beta_X \mathcal{D}_X)$, where $\mathcal{D}_X$ denotes the sheaf of analytic finite-order differential operators.

TEMPERED HOLOMORPHIC FUNCTIONS. Let X be a complex analytic manifold. One defines the ind-sheaf of tempered holomorphic functions as:

$$\mathcal{O}^t_X := R\mathcal{I}hom_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X_\mathbb{R}}),$$

where $\overline{X}$ denotes the complex conjugate manifold, $X_\mathbb{R}$ the underlying real analytic manifold, identified with the diagonal of $X \times \overline{X}$, and $\mathcal{D}_{\overline{X}}$ the sheaf of rings of holomorphic differential operators of finite order over $\overline{X}$. $\mathcal{O}^t_X$ is actually an object of $D^b(\beta_X\mathcal{D}_X)$ and it is not concentrated in degree 0 if dim $X > 1$. When X is a complex analytic curve, $\mathcal{O}^t_X$ is concentrated in degree 0. Moreover, $\mathcal{O}_X$ is ρ_*-acyclic and $\mathcal{O}^t_X$ is a sub-ind-sheaf of $\rho_*\mathcal{O}_X$.

We end this section by recalling two results of G. Morando, which will be useful in our proof.

THEOREM 1.3 ([2]). *Let X be an open subset of $\mathbb{C}$ and $f \in \mathcal{O}_{\mathbb{C}}(X)$. Let $U \in \mathrm{Op}^c_{X_{sa}}$ such that $f|_{\overline{U}}$ is an injective map. Let $h \in \mathcal{O}_{\mathbb{C}}(f(U))$. Then $h \circ f \in \mathcal{O}^t_X(U)$ if and only if $h \in \mathcal{O}^t_{\mathbb{C}}(f(U))$.*

PROPOSITION 1.4 ([2]). *Let $p \in z^{-1}\mathbb{C}[z^{-1}]$ and $U \in \mathrm{Op}^c_{\mathbb{C}_{sa}}$ with $0 \in \partial U$. The conditions below are equivalent.*

(i) $\exp(p(z)) \in \mathcal{O}^t_{\mathbb{C}}(U)$.
(ii) *There exists $A > 0$ such that* $\mathrm{Re}(p(z)) < A$, *for all $z \in U$.*

MICROSUPPORT AND REGULARITY FOR IND-SHEAVES. We refer to [8] for the equivalent definitions for the microsupport $SS(F)$ of an object $F \in D^b(\mathrm{I}(\boldsymbol{k}_X))$. We shall only recall the following useful properties of this closed conic subset of T^*X.

PROPOSITION 1.5. (i) *For $F \in D^b(\mathrm{I}(\boldsymbol{k}_X))$, one has $SS(F) \cap T^*_X X = \mathrm{supp}(F)$.*

(ii) *Let $F \in D^b(\boldsymbol{k}_X)$. Then $SS(\iota_X F) = SS(F)$.*
(iii) *Let $F \in D^b(\mathrm{I}(\boldsymbol{k}_X))$. Then $SS(\alpha_X(F)) \subset SS(F)$.*
(iii) *Let $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ be a distinguished triangle in $D^b(\mathrm{I}(\boldsymbol{k}_X))$. Then $SS(F_i) \subset SS(F_j) \cup SS(F_k)$, for $\{i, j, k\} = \{1, 2, 3\}$.*

Let J denotes the functor $J : D^b(\mathrm{I}(\boldsymbol{k}_X)) \to (D^b(\mathrm{Mod}^c(\boldsymbol{k}_X)))^\wedge$ (where $(D^b(\mathrm{Mod}^c(\boldsymbol{k}_X)))^\wedge$ denotes the category of functors from $D^b(\mathrm{Mod}^c(\boldsymbol{k}_X))^{\mathrm{op}}$ to **Set**) defined by:

$$J(F)(G) = \mathrm{Hom}_{D^b(\mathrm{I}(\boldsymbol{k}_X))}(G, F),$$

for every $F \in D^b(\mathrm{I}(\boldsymbol{k}_X))$ and $G \in D^b(\mathrm{Mod}^c(\boldsymbol{k}_X))$.

