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A Frictionless Contact Problem with Adhesion and Damage in Elasto-Viscoplasticity.

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ABSTRACT - We consider a model for the quasistatic, adhesive and frictionless contact problem for an elasto-viscoplastic material with damage. The adhesion process on the contact surface is modelled by a surface internal variable, the bonding field, and the tangential shear due to the bonding field is included. The problem is formulated as a system of a variational equality for the displacements, an inclusion of parabolic type for the damage field and an integro-differential equation for the bonding field. The existence of the weak solution for the problem is established by monotone operator and fixed-point arguments.

1. Introduction.

We investigate a mathematical model for the process of frictionless, adhesive and bilateral contact between an elasto-viscoplastic body and a rigid foundation. We assume that slowly varying time-dependent volume forces and surfaces tractions act on the body, thereby causing its mechanical state to evolve quasistatically. We model the mechanical properties of the material by an elasto-viscoplastic constitutive law

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma},\varepsilon(\boldsymbol{u}),\beta),$$

in which $\boldsymbol{\sigma} = (\sigma_{ij})$ denotes the stress tensor, $\boldsymbol{u} = (u_i)$ the displacement field and $\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ the linearized strain tensor, moreover, we use the dot above to indicate the derivative with respect to the time variable. $\mathcal{E}$ and $\mathcal{G}$ are the material constitutive functions. $\mathcal{E}$ is assumed to be linear and $\mathcal{G}$ nonlinear, β is an internal variable which may represent the damage of the material caused by plastic deformations, its evolution is governed by the

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$$\dot{\beta} - k \triangle \beta + \partial\varphi_K(\beta) \ni \phi(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}), \beta),$$

where k is a positive coefficient, $\partial\varphi_K$ denotes the subdifferential of the indicator function φ_K, where K denotes the set of admissible damage functions which satisfy $0 < \beta < 1$. When $\beta = 1$ the material is undamaged, when $\beta = 0$ the material is completely damaged, and for $0 < \beta < 1$ there is partial damage. ϕ is a given constitutive function which describes the sources of damage in the system. General models of mechanical damage, which were derived from thermodynamical considerations and the principle of virtual work, can be found in [7] and [8] and references therein. The models describe the evolution of the material damage which results from the excess tension or compression in the body as a result of applied forces and tractions. Mathematical analysis of quasistatic one-dimensional damage models can be found in [9]. The importance of this paper is to make the coupling of an elasto-viscoplastic problem with damage and a frictionless contact problem with adhesion.

Here, our purpose is to describe the delamination process when the frictional tangential traction is negligible in comparison with the traction due to adhesion. As in [5, 6] and [13, 16], we use the bonding field as an additional dependent variable, defined and evolving on the contact surface. We provide a variational formulation of the model and, using arguments of evolutionary equations in Banach spaces, we prove that the model has a unique weak solution.

The paper is structured as follows. The model is described in section 3 where the variational formulation is given. In section 4, we present our main result stated in Theorem 4.1 and its proof which is based on the construction of mappings between appropriate Banach spaces and a fixed-point arguments.

2. Notation and preliminaries.

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [1, 4] and [10, 12]. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while (.) and $|\ .\ |$ represent the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let υ denote the unit outer normal on Γ. We shall use the notation

$$H = L^2(\Omega)^d = \{ \boldsymbol{u} = (u_i) \ / \ u_i \in L^2(\Omega) \ \},$$
$$\mathcal{H} = \{ \boldsymbol{\sigma} = (\sigma_{ij}) \ / \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \ \},$$
$$H_1 = \{ \boldsymbol{u} = (u_i) \in H \ / \ \varepsilon(\boldsymbol{u}) \in \mathcal{H} \ \},$$
$$\mathcal{H}_1 = \{ \boldsymbol{\sigma} \in \mathcal{H} \ / \ Div \ \boldsymbol{\sigma} \in H \},$$

where $\varepsilon : H_1 \to \mathcal{H}$ and $Div : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \ Div \, \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Here and below, the indices i and j run between 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i \ dx \qquad \forall \, \boldsymbol{u}, \boldsymbol{v} \in H,$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij} \tau_{ij} \ dx \qquad \forall \, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H},$$
$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} \qquad \forall \, \boldsymbol{u}, \boldsymbol{v} \in H_1,$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div \ \boldsymbol{\sigma}, Div \ \boldsymbol{\tau})_H \ \ \forall \ \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathcal{H}_1.$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $| \, . \, |_H$, $| \, . \, |_{\mathcal{H}}$, $| \, . \, |_{H_1}$ and $| \, . \, |_{\mathcal{H}_1}$respectively. Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $\boldsymbol{v} \in H_1$, we also use the notation $\boldsymbol{v}$ to denote the trace $\gamma\boldsymbol{v}$ of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on the boundary Γ given by

(2.1) $$v_\nu = \boldsymbol{v}.\boldsymbol{\nu}, \ \ \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}.$$

Similarly, for a regular (say C^1) tensor field $\boldsymbol{\sigma} : \Omega \to S_d$ we define its normal and tangential components by

(2.2) $$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}).\boldsymbol{\nu}, \ \ \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu},$$

and we recall that the following Green's formula holds:

(2.3) $$(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (Div \ \boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}.\boldsymbol{v} \ da \qquad \forall \, \boldsymbol{v} \in H_1.$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leq p \leq +\infty$ and $k \geq 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$| \boldsymbol{f} |_{C(0,T;X)} = \max_{t\in[0,T]} | \boldsymbol{f}(t) |_X,$$

$$| \boldsymbol{f} |_{C^1(0,T;X)} = \max_{t\in[0,T]} | \boldsymbol{f}(t) |_X + \max_{t\in[0,T]} | \dot{\boldsymbol{f}}(t) |_X,$$

respectively. Moreover, for a real number r, we use r_+ to represent its positive part, that is $r_+ = \max\{0, r\}$. For the convenience of the reader, we recall the following version of the classical theorem of Cauchy-Lipschitz (see, e.g., [17] p. 60).

THEOREM 2.1. *Assume that* $(X, \ | \, . \, |_X \ )$ *is a real Banach space and* $T > 0$. *Let* $F(t,.) : X \to X$ *be an operator defined a.e. on* $(0,T)$ *satisfying the following conditions:*

1) $\exists \ L_F > 0$ *such that* $| F(t,x) - F(t,y) |_X \leq L_F \, | x - y |_X \ \forall x, y \in X$, *a.e.* $t \in (0,T)$.

2) $\exists \, p \geq 1$ *such that* $t \longmapsto F(t,x) \in L^p(0,T;X) \quad \forall x \in X$.

Then for any $x_0 \in X$, *there exists a unique function* $x \in W^{1,\,p}(0,T;X)$ *such that*

$$\dot{x}(t) = F(t, x(t)) \qquad a.e. \ t \in (0,T),$$

$$x(0) = x_0.$$

Theorem 2.1 will be used in section 4 to prove the unique solvability of the intermediate problem involving the bonding field.

Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1\times X_2}$.

3. Mechanical and variational formulations.

We describe the model for the process, we present its variational formulation. The physical setting is the following. An elasto-viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with outer Lipschitz surface Γ that is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. A surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2 \times (0, T)$ and a body force of density $\boldsymbol{f}_0$ acts in $\Omega \times (0, T)$. The body is in adhesive contact with a rigid obstacle, or foundation, over the contact surface Γ_3. Moreover the process is quasistatic, i.e. the inertial terms are neglected in the equation of motion.

To simplify the notation, we do not indicate explicitely the dependence of various functions on the variables $\boldsymbol{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$. Now, we describe the conditions on the contact surface Γ_3. We assume that the contact is bilateral, i.e., there is no separation between the body and the foundation during the process. Therefore, the normal displacement vanishes on $\Gamma_3 \times (0, T)$. We introduce the surface state variable α, which is a measure of the fractional intensity of adhesion between the surface and the foundation. This variable is restricted to have values $0 \leq \alpha \leq 1$, when $\alpha = 0$ all the bonds are severed and there are no active bonds, when $\alpha = 1$ all the bonds are active. When $0 < \alpha < 1$ it measures the fraction of active bonds, and partial adhesion takes place.

