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Huppert's Conjecture for Fi_{23}

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ABSTRACT - Let G denote a finite group and $\mathrm{cd}(G)$ the set of all irreducible character degrees of G. Bertram Huppert conjectured that if H is a finite nonabelian simple group such that $\mathrm{cd}(G) = \mathrm{cd}(H)$, then $G \cong H \times A$, where A is an abelian group. Huppert verified the conjecture for many of the sporadic simple groups. We illustrate the arguments by presenting the verification of Huppert's Conjecture for Fi_{23}.

1. Introduction and Notation

Let G be a finite group, $\mathrm{Irr}(G)$ the set of irreducible characters of G, and denote the set of character degrees of G by $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$. When context allows, the set of character degrees will be referred to as the set of degrees. In the late 1990s, Bertram Huppert posed a conjecture which, if true, would sharpen the connection between the character degree set of a nonabelian simple group and the structure of the group. In [4], he conjectured the following.

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HUPPERT'S CONJECTURE. *Let G be a finite group and H a finite nonabelian simple group such that the sets of character degrees of G and H are the same. Then $G \cong H \times A$, where A is an abelian group.*

Huppert verifies the conjecture for the Suzuki groups, the family of simple groups $\mathrm{PSL}_2(q)$ for $q \geq 4$, and many of the sporadic simple groups. Huppert's proofs rely upon the completion of the following five steps.

(1) Show $G' = G''$. Hence if G'/M is a chief factor of G, then $G'/M \cong S^k$, where S is a nonabelian simple group.

(2) Identify H as a chief factor G'/M of G.

(3) Show that if $\theta \in \mathrm{Irr}(M)$ and $\theta(1) = 1$, then θ is stable under G', which implies $[M, G'] = M'$.

(4) Show that $M = 1$.

(5) Show that $G = G' \times C_G(G')$. As $G/G' \cong C_G(G')$ is abelian and $G' \cong H$, Huppert's Conjecture is verified.

In addition to his work verifying his conjecture for many of the simple groups of Lie type, Huppert also verified the conjecture for many of the sporadic simple groups. Indeed, he demonstrated that his conjecture holds for the Mathieu groups, Janko groups, and many other sporadic simple groups in his preprints. The only sporadic simple groups not considered by Huppert are the Conway groups, Fischer groups, Monster, and Baby Monster. In [7], the authors establish Huppert's Conjecture for the Monster and Baby Monster. The authors have examined the remaining Fischer groups and Conway groups and have verified the conjecture for these families of simple groups using arguments similar to those presented in this paper. Thus, Huppert's Conjecture has been verified for all twenty-six sporadic simple groups.

If n is an integer then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a group, we will write $\pi(G)$ instead of $\pi(|G|)$ to denote the set of all prime divisors of the order of G. If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is denoted by $I_G(\theta)$. The set of all irreducible constituents of θ^G is denoted by $\mathrm{Irr}(G|\theta)$. Other notation is standard.

2. Preliminaries

In this section, we present some results that we will need for the proof of the Huppert's Conjecture.

LEMMA 2.1 ([4, Lemma 2]). *Suppose $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$.*

(*a*) *If $\chi_N = \theta_1 + \theta_2 + \cdots + \theta_k$ with $\theta_i \in \mathrm{Irr}(N)$, then k divides $|G/N|$. In particular, if $\chi(1)$ is prime to $|G/N|$ then $\chi_N \in \mathrm{Irr}(N)$.*

(*b*) *(Gallagher's Theorem) If $\chi_N \in \mathrm{Irr}(N)$, then $\chi\psi \in \mathrm{Irr}(G)$ for every $\psi \in \mathrm{Irr}(G/N)$.*

LEMMA 2.2 ([4, Lemma 3]). *Suppose $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$. Let $I = I_G(\theta)$.*

(*a*) *If $\theta^I = \sum_{i=1}^{k} \varphi_i$ with $\varphi_i \in \mathrm{Irr}(I)$, then $\varphi_i^G \in \mathrm{Irr}(G)$. In particular, $\varphi_i(1)|G : I| \in \mathrm{cd}(G)$.*