DEFINITION 1.6 ([8]). Let $F \in D^b(\mathrm{I}(\boldsymbol{k}_X))$, $\Lambda \subset T^*X$ be a locally closed conic subset and $p \in T^*X$. We say that F *is regular along Λ at p* if there exists F' isomorphic to F in a neighborhood of $\pi(p)$, an open neighborhood U of p with $\Lambda \cap U$ closed in U, a small and filtrant category I and a functor $\mathrm{I} \to D^{[a,b]}(\boldsymbol{k}_X); i \mapsto F_i$ such that $J(F') \simeq \text{``}\varinjlim_{i \in I}\text{''} J(F_i)$ and $SS(F_i) \cap U \subset \Lambda$. Otherwise, we say that F is irregular along Λ at p.

We say that F is regular at p if F is regular along $SS(F)$ at p. If F is regular at each $p \in SS(F)$, we say that F is regular.

PROPOSITION 1.7 ([8]). (i) *Let* $F \in D^b(\mathrm{I}(\boldsymbol{k}_X))$. *Then* F *is regular along any locally closed set* Λ *at each* $p \notin SS(F)$.

(ii) *Let* $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ *be a distinguished triangle in* $D^b(\mathrm{I}(\boldsymbol{k}_X))$. *If* F_j *and* F_k *are regular along* Λ, *so is* F_i, *for* $i,j,k \in \{1,2,3\}$, $j \neq k$.

(iii) *Let* $F \in D^b(\boldsymbol{k}_X)$. *Then* $\iota_X F$ *is regular.*

TEMPERATE HOLOMORPHIC SOLUTIONS OF $\mathcal{D}$-MODULES. Let X be a complex manifold and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Set

$$Sol(\mathcal{M}) = R\rho_* R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X),$$

$$Sol^t(\mathcal{M}) = R\mathcal{I}hom_{\beta_X \mathcal{D}_X}(\beta_X \mathcal{M}, \mathcal{O}^t_X).$$

The equality:

$$SS(Sol^t(\mathcal{M})) = Char(\mathcal{M}), \tag{1.1}$$

was obtained by M. Kashiwara and P. Schapira in [8], where these authors also proved that the natural morphism $Sol^t(\mathcal{M}) \to Sol(\mathcal{M})$ is an isomorphism, when $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module. This gives the "only if" part of the (K-S)-Conjecture.

2. Proof of the (K-S)-Conjecture in dimension one.

In this section, we consider $\mathbb{C}$ endowed with the holomorphic coordinate z and X will denote an open neighborhood of 0 in $\mathbb{C}$. We shall prove that, for every irregular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, $Sol^t(\mathcal{M})$ is irregular, using a similar argument as in the Example of [8].

We shall first reduce the proof to the case where $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$, for some $P \in \mathcal{D}_X$.

Let $\mathcal{M}$ be an irregular holonomic $\mathcal{D}_X$-module and let us denote by $Char(\mathcal{M})$ its characteristic variety. Since $\mathcal{M}$ is holonomic it is locally generated by one element and we may assume $\mathcal{M}$ is of the form $\mathcal{D}_X/\mathcal{I}$, for some coherent left ideal $\mathcal{I}$ of $\mathcal{D}_X$. We may also assume that, locally at $0 \in \mathbb{C}$, $Char(\mathcal{M}) \subset T^*_X X \cup T^*_{\{0\}} X$. Moreover, we may find $P \in \mathcal{I}$ such that the kernel of the surjective morphism

$$\mathcal{D}_X/\mathcal{D}_X P \to \mathcal{M} \to 0,$$

is isomorphic to a regular holonomic $\mathcal{D}_X$-module $\mathcal{N}$ (see, for example, Chapter VI of [10]). Therefore, we have an exact sequence

$$0 \to \mathcal{N} \to \mathcal{D}_X/\mathcal{D}_X P \to \mathcal{M} \to 0,$$

and we get the distinguished triangle

$$Sol^t(\mathcal{M}) \to Sol^t(\mathcal{D}_X/\mathcal{D}_X P) \to Sol^t(\mathcal{N}) \xrightarrow{+1} .$$

Since $Sol^t(\mathcal{N})$ is regular, by Proposition 1.7, $Sol^t(\mathcal{M})$ will be regular if and only if $Sol^t(\mathcal{D}_X/\mathcal{D}_X P)$ is. Therefore, we may assume from the beginning that $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$, for some $P \in \mathcal{D}_X$, having an irregular singularity at the origin.