We assume that the resistance to tangential motion is generated by the glue, in comparison to which the fractional traction can be neglected. A different assumption, taking friction into account, can be found in [13, 14, 15]. Thus, the tangential traction depends only on the intensity of adhesion, and the tangential displacement,

$$-\boldsymbol{\sigma}_\tau = \boldsymbol{p}_\tau(\alpha, \boldsymbol{u}_\tau) \text{ on } \Gamma_3 \times (0, T).$$

In particular, we may consider the case

$$\boldsymbol{p}_\tau(\alpha, \boldsymbol{r}) = \begin{cases} q_\tau(\alpha)\, \boldsymbol{r} & \text{if } \mid \boldsymbol{r} \mid \leq L_0, \\ q_\tau(\alpha)\, \dfrac{L_0}{\mid \boldsymbol{r} \mid} \boldsymbol{r} & \text{if } \mid \boldsymbol{r} \mid > L_0, \end{cases} \tag{3.1}$$

where $L_0 > 0$ is the limit bound length and q_τ is a nonnegative tangential stiffness function. A more general condition may be used, especially if the surface has intrinsic directions, such as grooves, on it. Then, one needs to replace q_τ with a two-dimensional tensor. As in [3], the evolution of the adhesion field is assumed to depend generally on α and $\boldsymbol{u}_\tau$. The whole process is assumed to be governed by the differential equation,

$$\dot{\alpha} = H_{ad}(\alpha, R(\mid \boldsymbol{u}_\tau \mid)) \text{ on } \Gamma_3 \times (0, T). \tag{3.2}$$

Here, H_{ad} is a general function discussed below, which vanishes when its first argument vanishes. The function $R : \mathbb{R}_+ \to \mathbb{R}_+$ is a truncation and is defined as

$$R(s) = \begin{cases} s & \text{if} \qquad 0 \le s \le L, \\ L & \text{if} \qquad s > L, \end{cases}$$

where $L > 0$ is a characteristic length of the bonds (see, e.g. [13]). We use it since usually, when the glue is streched beyond the limit L it does not contribute more to the bond strength. An example of such a function, used in [2], is

$$H_{ad}(\alpha, r) = -\gamma_\nu \alpha_+ r^2,$$

where γ_ν is the bonding energy constant, and $\gamma_\nu L$ is the maximal tensile normal traction that the adhesive can provide. We note that in this case only debonding is allowed. Then, the mechanical formulation of the frictionless, bilateral and adhesive problem may be stated as follows.

Problem P. Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to S_d$*, a damage field* $\beta : \Omega \times [0,T] \to \mathbb{R}$ *and a bonding field* $\alpha : \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that*

(3.3) $$\dot{\boldsymbol{\sigma}} = \mathcal{E}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}), \beta) \text{ in } \Omega \times (0,T),$$

(3.4) $$\dot{\beta} - k \triangle \beta + \partial\varphi_K(\beta) \ni \phi(\boldsymbol{\sigma}, \varepsilon(\boldsymbol{u}), \beta),$$

(3.5) $$Div\ \boldsymbol{\sigma} + \boldsymbol{f}_0 = 0 \qquad \text{in } \Omega \times (0,T),$$

(3.6) $$\boldsymbol{u} = 0 \qquad \text{on } \Gamma_1 \times (0,T),$$

(3.7) $$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2 \times (0,T),$$

(3.8) $$u_\nu = 0 \qquad \text{on } \Gamma_3 \times (0,T),$$

(3.9) $$-\boldsymbol{\sigma}_\tau = \boldsymbol{p}_\tau(\alpha, \boldsymbol{u}_\tau) \text{ on } \Gamma_3 \times (0,T),$$

(3.10) $$\dot{\alpha} = H_{ad}(\alpha, R(\mid \boldsymbol{u}_\tau \mid)) \text{ on } \Gamma_3 \times (0,T),$$

(3.11) $$\frac{\partial\beta}{\partial\nu} = 0 \text{ on } \Gamma \times (0,T),$$

(3.12) $$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0,\ \beta(0) = \beta_0, \text{ in } \Omega,$$

(3.13) $$\alpha(0) = \alpha_0 \text{ on } \Gamma_3.$$

The relation (3.3) represents the elasto-viscoplastic constitutive law with damage; the evolution of the damage field is governed by the inclusion of parabolic type given by the relation (3.4), where ϕ is the mechanical source of the damage growth, assumed to be rather general function of the strains and damage itself, $\partial\varphi_K$ is the subdifferential of the indicator function of the admissible damage functions set K. The relation (3.11) represents a homogeneous Newmann boundary condition where $\dfrac{\partial\beta}{\partial\upsilon}$ represents the normal derivative of β. In (3.12)-(3.13) $\boldsymbol{u}_0, \boldsymbol{\sigma}_0, \beta_0$ and α_0 represent the initial displacement, the initial stress, the initial damage field and the initial adhesion field, respectively. To obtain the variational formulation of the problem (3.3)-(3.13), we consider the following notations. We introduce the set of admissible damage functions defined by

$$K = \big\{\xi \in H^1(\Omega)\ /\ 0 \leq \xi \leq 1 \text{ a.e. in } \Omega\big\},$$

for the bonding field we need the set

$$Z = \big\{\theta : [0,T] \to L^2(\Gamma_3)\ /\ 0 \leq \theta(t) \leq 1\ \forall t \in [0,T] \text{ a.e. on } \Gamma_3\big\},$$

and for the displacement field we need the closed subspace of H_1 defined by

$$V = \{\boldsymbol{v} \in H_1\ /\ \boldsymbol{v} = 0 \text{ on } \Gamma_1,\ v_\upsilon = 0 \text{ on } \Gamma_3\}.$$

Since $meas\ (\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_k > 0$, that depends only on Ω and Γ_1, such that

$$\mid \varepsilon(\boldsymbol{v}) \mid_{\mathcal{H}} \geq C_k \mid \boldsymbol{v} \mid_{H_1}\ \forall \boldsymbol{v} \in V.$$

A proof of Korn's inequality may be found in ([11] p. 79). On the space V we consider the inner product and the associated norm given by

$$(\boldsymbol{u},\boldsymbol{v})_V = (\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))_{\mathcal{H}},\quad \mid \boldsymbol{v} \mid_V = \mid \varepsilon(\boldsymbol{v}) \mid_{\mathcal{H}} \quad \forall \boldsymbol{u},\boldsymbol{v} \in V. \tag{3.14}$$

It follows that $\mid . \mid_{H_1}$ and $\mid . \mid_V$ are equivalent norms on V and therefore $(V, \mid . \mid_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.15), there exists a constant $C_0 > 0$, depending only on Ω, Γ_1 and Γ_3 such that

$$\mid \boldsymbol{v} \mid_{L^2(\Gamma_3)^d} \leq C_0 \mid \boldsymbol{v} \mid_V \qquad \forall \boldsymbol{v} \in V. \tag{3.15}$$

In the study of the mechanical problem (3.3)-(3.13), we assume that the viscosity operator $\mathcal{E} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} (a)\ \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times S_d \to S_d. \\ (b)\ \mathcal{E}_{ijkl} \in L^\infty(\Omega), \quad 1 \leq i,j,k,l \leq d. \\ (c)\ \mathcal{E}\boldsymbol{\sigma}.\boldsymbol{\tau} = \boldsymbol{\sigma}.\ \mathcal{E}\boldsymbol{\tau} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in S_d, \text{ a.e. in } \Omega. \\ (d)\ \text{There exists a constant } m_0 > 0 \text{ such that} \\ \mathcal{E}\boldsymbol{\tau}.\boldsymbol{\tau} \geq m_0 \mid \boldsymbol{\tau} \mid^2 \quad \forall \boldsymbol{\tau} \in S_d, \text{ a.e. in } \Omega. \end{cases} \tag{3.16}$$

The constitutive function $\mathcal{G}: \Omega \times S_d \times S_d \times \mathbb{R} \to S_d$ satisfies

$$(3.17)\quad \begin{cases} (a) \text{ There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad \mid \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \varepsilon_1, \beta_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}_2, \varepsilon_2, \beta_2) \mid \leq L_{\mathcal{G}}(\mid \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \mid + \mid \varepsilon_1 - \varepsilon_2 \mid \\ \quad + \mid \beta_1 - \beta_2 \mid) \ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \varepsilon_1, \varepsilon_2 \in S_d, \ \forall \beta_1, \beta_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\sigma}, \varepsilon \in S_d \text{ and } \beta \in \mathbb{R}, \ \boldsymbol{x} \to \mathcal{G}(\boldsymbol{x}, \boldsymbol{\sigma}, \varepsilon, \beta) \text{ is measurable on } \Omega. \\ (c) \text{ The mapping } \boldsymbol{x} \to \mathcal{G}(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}, 0) \in \mathcal{H}. \end{cases}$$