(*b*) *If θ extends to $\psi \in \mathrm{Irr}(I)$, then $(\psi\tau)^G \in \mathrm{Irr}(G)$ for all $\tau \in \mathrm{Irr}(I/N)$. In particular, $\theta(1)\tau(1)|G : I| \in \mathrm{cd}(G)$.*

(*c*) *If $\rho \in \mathrm{Irr}(I)$ such that $\rho_N = e\theta$, then $\rho = \theta_0\tau_0$, where θ_0 is a character of an irreducible projective representation of I of degree $\theta(1)$ while τ_0 is the character of an irreducible projective representation of I/N of degree e.*

The following lemma will be used to verify Step 1. This is [7, Lemma 3].

LEMMA 2.3. *Let G/N be a solvable factor group of G, minimal with respect to being nonabelian. Then two cases can occur.*

(*a*) *G/N is an r-group for some prime r. Hence there exists $\psi \in \mathrm{Irr}(G/N)$ such that $\psi(1) = r^b > 1$. If $\chi \in \mathrm{Irr}(G)$ and $r \nmid \chi(1)$, then $\chi\tau \in \mathrm{Irr}(G)$ for all $\tau \in \mathrm{Irr}(G/N)$.*

(*b*) *G/N is a Frobenius group with an elementary abelian Frobenius kernel F/N. Then $f = |G : F| \in \mathrm{cd}(G)$ and $|F/N| = r^a$ for some prime r, and F/N is an irreducible module for the cyclic group G/F, hence a is the smallest integer such that $r^a \equiv 1 (\mathrm{mod} f)$. If $\psi \in \mathrm{Irr}(F)$ then either $f\psi(1) \in \mathrm{cd}(G)$ or r^a divides $\psi(1)^2$. In the latter case, r divides $\psi(1)$.*

(1) *If no proper multiple of f is in $\mathrm{cd}(G)$, then $\chi(1)$ divides f for all $\chi \in \mathrm{Irr}(G)$ such that $r \nmid \chi(1)$, and if $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) \nmid f$, then $r^a \mid \chi(1)^2$.*

(2) *If $\chi \in \mathrm{Irr}(G)$ such that no proper multiple of $\chi(1)$ is in $\mathrm{cd}(G)$, then either f divides $\chi(1)$ or r^a divides $\chi(1)^2$. Moreover if $\chi(1)$ is divisible by no nontrivial proper character degree in G, then $f = \chi(1)$ or $r^a \mid \chi(1)^2$.*

Let $\chi \in \mathrm{Irr}(G)$. We say that χ is *isolated* in G if $\chi(1)$ is divisible by no proper nontrivial character degree of G, and no proper multiple of $\chi(1)$ is a character degree of G. In this situation, we also say that $\chi(1)$ is an *isolated degree* of G.

The next two lemmas will be used to verify Steps 2 and 4. The first lemma appears in [1, Theorems $2, 3, 4$].

LEMMA 2.4. *If S is a nonabelian simple group, then there exists a nontrivial irreducible character θ of S that extends to* $\mathrm{Aut}(S)$. *Moreover the following holds:*

(i) *if S is an alternating group of degree at least* 7, *then S has two consecutive characters of degrees $n(n-3)/2$ and $(n-1)(n-2)/2$ that both extend to* $\mathrm{Aut}(S)$.

(ii) *if S is a sporadic simple group or the Tits group, then S has two nontrivial irreducible characters of coprime degrees which both extend to* $\mathrm{Aut}(S)$.

(iii) *if S is a simple group of Lie type then the Steinberg character St_S of S of degree $|S|_p$ extends to* $\mathrm{Aut}(S)$.

LEMMA 2.5 ([1, Lemma 5]). *Let N be a minimal normal subgroup of G so that $N \cong S^k$, where S is a nonabelian simple group. If $\theta \in \mathrm{Irr}(S)$ extends to $\mathrm{Aut}(S)$, then $\theta^k \in \mathrm{Irr}(N)$ extends to G.*

The following Lemma will be used to verify Step 4.