Let us now recall the following result, due to G. Morando:

THEOREM 2.1 ([2]). *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. The natural morphism*

$$H^1(Sol^t(\mathcal{M})) \to H^1(Sol(\mathcal{M})),$$

is an isomorphism.

The Theorem above together with the results in [4] entails that:

$$H^1(Sol^t(\mathcal{M})) \simeq H^1(Sol(\mathcal{M})) \simeq \mathbb{C}^m_{\{0\}},$$

for some $m \in \mathbb{N}$. Then $H^1(Sol^t(\mathcal{M}))$ is regular and $SS(H^1(Sol^t(\mathcal{M}))) = = T^*_{\{0\}}X$.

As in [8], let us set for short

$$\mathcal{S}^t := H^0(Sol^t(\mathcal{M})) \simeq \mathcal{I}hom_{\beta_X \mathcal{D}_X}(\beta_X \mathcal{M}, \mathcal{O}^t_X),$$

$$\mathcal{S} := H^0(Sol(\mathcal{M})) \simeq \rho_* \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq ker(\rho_* \mathcal{O}_X \xrightarrow{P} \rho_* \mathcal{O}_X).$$

Remark that, since $\dim X = 1$, one has a monomorphism $\mathcal{S}^t \to \mathcal{S}$. Moreover, we have the following distinguished triangle:

$$\mathcal{S}^t \to Sol^t(\mathcal{M}) \to H^1(Sol^t(\mathcal{M}))[-1] \xrightarrow{+1} .$$

Therefore, one has

$$SS(\mathcal{S}^t) \subset Char(\mathcal{M}) \cup T^*_{\{0\}}X \subset T^*_X X \cup T^*_{\{0\}}X,$$

and $\mathcal{S}^t$ will be irregular if and only if $Sol^t(\mathcal{M})$ is.

The problem is then reduced to prove the irregularity of $\mathcal{S}^t$, for an irregular holonomic $\mathcal{D}_X$-module of the form $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$, with $P \in \mathcal{D}_X$.

Moreover, we may assume P is of the form $P = z^N\partial_z^m + \sum_{k=0}^{m-1} a_k(z)\partial_z^k$, for some $N, m \in \mathbb{N}$.

Let U be an open neighborhood of the origin in $\mathbb{C}$. The problem of finding the solutions of the differential equation $Pu = 0$ in $\mathcal{O}_X(U)$ is equivalent to the one of finding the solutions in $\mathcal{O}_X(U)^m$ of a system of ordinary differential equations defined by a matrix of differential operators of the form

$$z^N\partial_z I_m + A(z),$$

where $m, N \in \mathbb{N}$, I_m is the identity matrix of order m and $A \in \mathrm{M}_m(\mathcal{O}_X(U))$(1). From now on we denote by P the system

$$P = z^N\partial_z I_m + A(z), \tag{2.1}$$

and we reduce to the case where $\mathcal{M} = \mathcal{D}_X^m/\mathcal{D}_X^m P$, so that

$$\mathcal{S} \simeq \rho_*\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X^m/\mathcal{D}_X^m P, \mathcal{O}_X),$$

and

$$\mathcal{S}^t \simeq \mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X(\mathcal{D}_X^m/\mathcal{D}_X^m P), \mathcal{O}_X^t).$$

Let $\theta_0, \theta_1, R \in \mathbb{R}$, with $\theta_0 < \theta_1$ and $R > 0$. We denote the open set

$$\{z \in \mathbb{C}; \theta_0 < \arg z < \theta_1, 0 < |z| < R\},$$

by $S(\theta_0, \theta_1, R)$ and call it *open sector of amplitude* $\theta_1 - \theta_0$ *and radius* R.

Let $l \in \mathbb{N}$. By choosing a branch, we consider $z^{1/l}$ as a holomorphic function on subsets of open sectors of amplitude smaller than 2π.

The next goal is to calculate the ind-sheaf $\mathcal{S}^t$. As an essential step we recall the following classical result that gives the characterization of the holomorphic solutions of the matrix of differential operators $z^N\partial_z I_m + A(z)$ in some open sectors.