The source damage function ϕ: $\Omega \times S_d \times S_d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$(3.18)\quad \begin{cases} (a) \text{ There exists a constant } L_{\phi} > 0 \text{ such that} \\ \quad \mid \phi(\boldsymbol{x}, \boldsymbol{\sigma}_1, \varepsilon_1, \beta_1) - \phi(\boldsymbol{x}, \boldsymbol{\sigma}_2, \varepsilon_2, \beta_2) \mid \leq L_{\phi}(\mid \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \mid + \mid \varepsilon_1 - \varepsilon_2 \mid \\ \quad + \mid \beta_1 - \beta_2 \mid) \ \forall \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \varepsilon_1, \varepsilon_2 \in S_d, \ \forall \beta_1, \beta_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\sigma}, \varepsilon \in S_d \text{ and } \beta \in \mathbb{R}, \ \boldsymbol{x} \to \phi(\boldsymbol{x}, \boldsymbol{\sigma}, \varepsilon, \beta) \text{ is measurable on } \Omega. \\ (c) \text{ The mapping } \boldsymbol{x} \to \phi(\boldsymbol{x}, \boldsymbol{0}, \boldsymbol{0}, 0) \in \mathcal{H}. \end{cases}$$

The tangential contact function $\boldsymbol{p}_\tau : \Gamma_3 \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$(3.19)\quad \begin{cases} (a) \text{ There exists a constant } L_\tau > 0 \text{ such that} \\ \quad \mid \boldsymbol{p}_\tau(\boldsymbol{x}, d_1, \boldsymbol{r}_1) - \boldsymbol{p}_\tau(\boldsymbol{x}, d_2, \boldsymbol{r}_2) \mid \leq L_\tau(\mid d_1 - d_2 \mid + \mid \boldsymbol{r}_1 - \boldsymbol{r}_2 \mid) \\ \quad \forall d_1, d_2 \in \mathbb{R}, \ \boldsymbol{r}_1, \boldsymbol{r}_2 \in \mathbb{R}^d, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ (b) \text{ The map } \boldsymbol{x} \to \boldsymbol{p}_\tau(\boldsymbol{x}, d, \boldsymbol{r}) \text{ is Lebesgue measurable on } \Gamma_3 \\ \quad \forall d \in \mathbb{R}, \boldsymbol{r} \in \mathbb{R}^d. \\ (c) \text{ The map } \boldsymbol{x} \to \boldsymbol{p}_\tau(\boldsymbol{x}, 0, \boldsymbol{0}) \in L^2(\Gamma_3)^d. \\ (d) \ \boldsymbol{p}_\tau(\boldsymbol{x}, d, \boldsymbol{r}).\upsilon(\boldsymbol{x}) = 0 \ \forall \boldsymbol{r} \in \mathbb{R}^d \text{ such that } \boldsymbol{r}.\upsilon(\boldsymbol{x}) = 0, \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases}$$

Clearly, if $q_\tau : \mathbb{R} \to \mathbb{R}$ is a bounded Lipschitz continuous function, then the tangential contact function (3.1) satisfies condition (3.19). We conclude that our results below are valid for the corresponding contact problems. Next, the adhesion rate $H_{ad} : \Gamma_3 \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$(3.20)\quad \begin{cases} (a) \text{ There exists a constant } L_{ad} > 0 \text{ such that} \\ \quad \mid H_{ad}(\boldsymbol{x}, b_1, r_1) - H_{ad}(\boldsymbol{x}, b_2, r_2) \mid \leq L_{ad}(\mid b_1 - b_2 \mid + \mid r_1 - r_2 \mid) \\ \quad \forall b_1, b_2 \in \mathbb{R}, \ \forall r_1, r_2 \in [0, L], \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ (b) \text{ The map } \boldsymbol{x} \to H_{ad}(\boldsymbol{x}, b, r) \text{ is Lebesgue measurable on } \Gamma_3 \\ \quad \forall b \in \mathbb{R}, \ r \in [0, L]. \\ (c) \text{ The map } (b, r) \to H_{ad}(\boldsymbol{x}, b, r) \text{ is continuous on } \mathbb{R} \times [0, L], \\ \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ (d) \ H_{ad}(\boldsymbol{x}, 0, r) = 0 \ \forall r \in [0, L], \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ (e) \ H_{ad}(\boldsymbol{x}, b, r) \geq 0 \ \forall b \leq 0, \ r \in [0, L], \ \text{a.e. } \boldsymbol{x} \in \Gamma_3 \text{ and} \\ \quad H_{ad}(\boldsymbol{x}, b, r) \leq 0 \ \forall b \geq 1, \ r \in [0, L], \ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases}$$

We suppose that the body forces and surface tractions satisfy

(3.21) $$\boldsymbol{f}_0 \in W^{1,\infty}(0,T;H), \quad \boldsymbol{f}_2 \in W^{1,\infty}(0,T;L^2(\Gamma_2)^d).$$

Finally, we assume that the initial data satisfy the conditions

(3.22) $$\boldsymbol{u}_0 \in V, \boldsymbol{\sigma}_0 \in \mathcal{H},$$

(3.23) $$\beta_0 \in K,$$

(3.24) $$\alpha_0 \in L^2(\Gamma_3), 0 \le \alpha_0 \le 1 \text{ a.e. in } \Gamma_3,$$

(3.25) $$(\boldsymbol{\sigma}_0, \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + j(\alpha_0, \boldsymbol{u}_0, \boldsymbol{v}) = (\boldsymbol{f}(0), \boldsymbol{v})_V \forall \boldsymbol{v} \in V.$$

We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

(3.26) $$a(\xi, \varphi) = k \int_\Omega \nabla\xi \,.\, \nabla\, \varphi \; dx.$$

Next, we denote by $\boldsymbol{f} : [0,T] \to V$ the functional defined by

(3.27) $$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t).\boldsymbol{v} \; dx \; + \int_{\Gamma_2} \boldsymbol{f}_2(t).\boldsymbol{v} \; dx \;\; \forall \boldsymbol{v} \in V, \;\; t \in [0,T].$$

Let $j : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ be the adhesion functional given by

(3.28) $$j(\alpha, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} \boldsymbol{p}_\tau(\alpha, \boldsymbol{u}_\tau).\boldsymbol{v}_\tau \; da \qquad \forall \alpha \in L^\infty(\Gamma_3), \; \forall \boldsymbol{u}, \boldsymbol{v} \in V.$$

Keeping in mind (3.19), we observe that the integral (3.28) is well defined and we note that conditions (3.21) imply

(3.29) $$\boldsymbol{f} \in W^{1,\infty}(0,T;V).$$

Using standard arguments we obtain the variational formulation of the mechanical problem (3.3)-(3.13).

Problem PV. Find a displacement field $\boldsymbol{u} : [0,T] \to V$ *, a stress field* $\boldsymbol{\sigma} : [0,T] \to \mathcal{H}$*, a damage field* $\beta : [0,T] \to H^1\,(\Omega)$ *and a bonding field* $\alpha : [0,T] \to L^\infty\;(\Gamma_3)$ *such that*

(3.30) $$\dot{\boldsymbol{\sigma}}\,(t) = \mathcal{E}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{u}(t)), \beta(t)) \text{ a.e. } t \in (0,T),$$

$$\beta(t) \in K \text{ for all } t \in [0,T], (\dot{\beta}(t), \xi - \beta(t))_{L^2(\Omega)} + a(\beta(t), \xi - \beta(t))$$

(3.31) $$\ge (\phi(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{u}(t)), \beta(t)), \xi - \beta(t))_{L^2(\Omega)} \forall \xi \in K,$$

(3.32) $$(\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + j(\alpha(t), \boldsymbol{u}(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_V \ \forall \boldsymbol{v} \in V, \ \text{a.e. } t \in (0, T),$$

(3.33) $$\dot{\alpha}(t) = H_{ad}(\alpha(t), R(\mid \boldsymbol{u}_\tau(t) \mid)),$$

(3.34) $$\boldsymbol{u}(0) = \boldsymbol{u}_0, \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \ \beta(0) = \beta_0, \ \alpha(0) = \alpha_0.$$

The existence of the unique solution of problem PV is stated and proved in the next section. First, we note that the functional j is linear with respect to the last argument and, therefore,

(3.35) $$j(\alpha, \boldsymbol{u}, -\boldsymbol{v}) = -\, j(\alpha, \boldsymbol{u}, \boldsymbol{v}).$$