LEMMA 2.6 ([4, Lemma 6]). *Suppose that $M \trianglelefteq G' = G''$ and that for any $\lambda \in \mathrm{Irr}(M)$ with $\lambda(1) = 1$, $\lambda^g = \lambda$ for all $g \in G'$. Then $M' = [M, G']$ and $|M/M'|$ divides the order of the Schur multiplier of G'/M.*

3. The sporadic simple group Fi_{23}

LEMMA 3.1. *Let H be the sporadic simple group Fi_{23}.*

(i) *The following degrees are isolated degrees of H:*

$$2^{18} \cdot 5^2 \cdot 7 \cdot 11 \quad 2^5 \cdot 3^{12} \cdot 17 \quad 3^{13} \cdot 13 \cdot 23$$
$$2^{16} \cdot 5 \cdot 13 \cdot 23 \quad 3^{12} \cdot 17 \cdot 23 \quad 3^{12} \cdot 5 \cdot 11 \cdot 17.$$

(ii) *Let* $1 \neq \chi(1) \in \mathrm{cd}(H)$. *Then* $(11 \cdot 17 \cdot 23, \chi(1)) > 1$ *and* $(\chi(1), 11 \cdot 13 \cdot 17) > 1$.

(iii) *H has no proper power degrees nor consecutive degrees.*

(iv) *If K is a maximal subgroup of H such that $|H : K|$ divides some character degree $\chi(1)$ of H then one of the following cases holds.*

(*a*) $K \cong 2^{\cdot}Fi_{22}$ *and* $\chi(1)/|H:K|$ *is one of the numbers in the set* $\mathcal{A}_1$, *consisting of the following numbers:*

$$\begin{array}{ccc} 2^3\cdot 3^3\cdot 7\cdot 11 & 2^4\cdot 5\cdot 11\cdot 13 & 2^2\cdot 3^3\cdot 7\cdot 11 \\ 3\cdot 5^2\cdot 7\cdot 13 & 3^8 & 3^2\cdot 5\cdot 11\cdot 13 \\ 2^5\cdot 5\cdot 13 & 2^5\cdot 5\cdot 11 & 5\cdot 11\cdot 13 \\ 2\cdot 3^3\cdot 11\cdot 13 & 2^2\cdot 3^4\cdot 13 & 2\cdot 3^3\cdot 11. \end{array}$$

(*b*) $K \cong O_8^+(3):S_3$ *and* $\chi(1)/|H:K|$ *is one of the numbers in the set* $\mathcal{A}_2$, *consisting of the following numbers:*

$$\begin{array}{ccc} 2^6\cdot 3\cdot 13 & 2^2\cdot 3^2\cdot 5\cdot 13 & 5^2\cdot 7\cdot 13 \\ 2^2\cdot 3\cdot 5^2\cdot 7 & 2^5\cdot 5\cdot 13 & 3^2\cdot 5^2\cdot 7 \\ 3^4\cdot 5 & 3^3\cdot 13 & 3\cdot 7\cdot 13 \\ 2\cdot 5\cdot 13 & 3\cdot 5\cdot 13 & 2^5\cdot 13. \end{array}$$

(*v*) *The outer automorphism group and the Schur multiplier of H are trivial.*

Proof. The list of maximal subgroups and their indices of the Fischer group Fi_{23} is given in Table 1. This table is taken from [6]. The other information can be found in [3]. □

Lemma 3.2. *Let* $L = O_8^+(3)$ *be the simple orthogonal group. The only nontrivial degree of a proper irreducible projective representation of L which divides one of the numbers in* $\mathcal{A}_2$ *is* 520.