THEOREM 2.2 (see [12]). *Let us denote by P the matrix of differential operators $z^N\partial_z I_m + A(z)$. There exist $l \in \mathbb{N}$, a diagonal matrix $\Lambda(z) \in \mathrm{M}_m(z^{-1/l}\mathbb{C}[z^{-1/l}])$ and, for any real number θ, there exist $R > 0$, $\theta_1 > \theta > \theta_0$ and $F_\theta \in \mathrm{GL}_m(\mathcal{O}_X(S(\theta_0, \theta_1, R)) \cap C^0(\overline{S(\theta_0, \theta_1, R)}\setminus\{0\}))$, such that the m-columns of the matrix $F_\theta(z)\exp(-\Lambda(z))$ are $\mathbb{C}$-linearly independent holomorphic solutions of the system $Pu = 0$. Moreover, for each*

(1) For a commutative ring R we denote by $\mathrm{M}_m(R)$ the ring of $m \times m$ matrices and by $\mathrm{GL}_m(R)$ the group of invertible $m \times m$ matrices.

θ there exist constants $C, M > 0$ so that F_θ has the estimate

(2.2) $$C^{-1}|z|^M < |F_\theta(z)| < C|z|^{-M}, \text{ for any } z \in S(\theta_0, \theta_1, R).$$

If there is no risk of confusion we shall write $F(z)$ instead of F_θ.

DEFINITION 2.3. We call the matrix $F(z)\exp(-\Lambda(z))$, given in Theorem 2.2, a *fundamental solution of P on $S(\theta_0, \theta_1, R)$.*

Let us point out that Theorem 2.2 gives a characterization of the holomorphic solutions of the systems of differential operators of the form $z^N \partial_z I_m + A(z)$, not necessarily irregular. However, it follows by Theorem 5.1 of [5] that the matrix Λ given by Theorem 2.2 will be non-zero if and only if P is irregular.

Let $l \in \mathbb{N}$ and $\Lambda(z)$ be the diagonal matrix given in Theorem 2.2 for the operator (2.1). For each $1 \leq j \leq m$, let $\Lambda_j(z) = \sum_{k=1}^{n_j} \alpha_k^j z^{-k/l}$ be the (j, j) entry of $\Lambda(z)$, with $n_j \in \mathbb{N}$, $\alpha_1^j, ..., \alpha_{n_j}^j \in \mathbb{C}$.

COROLLARY 2.4. *Let $V \in \mathrm{Op}^c_{X_{sa}}$ and let us suppose P has a fundamental solution $F(z)\exp(-\Lambda(z))$ on V. Then, $\Gamma(V; \mathcal{S}^t) \simeq \mathbb{C}^{n(V)}$, where $n(V)$ is the cardinality of the set:*

$$J(V) := \{j \in \{1, ..., m\}; \exp(-\Lambda_j(z))|_V \in \mathcal{O}^t_X(V)\}.$$

PROOF. By hypothesis, $\Gamma(V; \mathcal{S})$ is the m-dimensional $\mathbb{C}$-vector space generated by the m-columns of the matrix $F(z)\exp(-\Lambda(z))$. Let k be the dimension of the $\mathbb{C}$-vector space $\Gamma(V; \mathcal{S}^t)$. Clearly $n(V) \leq k$. Let us prove that $k \leq n(V)$.

Let $G_1, ..., G_k$ be a $\mathbb{C}$-basis of $\Gamma(V; \mathcal{S}^t)$. Clearly, for $h = 1, ..., k$, there exists $C_h \in \mathbb{C}^m$ such that $G_h = F(z)\exp(-\Lambda(z))C_h$. In particular, the j-th coordinate of $F^{-1}G_h$ is a complex multiple of $\exp(-\Lambda_j)$. Further, since F^{-1} is a matrix of tempered holomorphic functions, $F^{-1}G_1, ..., F^{-1}G_k$ are $\mathbb{C}$-linearly independent vectors in $\mathcal{O}^t_X(V)^m$. It follows that there exists $\{j_1, ..., j_k\} \subset \{1, ..., m\}$ such that $\exp(-\Lambda_{j_1}(z)), ..., \exp(-\Lambda_{j_k}(z)) \in \mathcal{O}^t_X(V)$. The conclusion follows. q.e.d.