Next, using (3.15), (3.19) and (3.28) we find

$$\begin{aligned}
& j(\alpha_1, \boldsymbol{u}_1, \boldsymbol{v}) - j(\alpha_2, \boldsymbol{u}_2, \boldsymbol{v}) \\
&= \int_{\Gamma_3} (\boldsymbol{p}_\tau(\alpha_1, \boldsymbol{u}_{1\tau}) - \boldsymbol{p}_\tau(\alpha_2, \boldsymbol{u}_{2\tau})).\boldsymbol{v}_\tau \, da \\
&\leq L_\tau \int_{\Gamma_3} (|\alpha_1 - \alpha_2 \mid + \mid \boldsymbol{u}_1 - \boldsymbol{u}_2 \mid) \mid \boldsymbol{v} \mid da \\
&\leq L_\tau (|\alpha_1 - \alpha_2 \mid_{L^2(\Gamma_3)} + \mid \boldsymbol{u}_1 - \boldsymbol{u}_2 \mid_{L^2(\Gamma_3)^d}) \mid \boldsymbol{v} \mid_{L^2(\Gamma_3)^d} \\
&\leq L_\tau C_0 (|\alpha_1 - \alpha_2 \mid_{L^2(\Gamma_3)} + C_0 \mid \boldsymbol{u}_1 - \boldsymbol{u}_2 \mid_V) \mid \boldsymbol{v} \mid_V \\
&\leq L_\tau C_0 |\alpha_1 - \alpha_2 \mid_{L^2(\Gamma_3)} \mid \boldsymbol{v} \mid_V + L_\tau C_0^2 \mid \boldsymbol{u}_1 - \boldsymbol{u}_2 \mid_V \mid \boldsymbol{v} \mid_V.
\end{aligned}$$

We now choose $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ in the previous inequality to find

$$j(\alpha_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - j(\alpha_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)$$

(3.36) $$\leq L_\tau C_0 |\alpha_1 - \alpha_2 \mid_{L^2(\Gamma_3)} \mid \boldsymbol{u}_1 - \boldsymbol{u}_2 |_V + L_\tau C_0^2 \mid \boldsymbol{u}_1 - \boldsymbol{u}_2 \mid_V^2 .$$

We choose again $\alpha_1 = \alpha_2 = \alpha$ in (3.36) to obtain

$$j(\alpha, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - j(\alpha, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)$$

(3.37) $$\leq L_\tau C_0^2 \mid \boldsymbol{u}_1 - \boldsymbol{u}_2 \mid_V^2 .$$

Inequalities (3.36)-(3.37) and equality (3.35) will be used in various places in the rest of the paper.

4. A well posedness of the problem.

Now, we propose our existence and uniqueness result.

THEOREM 4.1. *Assume that* (3.16)-(3.25) *hold and, assume that*

$L_\tau < \dfrac{m_0}{C_0^2}$. *Then there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \beta, \alpha\}$ *to problem PV. Moreover, the solution satisfies*

$$\boldsymbol{u} \in W^{1,\infty}(0,T;V),$$
$$\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H}_1),$$
$$\beta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$
$$\alpha \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z. \tag{4.1}$$

We conclude that, under the assumptions (3.16)-(3.25), the mechanical problem (3.3)-(3.13) has a unique weak solution satisfying (4.1). The proof of Theorem 4.1 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 4.1 hold and we consider that C is a generic positive constant which depends on $\Omega, \Gamma_1, \Gamma_3, L$ and may change from place to place. Let $\mathcal{Z}$ denote the closed subset of $C(0,T;L^2(\Gamma_3))$ defined by

$$\mathcal{Z} = \big\{\theta \in C(0,T;L^2(\Gamma_3)) \cap Z \ / \ \theta(0) = \alpha_0\big\}. \tag{4.2}$$

Let $\alpha \in \mathcal{Z}$ and let $\eta = (\eta^1, \eta^2) \in C(0,T;\mathcal{H} \times L^2(\Omega))$. We define a function $\boldsymbol{z}_\eta \in C^1(0,T;\mathcal{H})$ by

$$\boldsymbol{z}_\eta(t) = \int_0^t \eta^1(s)\, ds + \boldsymbol{z}_0, \tag{4.3}$$

where

$$\boldsymbol{z}_0 = \boldsymbol{\sigma}_0 - \mathcal{E}\varepsilon(\boldsymbol{u}_0), \tag{4.4}$$

and in the first step we consider the following variational problem.

Problem $PV_{\alpha\eta}$. *Find a displacement field* $\boldsymbol{u}_{\alpha\eta} : [0,T] \to V$ *such that for all* $\boldsymbol{v} \in V$ *and all* $t \in [0,T]$

$$(\mathcal{E}\varepsilon(\boldsymbol{u}_{\alpha\eta}(t)), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{z}_\eta(t), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + j(\alpha(t), \boldsymbol{u}_{\alpha\eta}(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_V. \tag{4.5}$$

We have the following result for the problem.

LEMMA 4.2. *There exists a unique solution to problem* $PV_{\alpha\eta}$. *The solution satisfies* $\boldsymbol{u}_{\alpha\eta} \in C(0,T;V)$ *and*

$$\boldsymbol{u}_{\alpha\eta}(0) = \boldsymbol{u}_0. \tag{4.6}$$

PROOF. Let $t \in [0,T]$ and let $A_t : V \to V$ be the operator given by

$$(A_t\boldsymbol{u},\boldsymbol{v})_V = (\mathcal{E}\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))_{\mathcal{H}} + j(\alpha(t),\boldsymbol{u},\boldsymbol{v}) \quad \forall \boldsymbol{u},\boldsymbol{v} \in V.$$

We use (3.16) and the properties (3.36)-(3.37) on the adhesion functional j to show that the operator A_t is strongly monotone if

$$L_\tau < \frac{m_0}{C_0^2},$$

and Lipschitz continuous and, therefore, invertible and its inverse is also strongly monotone Lipschitz continuous on V. Moreover using Riesz Representation theorem we may define an element $\boldsymbol{f}_\eta : [0,T] \to V$ by

$$(\boldsymbol{f}_\eta(t),\boldsymbol{v})_V = (\boldsymbol{f}(t),\boldsymbol{v})_V - (\boldsymbol{z}_\eta(t),\varepsilon(\boldsymbol{v}))_{\mathcal{H}}.$$

It follows now from classical result (see for example [4]) that there exists a unique element $\boldsymbol{u}_{\alpha\eta}(t) \in V$ which solves

$$A_t\boldsymbol{u}_{\alpha\eta}(t) = \boldsymbol{f}_\eta(t) \qquad \text{a.e. } t \in [0,T].$$

Moreover, we use assumptions (3.22) and (3.25) to prove that the solution satisfies condition (4.6). We let $t_1, t_2 \in [0,T]$ and use the notation $\boldsymbol{u}_{\alpha\eta}(t_i) = \boldsymbol{u}_i$, $\boldsymbol{z}_\eta(t_i) = \boldsymbol{z}_i$, $\alpha(t_i) = \alpha_i$, $\boldsymbol{f}(t_i) = \boldsymbol{f}_i$, for $i = 1,2$. We use standard arguments in (4.5) to find

$$\begin{aligned}(\mathcal{E}\varepsilon(\boldsymbol{u}_1-\boldsymbol{u}_2),\varepsilon(\boldsymbol{u}_1-\boldsymbol{u}_2))_{\mathcal{H}} &= (\boldsymbol{z}_1-\boldsymbol{z}_2,\varepsilon(\boldsymbol{u}_2-\boldsymbol{u}_1))_{\mathcal{H}} + (\boldsymbol{f}_1-\boldsymbol{f}_2,\boldsymbol{u}_1-\boldsymbol{u}_2)_V \\ &\quad + j(\alpha_1,\boldsymbol{u}_1,\boldsymbol{u}_2-\boldsymbol{u}_1) + j(\alpha_2,\boldsymbol{u}_2,\boldsymbol{u}_1-\boldsymbol{u}_2),\end{aligned}$$

and, by using (3.16) and (3.36), we obtain

$$(4.7) \qquad |\,\boldsymbol{u}_1-\boldsymbol{u}_2\,|_V \le C(\,|\,\boldsymbol{z}_1-\boldsymbol{z}_2\,|_{\mathcal{H}} + |\,\boldsymbol{f}_1-\boldsymbol{f}_2\,|_V + |\,\alpha_1-\alpha_2\,|_{L^2(\Gamma_3)}\,).$$

This inequality and the regularity of the functions α, $\boldsymbol{f}$, and $\boldsymbol{z}_\eta$ show that $\boldsymbol{u}_{\alpha\eta} \in C(0,T;V)$. Thus we conclude the existence part in lemma 4.2 and we note that the uniqueness of the solution follows from the unique solvability of (4.5) for every $t \in [0,T]$. Next to prove (4.6) we write (4.5) at $t = 0$ and use the initial values $\boldsymbol{z}_\eta(0) = \boldsymbol{\sigma}_0 - \mathcal{E}\varepsilon(\boldsymbol{u}_0)$, $\alpha(0) = \alpha_0$ to obtain

$$\begin{aligned}&(\mathcal{E}\varepsilon(\boldsymbol{u}_{\alpha\eta}(0)),\varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{\sigma}_0 - \mathcal{E}\varepsilon(\boldsymbol{u}_0),\varepsilon(\boldsymbol{v}))_{\mathcal{H}} \\ (4.8) \qquad &+ j(\alpha_0,\boldsymbol{u}_{\alpha\eta}(0),\boldsymbol{v}) = (\boldsymbol{f}(0),\boldsymbol{v})_V \forall \boldsymbol{v} \in V.\end{aligned}$$