Proof. By [3], the Schur multiplier of L is elementary abelian of order 4. Let $\tilde{L}$ be the full covering group of L. Then $Z(\tilde{L}) \cong \mathbf{Z}_2^2$. Hence $Z(\tilde{L})$ is noncyclic and hence $\tilde{L}$ has no faithful irreducible characters. Thus if $\chi \in \mathrm{Irr}(\tilde{L})$, then χ must be an irreducible character of one of the following groups: $O_8^+(3), 2\cdot O_8^+(3), 2'\cdot O_8^+(3)$, or $2''\cdot O_8^+(3)$. We observe that if χ is a nontrivial proper projective irreducible character of L, then χ is an ordinary character of $2\cdot O_8^+(3), 2'\cdot O_8^+(3)$, or $2''\cdot O_8^+(3)$. We have that $\Omega_8^+(3)\cong 2\cdot O_8^+(3)$ and the three double covers of L are isomorphic via the graph automorphism of order 3 of L. Hence it suffices to find the faithful irreducible characters of $\Omega_8^+(3)$ which divide one of the numbers in $\mathcal{A}_2$. We see that the largest number in $\mathcal{A}_2$ is 2496. Using [2], the faithful irreducible character degrees of $\Omega_8^+(3)$ which are less than 2496 are $520, 560$ and 1456. However only 520 divides one of the numbers in $\mathcal{A}_2$. The proof is now complete. □

4. Verifying Huppert's Conjecture for Fi_{23}

We assume that $H \cong Fi_{23}$ and G is a group such that $\mathrm{cd}(G) = \mathrm{cd}(H)$. In this section we will show that $G \cong H \times A$, where A is an abelian group, which confirms Huppert's Conjecture for the sporadic simple group Fi_{23}. We will follow five steps of Huppert's method described in the introduction.

4.1 – *Verifying Step 1*

By way of contradiction, suppose that $G' \neq G''$. Then there exists a normal subgroup $N \trianglelefteq G$ such that G/N is solvable and minimal with respect to being non-abelian. By Lemma 2.3, G/N is an r-group for some prime r or G/N is a Frobenius group.

Case 1. G/N is an r-group. Then there exists $\psi \in \mathrm{Irr}(G/N)$ such that $\psi(1) = r^b > 1$. However by Lemma 3.1(iii), G has no nontrivial prime power degrees. Hence this case cannot happen.

Case 2. G/N is a Frobenius group with Frobenius kernel F/N, $|F/N| = r^a$, $1 < f = |G : F| \in \mathrm{cd}(G)$ and $r^a \equiv 1 (\mathrm{mod}\ f)$. By Lemma 2.3$(b)$, if $\chi \in \mathrm{Irr}(G)$ such that $\chi(1)$ is isolated then either $f = \chi(1)$ or $r \mid \chi(1)$. We observe that there is no prime which divides all the isolated degrees listed in Lemma 3.1(i). Thus f must be one of the isolated degrees in Lemma 3.1(i). Hence f is isolated in G. By Lemma 2.3(b) again, if $\chi \in \mathrm{Irr}(G)$ with $r \nmid \chi(1)$ then $\chi(1) \mid f$. As f is isolated we deduce that r must divide every nontrivial degree $\chi(1)$ of G such that $\chi(1) \neq f$.

Assume first that $f = 3^{12} \cdot 17 \cdot 23$. Then r must divide all of the remaining isolated degrees in Lemma 3.1(i). However $(2^{18} \cdot 5^2 \cdot 7 \cdot 11, 3^{13} \cdot 13 \cdot 23) = 1$, which is a contradiction. Hence $f \neq 3^{12} \cdot 17 \cdot 23$ so that $r \in \{3, 17, 23\}$. Suppose that $r = 3$. Then $2^{18} \cdot 5^2 \cdot 7 \cdot 11$ and $2^{16} \cdot 5 \cdot 13 \cdot 23$ are two isolated degrees of H, which are both coprime to r, so that f must be equal to both of them by Lemma 2.3(b)(2), which is impossible. Hence $r \neq 3$. Assume next that $r = 17$. Then $2^{18} \cdot 5^2 \cdot 7 \cdot 11$ and $3^{13} \cdot 13 \cdot 23$ are isolated degrees of H, which are both relatively prime to $r = 17$, hence we obtain a contradiction as in the previous case. Finally, assume $r = 23$. Then $2^{18} \cdot 5^2 \cdot 7 \cdot 11$ and $2^5 \cdot 3^{12} \cdot 17$ are isolated degrees of H and both are coprime to $r = 23$, which leads to a contradiction as before. Thus $G' = G''$.