LEMMA 2.5. *Let S be an open sector of amplitude smaller than 2π, $p \in z^{-1}\mathbb{C}[z^{-1}]$, $l \in \mathbb{N}$ and $V \in \mathrm{Op}^c_{X_{sa}}$, with $V \subset S$ and $0 \in \partial V$. Then $\exp(p(z^{1/l})) \in \mathcal{O}^t_X(V)$ if and only if there exists $A > 0$ such that $\mathrm{Re}(p(z^{1/l})) < A$, for all $z \in V$.*

PROOF. Let $\theta_0, \theta_1, R \in \mathbb{R}$ such that $0 < \theta_1 - \theta_0 < 2\pi$ and $S = S(\theta_0, \theta_1, R)$, and let us denote by U the open sector $S\left(\frac{\theta_0}{l}, \frac{\theta_1}{l}, R^{1/l}\right)$. Let $f : X \to X$ be the holomorphic function defined by $f(z) = z^l$. Since $\theta_1 - \theta_0 < 2\pi$, we may easily check that $f|_{\overline{U}}$ is an injective map. Moreover, $f(U) = S$ and $f|_U : U \to S$ is bijective. Set $V' = f^{-1}(V) \cap U$ and let h denotes the holomorphic function defined for each $z \in S$ by $h(z) = \exp(p(z^{1/l}))$. By Theorem 1.3, we have $h \circ f \in \mathcal{O}^t_X(V')$ if and only if $h \in \mathcal{O}^t_X(V)$. On the other hand, one has $\exp(p)|_{V'} = h \circ f|_{V'}$ and, by Proposition 1.4, $h \circ f \in \mathcal{O}^t_X(V')$ if and only if there exists $A > 0$ such that $\mathrm{Re}(p(z)) < A$, for all $z \in V'$. Combining these two facts, we conclude that $\exp(p(z^{1/l})) \in \mathcal{O}^t_X(V)$ if and only if there exists $A > 0$ such that $\mathrm{Re}(p(z^{1/l})) < A$, for all $z \in V$, as desired. q.e.d.

PROPOSITION 2.6. *With the notation above, there exist an open sector S, with amplitude smaller than 2π and radius $R > 0$, and a non-empty subset* I *of* $\{1, ..., m\}$ *such that, for each* $j \in \mathrm{I}$ *and each open subanalytic subset* $V \subset S$, *the conditions below are equivalent:*

(i) *there exists* $A > 0$ *such that* $\mathrm{Re}(-\Lambda_j(z)) < A$ *for all* $z \in V$,
(ii) *there exists* $0 < \delta < R$ *such that* $V \subset \{z \in S; |z| > \delta\}$.

Moreover, for each $j \in \{1, ..., m\} \setminus \mathrm{I}$, *there exists* $A > 0$ *such that, for every* $z \in S$, $\mathrm{Re}(-\Lambda_j(z)) < A$.

PROOF. For each $j = 1, ..., m$, if $z = \rho \exp(i\theta)$, one has:

$$\mathrm{Re}(-\Lambda_j(z)) = \sum_{k=1}^{n_j} \alpha^j_k \rho^{-k/l} \cos(\phi^j_k - k/l\theta),$$

where $\alpha^j_k = |a^j_k|$ and $\phi^j_k = \arg(-a^j_k)$, for every $k = 1, ..., n_j$.

Since $\Lambda \neq 0$, we may assume from the beginning that $\alpha^1_{n_1} \neq 0$. For each $j = 1, ..., m$ and $\theta \in \mathbb{R}$, set $c_j(\theta) = \cos(\phi^j_{n_j} - n_j/l\theta)$. Pick $\theta' \in \mathbb{R}$ such that $c_j(\theta') \neq 0$, for $j = 1, ..., m$, and $c_1(\theta') > 0$. By continuity, these conditions hold in a neighborhood $[\theta_0, \theta_1]$ of θ'. Moreover, we may assume that one has $0 < \theta_1 - \theta_0 < 2\pi$.