We consider now the difference between (4.8) and (3.25) to deduce

$$(4.9) \qquad (\mathcal{E}\varepsilon(\boldsymbol{u}_{\alpha\eta}(0)-\boldsymbol{u}_0),\varepsilon(\boldsymbol{v}))_{\mathcal{H}} + j(\alpha_0,\boldsymbol{u}_{\alpha\eta}(0),\boldsymbol{v}) - j(\alpha_0,\boldsymbol{u}_0,\boldsymbol{v}) = 0.$$

We now choose $\boldsymbol{v} = \boldsymbol{u}_{\alpha\eta}(0) - \boldsymbol{u}_0$ in (4.9) to obtain

$$\begin{aligned}&(\mathcal{E}\varepsilon(\boldsymbol{u}_{\alpha\eta}(0) - \boldsymbol{u}_0), \varepsilon(\boldsymbol{u}_{\alpha\eta}(0) - \boldsymbol{u}_0))_{\mathcal{H}}\\ &= j(\alpha_0, \boldsymbol{u}_{\alpha\eta}(0), \boldsymbol{u}_{\alpha\eta}(0) - \boldsymbol{u}_0) - j(\alpha_0, \boldsymbol{u}_0, \boldsymbol{u}_{\alpha\eta}(0) - \boldsymbol{u}_0).\end{aligned}$$

In this equality we use the assumption (3.16) on the elasticity tensor $\mathcal{E}$ and property (3.37) on the functional j and obtain

$$| \boldsymbol{u}_{\alpha\eta}(0) - \boldsymbol{u}_0 |_V^2 \leq 0.$$

This inequality gives us (4.6), which concludes the proof. □

Next, for a given $\alpha \in \mathcal{Z}$ and for every $\eta = (\eta^1, \eta^2) \in C(0, T; \mathcal{H} \times L^2(\Omega))$ we denote by $\boldsymbol{u}_{\alpha\eta}$ the solution of problem $PV_{\alpha\eta}$ obtained in lemma 4.2 and we define a function $\boldsymbol{\sigma}_{\alpha\eta} \in C(0, T; \mathcal{H})$ by

(4.10) $$\boldsymbol{\sigma}_{\alpha\eta}(t) = \mathcal{E}\varepsilon(\boldsymbol{u}_{\alpha\eta}(t)) + \boldsymbol{z}_\eta(t) \ \forall t \in [0, T].$$

We suppose that the assumptions of Theorem 4.1 hold and we consider the following intermediate problem for the damage field.

Problem PV_β. *Find a damage field* $\beta_{\alpha\eta} : [0, T] \to H^1(\Omega)$ *such that* $\beta_{\alpha\eta}(t) \in K$, *for all* $t \in [0, T]$ *and*

(4.11) $$\begin{aligned}&(\dot{\beta}_{\alpha\eta}(t), \xi - \beta_{\alpha\eta}(t))_{L^2(\Omega)} + a(\beta_{\alpha\eta}(t), \xi - \beta_{\alpha\eta}(t))\\ &\geq (\eta^2(t), \xi - \beta_{\alpha\eta}(t))_{L^2(\Omega)} \forall \xi \in K \text{ a.e. } t \in (0, T),\end{aligned}$$

(4.12) $$\beta_{\alpha\eta}(0) = \beta_0.$$

LEMMA 4.3. *Problem* PV_β *has a unique solution* $\beta_{\alpha\eta}$ *such that*

(4.13) $$\beta_{\alpha\eta} \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)).$$

PROOF. We use (3.23), (3.26) and a classical existence and uniqueness result on parabolic equations (see for instance [16]). □

We also use the properties (3.17) of the constitutive function $\mathcal{G}$ and (3.18) of the function ϕ to define the operator $\Lambda_\alpha : C(0, T; \mathcal{H} \times L^2(\Omega)) \to$ $\to C(0, T; \mathcal{H} \times L^2(\Omega))$ by

(4.14) $$\Lambda_\alpha \eta = (\mathcal{G}(\boldsymbol{\sigma}_{\alpha\eta}, \varepsilon(\boldsymbol{u}_{\alpha\eta}), \beta_{\alpha\eta}), \phi(\boldsymbol{\sigma}_{\alpha\eta}, \varepsilon(\boldsymbol{u}_{\alpha\eta}), \beta_{\alpha\eta})),$$

for all $\eta \in C(0, T; \mathcal{H} \times L^2(\Omega))$. Then we have.

LEMMA 4.4. *The operator Λ_α has a unique fixed-point*

$$\eta^*_\alpha \in C(0,T;\mathcal{H}\times L^2(\Omega)).$$

PROOF. Let $\eta_i = (\eta^1_i, \eta^2_i) \in C(0,T;\mathcal{H}\times L^2(\Omega))$, let $t\in[0,T]$ and use the notation $\boldsymbol{u}_{\alpha\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\alpha\eta_i} = \boldsymbol{\sigma}_i$, $\beta_{\alpha\eta_i} = \beta_i$ and $\boldsymbol{z}_{\eta_i} = \boldsymbol{z}_i$ for $i=1,2$. Taking into account the relations (3.17), (3.18) and (4.14), we deduce that

$$| \Lambda_\alpha\eta_1(t) - \Lambda_\alpha\eta_2(t) |_{\mathcal{H}\times L^2(\Omega)}$$

$$\le C(| \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t) |_{\mathcal{H}} + | \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) |_V + | \beta_1(t) - \beta_2(t) |_{L^2(\Omega)}). \tag{4.15}$$

Using (4.5) we obtain

$$(\mathcal{E}\varepsilon(\boldsymbol{u}_1 - \boldsymbol{u}_2), \varepsilon(\boldsymbol{u}_1 - \boldsymbol{u}_2))_{\mathcal{H}} = j(\alpha, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) - j(\alpha, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2)$$

$$+(\boldsymbol{z}_2 - \boldsymbol{z}_1, \varepsilon(\boldsymbol{u}_1 - \boldsymbol{u}_2))_{\mathcal{H}} \text{ a.e. } t\in(0,T). \tag{4.16}$$

Keeping in mind (3.15), (3.16) and (3.37) we find

$$| \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) |_V \le C \,| \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t) |_{\mathcal{H}} . \tag{4.17}$$

We use (4.10) and (4.17) to obtain

$$| \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t) |_{\mathcal{H}} \le C \,| \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t) |_{\mathcal{H}} . \tag{4.18}$$

We add the two previous inequalities and we use (4.3) and (4.4) to find

$$| \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) |_V + | \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t) |_{\mathcal{H}} \le C \int_0^t | \eta^1_1(s) - (\eta^1_2(s) |_{\mathcal{H}}\, ds. \tag{4.19}$$

From (4.11) we deduce that

$$(\dot\beta_1, \beta_2 - \beta_1)_{L^2(\Omega)} + a(\beta_1, \beta_2 - \beta_1)$$

$$\ge (\eta^2_1, \beta_2 - \beta_1)_{L^2(\Omega)} \text{ a.e. } t\in(0,T),$$

and,

$$(\dot\beta_2, \beta_1 - \beta_2)_{L^2(\Omega)} + a(\beta_2, \beta_1 - \beta_2)$$

$$\ge (\eta^2_2, \beta_1 - \beta_2)_{L^2(\Omega)} \text{ a.e. } t\in(0,T).$$

Adding the previous inequalities we obtain

$$(\dot\beta_1 - \dot\beta_2, \beta_1 - \beta_2)_{L^2(\Omega)} + a(\beta_1 - \beta_2, \beta_1 - \beta_2)$$

$$\le (\eta^2_1 - \eta^2_2, \beta_1 - \beta_2)_{L^2(\Omega)} \text{ a.e. } t\in(0,T),$$

which implies that

$$(\dot{\beta_1} - \dot{\beta_2}, \beta_1 - \beta_2)_{L^2(\Omega)} + a(\beta_1 - \beta_2, \beta_1 - \beta_2)$$
$$\leq | \eta_1^2 - \eta_2^2 |_{L^2(\Omega)} | \beta_1 - \beta_2 |_{L^2(\Omega)} \text{ a.e. } t \in (0, T).$$