4.2 – *Verifying Step 2*

Let $M \leq G'$ be a normal subgroup of G such that G'/M is a chief factor of G. As G' is perfect, G'/M is non-abelian so that $G'/M \cong S^k$ for some non-abelian simple group S and some integer $k \geq 1$.

Claim 1. $k = 1$. By way of contradiction, assume that $k \geq 2$. By Lemma 2.4, S possesses a nontrivial irreducible character θ, which is extendible to $\mathrm{Aut}(S)$ and so by Lemma 2.5, $\theta^k \in \mathrm{Irr}(G'/M)$ extends to G/M, hence $\theta(1)^k \in \mathrm{cd}(G)$, which contradicts Lemma 3.1(iii). This shows that $k = 1$. Hence $G'/M \cong S$.

Claim 2. S is not an alternating group of degree at least 7. By way of contradiction, assume that $S = A_n, n \geq 7$. By Lemma 2.4, S has two nontrivial irreducible characters θ_1, θ_2 with $\theta_1(1) = n(n-3)/2$, $\theta_2(1) = \theta_1(1) + 1 = (n-1)(n-2)/2$ and both θ_i extend to $\mathrm{Aut}(S)$, so that G possesses two consecutive nontrivial character degrees contradicting Lemma 3.1(iii).

Claim 3. S is not a simple group of Lie type. If S is a simple group of Lie type in characteristic p, and $S \neq {}^2F_4(2)'$, then the Steinberg character of S of degree $|S|_p$ extends to $\mathrm{Aut}(S)$ so that G possesses a nontrivial prime power degree, which contradicts Lemma 3.1(iii).

Claim 4. $S \cong Fi_{23}$. By Claims 1, 2, and 3, S is a sporadic simple group or the Tits group. We will eliminate all other possibilities for S and hence the claim will follow. By Ito-Michler Theorem (see [4, Lemma 1]) we deduce that every prime divisor of S must divide some character degree of S, and since every character degree of $S \cong G'/M$ divides some character degree of H, we obtain that every prime divisor of S is also a prime divisor of H, so that $\pi(S) \subseteq \pi(H) = \{2, 3, 5, 7, 11, 13, 17, 23\}$. Hence we only need to consider the simple groups in Table 2. For each sporadic simple group or the Tits group S in Table 2, we exhibit a nontrivial irreducible character θ of S such that θ extends to $\mathrm{Aut}(S)$ and either $(11 \cdot 13 \cdot 17, \theta(1)) = 1$ or $(11 \cdot 23 \cdot 17, \theta(1)) = 1$, which contradicts Lemma 3.1(ii). This finishes the proof of Step 2.

4.3 – *Verifying Step 3*

If $\theta \in \mathrm{Irr}(M)$, $\theta(1) = 1$, then $I_{G'}(\theta) = G'$. Let $\theta \in \mathrm{Irr}(M)$ and $I = I_{G'}(\theta)$. Assume that $I < G'$ and $\theta^I = \sum_{i=1}^{s} e_i\phi_i$, where $\phi_i \in \mathrm{Irr}(I), i = 1, 2, \ldots, s$. Let

U/M be a maximal subgroup of G'/M containing I/M and let $t = |U : I|$. Then $\phi_i(1)|G' : I|$ is a character degree of G' by Lemma 2.2(a), so it divides some character degree of G. Thus $t\phi_i(1)|G' : U|$ divides some character degree of G and so the index $|G' : U|$ must divide some character degree of H. By Lemma 3.1(iv), one of the following cases holds.