Let us set:

$$\mathrm{J} := \{j \in \{1, ..., m\}; c_j(\theta) < 0,\ \forall \theta \in [\theta_0, \theta_1]\} \cup \{j \in \{1, ..., m\}; \Lambda_j = 0\}.$$

Let $j \in \mathrm{J}$, with $\Lambda_j(z) \neq 0$, and choose $C_j > 0$ such that $c_j(\theta) \leq -C_j$, for all $\theta \in [\theta_0, \theta_1]$. We may assume $\alpha^j_{n_j} \neq 0$. Then, for each $\theta \in [\theta_0, \theta_1]$ and $\rho > 0$, one has:

$$\operatorname{Re}(-\Lambda_j(\rho\exp(i\theta))) =$$
$$= \rho^{-n_j/l}\left[\sum_{k=1}^{n_j-1}\alpha_k^j\rho^{(n_j-k)/l}\cos(\phi_k^j - k/l\theta) + \alpha_{n_j}^j\cos(\phi_{n_j}^j - n_j/l\theta)\right] \leq$$
$$\leq \rho^{-n_j/l}\left[\sum_{k=1}^{n_j-1}\alpha_k^j\,\rho^{(n_j-k)/l} - \alpha_{n_j}^j C_j\right],$$

and

$$\lim_{\rho\to 0^+}\rho^{-n_j/l}\left[\sum_{k=1}^{n_j-1}\alpha_k^j\rho^{(n_j-k)/l} - \alpha_{n_j}^j C_j\right] = -\infty.$$

Hence, for each $j \in \mathrm{J}$, there exists $R_j > 0$ such that $\operatorname{Re}(-\Lambda_j(\rho\exp(i\theta))) \leq 0$, for every $0 < \rho < R_j$ and $\theta_0 \leq \theta \leq \theta_1$. Therefore, setting $R = \min\{R_j; j \in \mathrm{J}\}$, one gets that $\operatorname{Re}(-\Lambda_j(z)) < A$, for every $A > 0$, $z \in S(\theta_0, \theta_1, R)$ and $j \in \mathrm{J}$.

Let us now set

$$\mathrm{I} := \{j \in \{1, ..., m\}; c_j(\theta) > 0, \forall\theta \in [\theta_0, \theta_1], \Lambda_j(z) \neq 0\}.$$

Let $j \in \mathrm{I}$ and $C_j > 0$ such that $c_j(\theta) > C_j$, for all $\theta \in [\theta_0, \theta_1]$. We may assume $\alpha_{n_j}^j \neq 0$. Let V be an open subanalytic subset of the sector $S(\theta_0, \theta_1, R)$ and suppose that there exists $A > 0$ such that $\operatorname{Re}(-\Lambda_j(z)) < A$, for every $z \in V$, and that, for each $0 < \delta < R$, there exists $z_\delta \in V$ with $|z_\delta| \leq \delta$. For each $0 < \delta < R$, let us denote: $\rho_\delta = |z_\delta|$ and $\theta_\delta = \arg(z_\delta)$. The sequence $\{\rho_\delta\}_\delta$ converges to 0 and one has:

$$\lim_{\delta\to 0^+}\operatorname{Re}(-\Lambda_j(\rho_\delta\exp(i\theta_\delta))) =$$
$$= \lim_{\delta\to 0^+}\rho_\delta^{-n_j/l}\left[\sum_{k=1}^{n_j-1}\alpha_k^j\rho_\delta^{(n_j-k)/l}\cos(\phi_k^j - k/l\theta_\delta) + \alpha_{n_j}^j\cos(\phi_{n_j}^j - n_j/l\theta_\delta)\right] \geq$$
$$\geq \lim_{\delta\to 0^+}\rho_\delta^{-n_j/l}\left[-\sum_{k=1}^{n_j-1}\alpha_k^j\rho_\delta^{(n_j-k)/l} + \alpha_{n_j}^j C_j\right] = +\infty,$$

which is a contradiction. Conversely, if V is an open subanalytic subset of the set $\{z \in S(\theta_0, \theta_1, R); |z| > \delta\}$, for some $0 < \delta < R$, then V is contained on the compact set $\{z \in \mathbb{C}; \theta_0 \leq \arg z \leq \theta_1, \delta \leq |z| \leq R\}$, and $\operatorname{Re}(-\Lambda_j(z))$ is obviously bounded on V. We conclude that I is the desired subset of $\{1, ..., m\}$, with $\{1, ..., m\}\setminus\mathrm{I} = \mathrm{J}$. q.e.d.

We shall now describe the ind-sheaf of temperate holomorphic solutions of the differential system $Pu = 0$ in the open sector given by Proposition 2.6, where P is the operator (2.1).