Integrating the previous inequality on $[0, t]$, after some manipulations we obtain

$$\frac{1}{2} | \beta_1(t) - \beta_2(t) |^2_{L^2(\Omega)} \leq C \int_0^t | \eta_1^2(s) - \eta_2^2(s) |_{L^2(\Omega)} | \beta_1(s) - \beta_2(s) |_{L^2(\Omega)} \, ds$$
$$+ C \int_0^t | \beta_1(s) - \beta_2(s) |^2_{L^2(\Omega)} \, ds.$$

Applying Gronwall's inequality to the previous inequality yields

$$(4.20) \qquad | \beta_1(t) - \beta_2(t) |_{L^2(\Omega)} \leq C \int_0^t | \eta_1^2(s) - \eta_2^2(s) |_{L^2(\Omega)} \, ds.$$

Substituting (4.19) and (4.20) in (4.15), we obtain

$$| \Lambda_\alpha \eta_1(t) - \Lambda_\alpha \eta_2(t) |_{\mathcal{H} \times L^2(\Omega)}$$
$$\leq C \int_0^t | \eta_1(s) - (\eta_2(s) |_{\mathcal{H} \times L^2(\Omega)} \, ds.$$

Reiterating this inequality m times yields

$$(4.22) \qquad | \Lambda_\alpha^m \eta_1 - \Lambda_\alpha^m \eta_2 |_{C(0,T;\mathcal{H} \times L^2(\Omega))} \leq \frac{C^m T^m}{m!} | \eta_1 - \eta_2 |_{C(0,T;\mathcal{H} \times L^2(\Omega))},$$

which implies that for m sufficiently large a power Λ_α^m of Λ_α is a contraction in the Banach space $C(0, T; \mathcal{H} \times L^2(\Omega))$. Then from the Banach fixed-point theorem Λ_α has a unique fixed-point $\eta_\alpha^* \in C(0, T; \mathcal{H} \times L^2(\Omega))$. □

In the next step, for a given $\alpha \in \mathcal{Z}$, we consider the following variational problem.

Problem PV_α. Find a displacement field $\boldsymbol{u}_\alpha : [0, T] \to V$, a stress field $\boldsymbol{\sigma}_\alpha : [0, T] \to \mathcal{H}$ and a damage field $\beta_\alpha : [0, T] \to H^1(\Omega)$ such that for all

$t \in [0, T]$

$$\boldsymbol{\sigma}_\alpha(t) = \mathcal{E}\varepsilon(\boldsymbol{u}_\alpha(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_\alpha(s), \varepsilon(\boldsymbol{u}_\alpha(s)), \beta_\alpha(s))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\varepsilon(\boldsymbol{u}_0), \tag{4.23}$$

$$\beta_\alpha(t) \in K \text{ for all } t \in [0, T], (\dot{\beta}_\alpha(t), \xi - \beta_\alpha(t))_{L^2(\Omega)} + a(\beta_\alpha(t), \xi - \beta_\alpha(t))$$

$$\geq (\phi(\boldsymbol{\sigma}_\alpha(t), \varepsilon(\boldsymbol{u}_\alpha(t)), \beta_\alpha(t)), \xi - \beta_\alpha(t))_{L^2(\Omega)} \forall \xi \in K, \tag{4.24}$$

$$(\boldsymbol{\sigma}_\alpha(t), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + j(\alpha(t), \boldsymbol{u}_\alpha(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_V \forall \boldsymbol{v} \in V, \tag{4.25}$$

$$\boldsymbol{u}_\alpha(0) = \boldsymbol{u}_0, \boldsymbol{\sigma}_\alpha(0) = \boldsymbol{\sigma}_0, \ \beta_\alpha(0) = \beta_0. \tag{4.26}$$

We have the following result concerning this problem.

LEMMA 4.5. *There exists a unique solution to problem PV_α, and it satisfies* $\boldsymbol{u}_\alpha \in C(0, T; V), \boldsymbol{\sigma}_\alpha \in C(0, T; \mathcal{H})$ *and* $\beta_\alpha \in W^{1,2}(0, T; L^2(\Omega)) \cap \cap L^2(0, T; H^1(\Omega))$.

PROOF. Existence. We let $\eta_\alpha = (\eta_\alpha^1, \eta_\alpha^2) \in C(0, T; \mathcal{H} \times L^2(\Omega))$ be the fixed-point of Λ_α and denote $\boldsymbol{u}_\alpha = \boldsymbol{u}_{\alpha\eta_\alpha}$, $\boldsymbol{\sigma}_\alpha = \boldsymbol{\sigma}_{\alpha\eta_\alpha}$, $\boldsymbol{z}_\alpha = \boldsymbol{z}_{\eta_\alpha}$ and $\beta_\alpha = \beta_{\alpha\eta_\alpha}$. We let $\eta = \eta_\alpha$ in (4.10)-(4.11) and obtain

$$\boldsymbol{\sigma}_\alpha(t) = \mathcal{E}\varepsilon(\boldsymbol{u}_\alpha(t)) + \boldsymbol{z}_\alpha(t) \qquad \forall t \in [0, T], \tag{4.27}$$

$$(\dot{\beta}_\alpha(t), \xi - \beta_\alpha(t))_{L^2(\Omega)} + a(\beta_\alpha(t), \xi - \beta_\alpha(t))$$

$$\geq (\eta_\alpha^2(t), \xi - \beta_\alpha(t))_{L^2(\Omega)} \forall \xi \in K \text{a.e. } t \in (0, T), \tag{4.28}$$

and we use (4.3)-(4.4) to find that

$$\boldsymbol{\sigma}_\alpha(t) = \mathcal{E}\varepsilon(\boldsymbol{u}_\alpha(t)) + \int_0^t \eta_\alpha^1(s)ds + \boldsymbol{\sigma}_0 - \mathcal{E}\varepsilon(\boldsymbol{u}_0) \qquad \forall t \in [0, T], \tag{4.29}$$

Since

$$\begin{aligned}\eta_\alpha = \Lambda_\alpha \eta_\alpha &= (\mathcal{G}(\boldsymbol{\sigma}_{\alpha\eta_\alpha}, \varepsilon(\boldsymbol{u}_{\alpha\eta_\alpha}), \beta_{\alpha\eta_\alpha}), \phi(\boldsymbol{\sigma}_{\alpha\eta_\alpha}, \varepsilon(\boldsymbol{u}_{\alpha\eta_\alpha}), \beta_{\alpha\eta_\alpha}))\\ &= (\mathcal{G}(\boldsymbol{\sigma}_\alpha, \varepsilon(\boldsymbol{u}_\alpha), \beta_\alpha), \phi(\boldsymbol{\sigma}_\alpha, \varepsilon(\boldsymbol{u}_\alpha), \beta_\alpha)),\end{aligned}$$

we see that (4.27) and (4.29) imply (4.23), (4.28) implies (4.24). Next, we let $\eta = \eta_\alpha$ in (4.5) and we use (4.27) to obtain (4.25) and, finally, (4.26) follows from (4.6), (4.12) and (4.29). This concludes the existence part of the lemma 4.5 since the functions $\boldsymbol{u}_\alpha$, $\boldsymbol{\sigma}_\alpha$ and β_α satisfy $\boldsymbol{u}_\alpha \in C(0, T; V)$, $\boldsymbol{\sigma}_\alpha \in C(0, T; \mathcal{H})$ and $\beta_\alpha \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed-point of the operator Λ_α defined in (4.14). Indeed, let $(\boldsymbol{u}_\alpha, \boldsymbol{\sigma}_\alpha, \beta_\alpha)$ be a solution of problem PV_α which satisfies $\boldsymbol{u}_\alpha \in C(0,T;V)$, $\boldsymbol{\sigma}_\alpha \in C(0,T;\mathcal{H})$ and $\beta_\alpha \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and let $\eta \in C(0,T;\mathcal{H} \times L^2(\Omega))$ be the function given by

$$\eta = (\mathcal{G}(\boldsymbol{\sigma}_\alpha, \varepsilon(\boldsymbol{u}_\alpha), \beta_\alpha), \phi(\boldsymbol{\sigma}_\alpha, \varepsilon(\boldsymbol{u}_\alpha), \beta_\alpha)). \tag{4.30}$$

We denote by $\boldsymbol{z}_\eta$ the function defined by (4.3)-(4.4). We use (4.23) and (4.30) to find that

$$\boldsymbol{\sigma}_\alpha(t) = \mathcal{E}\varepsilon(\boldsymbol{u}_\alpha(t)) + \boldsymbol{z}_\eta(t) \quad \forall t \in [0,T], \tag{4.31}$$