CASE. $U/M \cong 2^{\cdot}Fi_{22}$. Then for each i, $t\phi_i(1)$ divides one of the numbers in $\mathcal{A}_1$. As U/M is perfect, the center of U/M lies in every maximal subgroup of U/M and so the indices of maximal subgroups of U/M and those of Fi_{22} are the same. By inspecting the list of maximal subgroups of Fi_{22} in [3], the index of a maximal subgroup of U/M divides no number in $\mathcal{A}_1$ so that $t = 1$ and hence $I = U$. Recall that $\theta^I = \sum_{i=1}^{s} e_i\phi_i$, where $\phi_i \in \mathrm{Irr}(I), i = 1, 2, \ldots, s$. Assume first that $e_j = 1$ for some j. Then θ extends to $\theta_o \in \mathrm{Irr}(I)$. Hence by Gallagher's Theorem, $\tau\theta_0$ is an irreducible constituent of θ^I for any $\tau \in \mathrm{Irr}(I/M)$ and then $\tau(1)\theta_0(1) = \tau(1)$ divides one of the numbers in $\mathcal{A}_1$. However we can choose $\tau \in \mathrm{Irr}(I/M) = \mathrm{Irr}(2^{\cdot}Fi_{22})$ with $\tau(1) = 2729376$ and obviously this degree divides none of the numbers in $\mathcal{A}_1$. Therefore $e_i > 1$ for all i. We deduce that for each i, e_i is the degree of a nontrivial proper irreducible projective representation of $2^{\cdot}Fi_{22}$. Moreover as $\phi_i(1) = e_i\theta(1) = e_i$, each e_i divides one of the numbers in $\mathcal{A}_1$. It follows that for each i, $e_i \leq 2^3 \cdot 3^3 \cdot 7 \cdot 11 = 16632$ and e_i is the degree of a nontrivial proper irreducible projective representation of $2^{\cdot}Fi_{22}$. Using [3], we obtain $e_i \in \{3^3 \cdot 13, 2 \cdot 3^3 \cdot 11 \cdot 13\}$ for all i. As $\theta^I = \sum_{i=1}^{s} e_i\phi_i$, and $\phi_i(1) = e_i$, we deduce that $|I/M| = \sum_{i=1}^{s} e_i^2$. Let a and b be the numbers of $e_i's$ which equal $3^3 \cdot 13$ and $2 \cdot 3^3 \cdot 11 \cdot 13$, respectively. We have

$$2^{18} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 = 3^6 \cdot 13^2 a + 2^2 \cdot 3^6 \cdot 11^2 \cdot 13^2 b = 13^2(3^6 a + 2^2 \cdot 3^6 \cdot 11^2 b),$$

which is impossible. Thus this case cannot happen.

CASE. $U/M \cong O_8^+(3) : S_3$. Then for each i, $t\phi_i(1)$ divides one of the numbers in $\mathcal{A}_2$. Let $M \trianglelefteq W \trianglelefteq U$ such that $W/M \cong O_8^+(3)$. We have that $M \trianglelefteq I \cap W \trianglelefteq I$ and $M \trianglelefteq I \cap W \leq W$. Assume $W \not\leq I$. Then $I \lneq WI \leq U$ and $t = |U : I| = |U : WI| \cdot |WI : I|$. Now $|WI : I| = |W : W \cap I|$ and hence t is divisible by $|W : W \cap I|$ and also as $W/M \cong O_8^+(3)$, t is divisible by the index of some maximal subgroup of $O_8^+(3)$. Thus some index of maximal subgroup of $O_8^+(3)$ divides one of the numbers in $\mathcal{A}_2$. However by inspecting the list of maximal subgroups of $O_8^+(3)$ in [3], we see that this is impossible and thus $W \leq I \leq U$. Write $\theta^W = f_1\mu_1 + f_2\mu_2 + \cdots + f_s\mu_s$, where $\mu_i \in \mathrm{Irr}(W|\theta)$. As $W \trianglelefteq I$, we obtain that for each i, $\mu_i(1)$ divides one of the members in $\mathcal{A}_2$. If $f_j = 1$ for some j then θ extends to $\theta_0 \in \mathrm{Irr}(W)$. Hence by Gallagher's Theorem, $\tau\theta_0$ is an irreducible constituent of θ^W for any $\tau \in \mathrm{Irr}(W/M)$ and then $\tau(1)\theta_0(1) = \tau(1)$ divides one of the numbers in $\mathcal{A}_2$. However we can choose $\tau \in \mathrm{Irr}(W/M)$ with $\tau(1) = 716800$ and obviously this degree divides none of the numbers in $\mathcal{A}_2$. Therefore $f_i > 1$ for all i. We deduce that for each i, f_i is the degree of a nontrivial proper irreducible projective representation of $O_8^+(3)$. Moreover as $\mu_i(1) = f_i\theta(1) = f_i$, each f_i divides some of the numbers in $\mathcal{A}_2$, and so by Lemma 3.2, we obtain that $f_i = 520 = 2^3 \cdot 5 \cdot 13$ for all i. It follows that

$$|W/M| = 2^{12} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 = \sum_{i=1}^{s} f_i^2 = s \cdot 2^6 \cdot 5^2 \cdot 13^2,$$

which is impossible. Thus we conclude that θ is G'-invariant.