THEOREM 2.7. *Let $\mathcal{M}$ be the $\mathcal{D}_X$-module $\mathcal{D}_X^m/\mathcal{D}_X^m P$, where P is the matrix of differential operators* (2.1). *If $\mathcal{M}$ is irregular, then there exist $n > 0$ and an open sector S such that:*

$$\text{(2.3)} \qquad \text{“}\varinjlim_{R>\delta>0}\text{”}(\mathbb{C}^n_{S_\delta} \oplus \mathbb{C}^{m-n}_S) \xrightarrow{\sim} \mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X\mathcal{M}, \mathcal{O}^t_X)_S,$$

where $S_\delta = \{z \in S; |z| > \delta\}$, for each $0 < \delta < R$.

PROOF. Let $S(\theta_0', \theta_1', R')$ and I be, respectively, the open sector and the subset of $\{1, ..., m\}$ given by Proposition 2.6 and let us choose $\theta_0, \theta_1, R \in \mathbb{R}$, with $\theta_0' < \theta_0 < \theta_1 < \theta_1'$ and $R > 0$, such that the matrix of differential operators P admits a fundamental solution $F(z)\exp(-\Lambda(z))$ on the open sector $S(\theta_0, \theta_1, R)$. Let us denote $S = S(\theta_0, \theta_1, R)$ and let n be the cardinality of the set I. Remark that $\mathrm{I} \neq \emptyset$ and so, $n > 0$.

Let V be a connected subanalytic open subset of S, relatively compact in X, with $0 \in \partial V$. By Lemma 2.5, for each $j = 1, ..., m$, one has $\exp(-\Lambda_j(z)) \in \mathcal{O}^t_X(V)$ if and only if there exists $A > 0$ such that $\mathrm{Re}(-\Lambda_j(z)) < A$ for each $z \in V$. Thus, by Proposition 2.6, $\exp(-\Lambda_j(z)) \in \mathcal{O}^t_X(V)$, for all $j \in \{1, ..., m\}\backslash \mathrm{I}$, and, for $j \in \mathrm{I}$ one has $\exp(-\Lambda_j(z)) \in \mathcal{O}^t_X(V)$ if and only if $V \subset S_\delta$, for some $0 < \delta < R$. Since $0 \in \partial V$, it follows that $V \not\subseteq S_\delta$, for all $0 < \delta < R$ and, hence, $\exp(-\Lambda_j(z)) \notin \mathcal{O}^t_X(V)$, for all $j \in \mathrm{I}$.

Therefore, by Corollary 2.4, given a connected subanalytic open subset V of S, relatively compact in X, either $V \subset S_\delta$, for some $0 < \delta < R$, and in this case $\Gamma(V; \mathcal{S}^t) \simeq \mathbb{C}^m$, or else $\Gamma(V; \mathcal{S}^t) \simeq \mathbb{C}^{m-n}$. By Theorem 1.2, we get the desired isomorphism (2). q.e.d.

Finally, we may conclude the irregularity of $\mathcal{S}^t$ arguing by contradiction. Let S (resp. n) be the open sector (resp. the positive integer) given by Theorem 2.7. Let us suppose that $\mathcal{S}^t$ is regular at $p = (0; 0)$. Then, there exist an open neighborhood U of 0, a small and filtrant category L and a functor $G : L \to D^{[a,b]}(\mathrm{Mod}^c(\mathbb{C}_X))$, $l \mapsto G_l$, such that $J_X(\mathcal{S}^t) \simeq$ “$\varinjlim_l$”G_l and $SS(G_l) \cap \pi^{-1}(U) \subset T^*_{\{0\}}X \cup T^*_X X$, for all $l \in L$. We may assume from the

(2) Notice that, for $F \in \mathrm{Mod}(\boldsymbol{k}_{X_{sa}})$, one has $F_S := i_{S!}i_S^{-1}F$, where $i_S : S_{X_{sa}} \to X_{sa}$ denotes the natural embedding (see [7]).

beginning that U is an open ball with center at the origin and that $S \subset U$. Since $0 \notin S$, we get:

$$SS(G_l) \cap \pi^{-1}(S) \subset T_X^*X, \ \forall l.$$

Then, for each l and $k \in \mathbb{Z}$, $H^k(G_l)_S$ is a constant sheaf, since S is contractible. In particular, we may find $M_l \in D^b(\mathrm{Mod}(\mathbb{C}))$ such that $(G_l)_S \simeq (M_l)_S$, for each l. Moreover, one has:

$$\text{``}\varinjlim_{R>\delta>0}\text{''}\mathbb{C}^n_{S_\delta} \oplus \mathbb{C}^{m-n}_S \simeq \text{``}\varinjlim_{l}\text{''}H^0(M_l)_S.$$