We use (4.24) and (4.30) to obtain

$$\begin{aligned} &(\dot{\beta}_\alpha(t), \xi - \beta_\alpha(t))_{L^2(\Omega)} + a(\beta_\alpha(t), \xi - \beta_\alpha(t)) \\ &\geq (\eta^2(t), \xi - \beta_\alpha(t))_{L^2(\Omega)} \forall \xi \in K, \end{aligned} \tag{4.32}$$

and, substituting the equality (4.31) in (4.25) we deduce that $\boldsymbol{u}_\alpha$ is a solution of problem $PV_{\alpha\eta}$. By the uniqueness part in lemma 4.2 it follows that this problem has a unique solution, denoted by $\boldsymbol{u}_{\alpha\eta}$ and, therefore, $\boldsymbol{u}_\alpha = \boldsymbol{u}_{\alpha\eta}$. Moreover, (4.31)-(4.32) and (4.10)-(4.11) imply that $\boldsymbol{\sigma}_\alpha = \boldsymbol{\sigma}_{\alpha\eta}$ and $\beta_\alpha = \beta_{\alpha\eta}$. We use now (4.30) and (4.14) to obtain that $\eta = \Lambda_\alpha \eta$. Then, by the uniqueness of the fixed-point of the operator Λ_α, guaranteed by lemma 4.4, it follows that $\eta = \eta_\alpha$. So, $\boldsymbol{u}_\alpha = \boldsymbol{u}_{\alpha\eta_\alpha}$, $\boldsymbol{\sigma}_\alpha = \boldsymbol{\sigma}_{\alpha\eta_\alpha}$ and $\beta_\alpha = \beta_{\alpha\eta_\alpha}$. We conclude that every solution $(\boldsymbol{u}_\alpha, \boldsymbol{\sigma}_\alpha, \beta_\alpha)$ of problem PV_α coincides with the solution $(\boldsymbol{u}_{\alpha\eta}, \boldsymbol{\sigma}_{\alpha\eta}, \beta_{\alpha\eta})$ obtained in the existence part, which implies uniqueness of the solution of problem PV_α. □

In the next step we use the displacement field $\boldsymbol{u}_\alpha$ obtained in lemma 4.5 and we consider the following initial-value problem.

Problem PV_θ. Find the adhesion field $\theta_\alpha : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\dot{\theta}_\alpha(t) = H_{ad}(\theta_\alpha(t), R(\mid \boldsymbol{u}_{\alpha\tau}(t) \mid)) \text{ a.e. } t \in (0,T), \tag{4.33}$$

$$\theta_\alpha(0) = \alpha_0. \tag{4.34}$$

We have the following result.

LEMMA 4.6. *There exists a unique solution to problem PV_θ and it satisfies $\theta_\alpha \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z$.*

PROOF. For the sake of simplicity we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3. Consider the mapping $F_\alpha : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\alpha(t,\theta) = H_{ad}(\theta(t), R(\mid \boldsymbol{u}_{\alpha\tau}(t) \mid)). \tag{4.35}$$

It is easy to check that F_α is Lipschitz continuous with respect to the second variable, uniformly in time, moreover, for all $\theta \in L^2(\Gamma_3)$, the mapping $t \to F_\alpha(t,\theta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using a version of Cauchy-Lipschitz theorem given in Theorem 2.1 we deduce that there exists a unique function $\theta_\alpha \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to the problem PV_θ. We prove that θ_α belongs to Z. To this end, we suppose that $\theta_\alpha(t_0) < 0$ for some $t_0 \in [0,T]$. Since $0 \le \alpha_0 \le 1$ we have $0 \le \theta_\alpha(0) \le 1$ and, since the mapping $t \to \theta_\alpha(t) : [0,T] \to \mathbb{R}$ is continuous, we can find $t_1 \in [0,t_0)$, such that $\theta_\alpha(t_1) = 0$. Now, let $t_2 = \sup\{t \in [t_1,t_0],\ \theta_\alpha(t) = 0\}$, then $t_2 < t_0$, $\theta_\alpha(t_2) = 0$ and $\theta_\alpha(t) < 0$, for $t \in (t_2,t_0]$. The assumption (3.20)(e) and (4.33) imply that $\dot\theta_\alpha(t) \ge 0$ for $t \in (t_2,t_0]$, therefore $\theta_\alpha(t_0) \ge \theta_\alpha(t_2) = 0$, which is a contradiction. A similar argument shows that $\theta_\alpha(t) \le 1$ for all $t \in [0,T]$. We conclude that $\theta_\alpha(t) \in Z$. □

It follows from lemma 4.6 that for all $\alpha \in \mathcal{Z}$ the solution θ_α of problem PV_θ belongs to $\mathcal{Z}$. Therefore, we may consider the operator $\Lambda : \mathcal{Z} \to \mathcal{Z}$ given by

$$\Lambda\alpha = \theta_\alpha. \tag{4.36}$$

We have the following result.

Lemma 4.7. *There exists a unique element $\alpha^* \in \mathcal{Z}$ such that $\Lambda\alpha^* = \alpha^*$.*

Proof. We show that, for a positive integer m, the mapping Λ^m is a contraction on $\mathcal{Z}$. To this end, we suppose that α_1 and α_2 are two functions in $\mathcal{Z}$ and denote $\boldsymbol{u}_{\alpha_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\alpha_i} = \boldsymbol{\sigma}_i$, $\beta_{\alpha_i} = \beta_i$ and $\theta_{\alpha_i} = \theta_i$ the functions obtained in lemmas 4.5 and 4.6, respectively, for $\alpha = \alpha_i$, $i = 1,2$. We also define by $\boldsymbol{z}_i$, for $i = 1,2$, the function

$$\boldsymbol{z}_i(t) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}_i(s), \varepsilon(\boldsymbol{u}_i(s)), \beta_i(s))ds + \boldsymbol{\sigma}_0 - \mathcal{E}\varepsilon(\boldsymbol{u}_0) \quad \forall t \in [0,T], \tag{4.37}$$

Let $t \in [0,T]$. We use (4.23) and (4.37) to obtain

$$\boldsymbol{\sigma}_i(t) = \mathcal{E}\varepsilon(\boldsymbol{u}_i(t)) + \boldsymbol{z}_i(t) \qquad i = 1,2, \tag{4.38}$$

We insert the inequality (4.38) in (4.25) and use arguments similar to those used in the proof of (4.7) to deduce that

$$|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)|_V\le C(|\boldsymbol{z}_1(t)-\boldsymbol{z}_2(t)|_{\mathcal{H}}+|\alpha_1(t)-\alpha_2(t)|_{L^2(\Gamma_3)}). \tag{4.39}$$

We use (4.37) to find

$$|\boldsymbol{z}_1(t)-\boldsymbol{z}_2(t)|_{\mathcal{H}} \le C\int_0^t(|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)|_{\mathcal{H}}+|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)|_V+|\beta_1(s)-\beta_2(s)|_{L^2(\Omega)})ds. \tag{4.40}$$

We substitute (4.40) in (4.39) to obtain

$$|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)|_V\le C|\alpha_1(t)-\alpha_2(t)|_{L^2(\Gamma_3)} + C\int_0^t(|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)|_{\mathcal{H}}+|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)|_V+|\beta_1(s)-\beta_2(s)|_{L^2(\Omega)})ds. \tag{4.41}$$

We use (4.38), (4.40) and (4.41) to find

$$|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|_{\mathcal{H}}\le C|\alpha_1(t)-\alpha_2(t)|_{L^2(\Gamma_3)} + C\int_0^t(|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)|_{\mathcal{H}}+|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)|_V+|\beta_1(s)-\beta_2(s)|_{L^2(\Omega)})ds. \tag{4.42}$$

We use now similar arguments to those used in the proof of (4.20) and the properties of the function ϕ to find

$$|\beta_1(t)-\beta_2(t)|_{L^2(\Omega)} \le C\int_0^t(|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)|_{\mathcal{H}}+|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)|_V+|\beta_1(s)-\beta_2(s)|_{L^2(\Omega)})ds. \tag{4.43}$$

We add the three previous inequalities to obtain

$$|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|_{\mathcal{H}}+|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)|_V+|\beta_1(t)-\beta_2(t)|_{L^2(\Omega)} \le C|\alpha_1(t)-\alpha_2(t)|_{L^2(\Gamma_3)} + C\int_0^t(|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)|_{\mathcal{H}}+|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)|_V+|\beta_1(s)-\beta_2(s)|_{L^2(\Omega)})ds. \tag{4.44}$$