Let V be a connected subanalytic open subset of S_ε, for some $R > \varepsilon > 0$, and assume V is contractible. Then:

$$\mathbb{C}^m \simeq \text{``}\varinjlim_{R>\delta>0}\text{''}\Gamma(V; \mathbb{C}^n_{S_\delta} \oplus \mathbb{C}^{m-n}_S) \simeq \varinjlim_{l} \Gamma(V; H^0(M_l)_S) \simeq \varinjlim_{l} H^0(M_l).$$

On the other hand:

$$\mathbb{C}^{m-n} \simeq \text{``}\varinjlim_{R>\delta>0}\text{''}\Gamma(S; \mathbb{C}^n_{S_\delta} \oplus \mathbb{C}^{m-n}_S) \simeq \varinjlim_{l} \Gamma(S; H^0(M_l)_S) \simeq \varinjlim_{l} H^0(M_l).$$

This entails that $\mathbb{C}^m \simeq \mathbb{C}^{m-n}$ and hence, $n = 0$, which is a contradiction.

REMARK 2.8. Let us consider the irregular holonomic $\mathcal{D}_X$-module

$$\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X(z^2\partial_z + 1).$$

In this case, $\exp(1/z)$ is a fundamental solution of the differential operator $z^2\partial_z + 1$ in $X\backslash\{0\}$. Arguing as in the proof of Proposition 2.6, we find $R > 0$ with the following property: given an open subanalytic subset V of the sector $S = S(0, \pi/4, R)$, then there exists $A > 0$ such that $\mathrm{Re}(-1/z) < A$, for every $z \in V$, if and only if there exists $0 < \delta < R$ such that $V \subset \{z \in S; |z| > \delta\}$. Moreover, by Proposition 2.7, one has the isomorphism below:

$$\text{``}\varinjlim_{R>\delta>0}\text{''}\mathbb{C}_{S_\delta} \xrightarrow{\sim} \mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X\mathcal{M}, \mathcal{O}^t_X)_S.$$

On the other hand, M. Kashiwara and P. Schapira proved in [8] the following isomorphism on X,

$$\text{``}\varinjlim_{\varepsilon>0}\text{''}\mathbb{C}_{U_\varepsilon} \sim \mathcal{I}hom_{\beta_X\mathcal{D}_X}(\beta_X\mathcal{M}, \mathcal{O}^t_X),$$

where $U_\varepsilon = X\backslash\overline{B_\varepsilon(\varepsilon, 0)}$, and $B_\varepsilon(\varepsilon, 0)$ denotes the open ball with center at $(\varepsilon, 0)$ and radius ε, for every $\varepsilon > 0$.

Let us check that

$$``\varinjlim_{R>\delta>0}"\mathbb{C}_{S_\delta} \simeq ``\varinjlim_{\varepsilon>0}"\mathbb{C}_{U_\varepsilon\cap S}.$$

It is enough to prove that, for every $0<\delta<R$, there exists $\varepsilon>0$ such that $S_\delta \subset U_\varepsilon \cap S$ and that, for every $\varepsilon>0$, there exists $0<\delta<R$ such that $U_\varepsilon \cap S \subset S_\delta$. In fact, for each $0<\delta<R$, if $x+iy \in S_\delta$, then $x^2+y^2>\delta^2$ and $x>y>0$. It follows that $2x^2>\delta^2$ and hence, $x^2+y^2>x^2>x(\delta/2)$, this is, $(x-\delta/4)^2+y^2>(\delta/4)^2$. Thus, $S_\delta \subset U_{\delta/4}\cap S$. Conversely, given $\varepsilon>0$ and $x+iy \in U_\varepsilon \cap S$, we have $x^2+y^2>2x\varepsilon$ and $x>y>0$. This gives $x>\varepsilon$ and so, $x^2+y^2>\varepsilon^2$. Therefore, $U_\varepsilon \cap S \subset S_\varepsilon$.

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