Next applying the Gronwall inequality given in [16] to (4.44) yields

$$| \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t) |_{\mathcal{H}} + | \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) |_V + | \beta_1(t) - \beta_2(t) |_{L^2(\Omega)}$$

$$\leq C(| \alpha_1(t) - \alpha_2(t) |_{L^2(\Gamma_3)} + \int_0^t | \alpha_1(s) - \alpha_2(s) |_{L^2(\Gamma_3)} \, ds),$$

which implies that

$$\int_0^t | \boldsymbol{u}_1(s) - \boldsymbol{u}_2(s) |_V \, ds \leq C \int_0^t | \alpha_1(s) - \alpha_2(s) |_{L^2(\Gamma_3)} \, ds. \tag{4.45}$$

On the other hand, from the Cauchy problem (4.33)-(4.34) we can write

$$\theta_i(t) = \alpha_0 + \int_0^t H_{ad}(\theta_i(s), R(| \boldsymbol{u}_{i\tau}(s) |)) \, ds, \ \ i = 1, 2. \tag{4.46}$$

The assumption (3.20) implies

$$| \theta_1(t) - \theta_2(t) |_{L^2(\Gamma_3)}$$

$$\leq C \int_0^t | \theta_1(s) - \theta_2(s) |_{L^2(\Gamma_3)} \, ds + C \int_0^t | \boldsymbol{u}_1(s) - \boldsymbol{u}_2(s) |_{L^2(\Gamma_3)^d} \, ds. \tag{4.47}$$

Next, we apply Gronwall's inequality to deduce

$$| \theta_1(t) - \theta_2(t) |_{L^2(\Gamma_3)} \leq C \int_0^t | \boldsymbol{u}_1(s) - \boldsymbol{u}_2(s) |_{L^2(\Gamma_3)^d} \, ds. \tag{4.48}$$

The relation (4.36), the estimate (4.48) and the relation (3.15) lead to

$$| \Lambda\alpha_1(t) - \Lambda\alpha_2(t) |_{L^2(\Gamma_3)} \leq C \int_0^t | \boldsymbol{u}_1(s) - \boldsymbol{u}_2(s) |_V \, ds. \tag{4.49}$$

We now combine (4.45) and (4.49) and see that

$$| \Lambda\alpha_1(t) - \Lambda\alpha_2(t) |_{L^2(\Gamma_3)} \leq C \int_0^t | \alpha_1(s) - \alpha_2(s) |_{L^2(\Gamma_3)} \, ds,$$

and reiterating this inequality m times we obtain

$$| \Lambda^m \alpha_1 - \Lambda^m \alpha_2 |_{C(0,T;L^2(\Gamma_3))} \leq \frac{C^m T^m}{m!} | \alpha_1 - \alpha_2 |_{C(0,T;L^2(\Gamma_3))} . \tag{4.50}$$

Recall that $\mathcal{Z}$ is a nonempty closed set in the Banach space $C(0,T;L^2(\Gamma_3))$ and note that (4.50) shows that for m sufficiently large the operator $\Lambda^m : \mathcal{Z} \to \mathcal{Z}$ is a contraction. Then by the Banach fixed-point theorem (see [16]) it follows that Λ has a fixed-point $\alpha^* \in \mathcal{Z}$. □

Now, we have all the ingredients to prove Theorem 4.1.

PROOF. Existence. Let $\alpha^* \in \mathcal{Z}$ be the fixed-point of Λ and let $(\boldsymbol{u}^*, \boldsymbol{\sigma}^*, \beta^*)$ be the solution of problem PV_α for $\alpha = \alpha^*$, i.e. $\boldsymbol{u}^* = \boldsymbol{u}_{\alpha^*}$, $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}_{\alpha^*}$ and $\beta^* = \beta_{\alpha^*}$. Consider the function $\boldsymbol{z}^*$ given by

$$\boldsymbol{z}^*(t) = \int_0^t \mathcal{G}(\boldsymbol{\sigma}^*(s), \varepsilon(\boldsymbol{u}^*(s)), \beta^*(s)) ds + \boldsymbol{\sigma}_0 - \mathcal{E}\varepsilon(\boldsymbol{u}_0) \quad \forall t \in [0,T], \tag{4.51}$$

and note that arguments similar to those used in the proof of (4.7) lead to

$$\begin{aligned} | \boldsymbol{u}^*(t_1) - \boldsymbol{u}^*(t_2) |_V \leq C(& | \boldsymbol{z}^*(t_1) - \boldsymbol{z}^*(t_2) |_{\mathcal{H}} \\ & + | \boldsymbol{f}(t_1) - \boldsymbol{f}(t_2) |_V + | \alpha^*(t_1) - \alpha^*(t_2) |_{L^2(\Gamma_3)}), \end{aligned} \tag{4.52}$$

for all $t_1, t_2 \in [0,T]$. We use now (4.23) and (4.51) to find that

$$\boldsymbol{\sigma}^*(t) = \mathcal{E}\varepsilon(\boldsymbol{u}^*(t)) + \boldsymbol{z}^*(t) \qquad \forall t \in [0,T], \tag{4.53}$$

and by (4.51), (4.53) and the properties (3.17) of the functional $\mathcal{G}$ we obtain

$$| \boldsymbol{\sigma}^*(t_1) - \boldsymbol{\sigma}^*(t_2) |_{\mathcal{H}} \leq C(| \boldsymbol{u}^*(t_1) - \boldsymbol{u}^*(t_2) |_V + | \boldsymbol{z}^*(t_1) - \boldsymbol{z}^*(t_2) |_{\mathcal{H}}), \tag{4.54}$$

for all $t_1, t_2 \in [0,T]$. Since $\alpha^* = \theta_{\alpha^*}$ it follows from lemma 4.6 that $\alpha^* \in W^{1,\infty}(0,T;L^2(\Gamma_3))$, recall also that $\boldsymbol{f} \in W^{1,\infty}(0,T;V)$ and the regularity $\boldsymbol{z} \in C^1(0,T;\mathcal{H})$. We use now (4.52) and (4.54) to deduce that $\boldsymbol{u}^* \in W^{1,\infty}(0,T;V)$, $\boldsymbol{\sigma}^* \in W^{1,\infty}(0,T;\mathcal{H})$. Next, we let $\alpha = \alpha^*$ in equality (4.23) and then differentiate it with respect to time we obtain (3.30). We employ $\alpha = \alpha^*$ in problem PV_α , PV_θ and use the equality $\alpha^* = \theta_{\alpha^*}$ and, as a result, we obtain that $(\boldsymbol{u}^*, \boldsymbol{\sigma}^*, \beta^*, \alpha^*)$ satisfies (3.31), (3.32), (3.33) and (3.34).

Choosing now $\boldsymbol{v} = \pm\, \omega$ in (3.32) where $\omega \in C_0^\infty(\Omega)^d$, yields

$$Div\ \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = 0 \quad \forall t \in [0,T],$$

and, by assumption (3.21), we obtain that $Div\ \boldsymbol{\sigma} \in W^{1,\infty}(0,T;H)$ and, therefore, $\boldsymbol{\sigma} \in W^{1,\infty}(0,T;\mathcal{H}_1)$. We conclude that $(\boldsymbol{u}^*, \boldsymbol{\sigma}^*, \beta^*, \alpha^*)$ is a solution of problem PV and it satisfies (4.1), which concludes the proof of the existence part of Theorem 4.1.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed-point of the operator Λ defined by (4.36) combined with the unique solvability of problem PV_α. Indeed, let $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta, \alpha)$ be a solution of problem PV which satisfies (4.1). Using (4.2) we deduce that $\alpha \in \mathcal{Z}$, and it also follows from (3.30), (3.31) and (3.32) that $(\boldsymbol{u}, \boldsymbol{\sigma}, \beta)$ is a solution to problem PV_α, moreover, since by lemma 4.5 this problem has a unique solution denoted $(\boldsymbol{u}_\alpha, \boldsymbol{\sigma}_\alpha, \beta_\alpha)$, we obtain

$$\boldsymbol{u} = \boldsymbol{u}_\alpha, \ \ \boldsymbol{\sigma} = \boldsymbol{\sigma}_\alpha, \ \ \beta = \beta_\alpha. \tag{4.55}$$

We replace $\boldsymbol{u} = \boldsymbol{u}_\alpha$ in (3.33) and use the initial condition $\alpha(0) = \alpha_0$ in (3.34) to deduce that α is a solution of problem PV_θ. It follows now that from lemma 4.6 that the last problem PV_θ has a unique solution, denoted by θ_α and, therefore,

$$\alpha = \theta_\alpha. \tag{4.56}$$

We use now (4.36) and (4.56) to see that $\Lambda\alpha = \alpha$, i.e. α is a fixed-point of the operator Λ. It follows now from lemma 4.7 that

$$\alpha = \alpha^*. \tag{4.57}$$

The uniqueness part of the theorem is now a consequence of equalities (4.55) and (4.57). □

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