4.4 – *Verifying Step 4*

$M = 1$. By Step 3 and Lemma 2.5, we have $M' = [G', M]$ and $|M : M'|$ divides the order of the Schur multiplier of H, which is trivial so that $|M : M'| = 1$. Thus $M = M'$. If M is abelian then we are done. Hence we assume that $M = M'$ is nonabelian. Let $N \leq M$ be a normal subgroup of G' such that M/N is a chief factor of G'. Then $M/N \cong S^k$ for some nonabelian simple group S. By Lemma 2.4, there exists an irreducible character $\tau \in \mathrm{Irr}(S)$ such that $\tau(1) > 1$ and τ extends to $\mathrm{Aut}(S)$. By Lemma 2.5, τ^k extends to $\psi \in \mathrm{Irr}(G')$, and so by Gallagher's Theorem, $\psi\beta \in \mathrm{Irr}(G')$ for all $\beta \in \mathrm{Irr}(G'/M)$. Let $\chi \in \mathrm{Irr}(G'/M) = \mathrm{Irr}(Fi_{23})$ such that $\chi(1)$ is the largest character degree of H. Then $\psi(1)\chi(1) = \tau^k(1)\chi(1)$ divides some character degree of G, which is impossible. Hence $M = 1$.

4.5 – *Verifying Step 5*

$G = G' \times C_G(G')$. It follows from Step 4 that $G' \cong H$ is a nonabelian simple group. Let $C = C_G(G')$. Then G/C is an almost simple group with socle G'. As $\mathrm{Out}(H) = 1$, we deduce that $G = G'C$. As G' is simple, we have $G' \cap C = 1$ so that $G = G' \times C = G' \times C_G(G')$. It follows that $C_G(G') \cong G/G'$ is abelian. The proof is now complete.

TABLE 1. – Maximal subgroups of the Fischer group Fi_{23}.

Group	Index
$2^{\cdot}Fi_{22}$	$3^4 \cdot 17 \cdot 23$
$2^{2\cdot}U_6(2) \cdot 2$	$3^7 \cdot 5 \cdot 13 \cdot 17 \cdot 23$
$(2^2 \times 2^{1+8})(3 \times U_4(2)) \cdot 2$	$3^8 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
$2^{6+8} \cdot (A_7 \times S_3)$	$3^{10} \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
$2^{11} \cdot M_{23}$	$2^8 \cdot 3^3 \cdot 5 \cdot 11 \cdot 17 \cdot 23$
$O_8^+(3) : S_3$	$2^5 \cdot 11 \cdot 17 \cdot 23$
$S_3 \times \Omega_7(3)$	$2^8 \cdot 3^3 \cdot 5 \cdot 11 \cdot 17 \cdot 23$
$S_4 \times Sp_6(2)$	$2^6 \cdot 3^8 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
$Sp_8(2)$	$2^2 \cdot 3^8 \cdot 11 \cdot 13 \cdot 23$
S_{12}	$2^8 \cdot 3^8 \cdot 13 \cdot 17 \cdot 23$
$S_4(4) : 4$	$2^8 \cdot 3^{11} \cdot 7 \cdot 11 \cdot 13 \cdot 23$
$L_2(23)$	$2^{15} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$
$3_+^{1+8} : 2_-^{1+6} : 3_+^{1+2} : 2S_4$	$2^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
$3^{3+1+3+3} \cdot (2 \times L_3(3))$	$2^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 23$

TABLE 2. – Character degrees of sporadic simple groups and Tits group.

Group	Character	Degree	Group	Character	Degree
M_{11}	χ_9	$3^2 \cdot 5$	J_2	χ_6	$2^2 \cdot 3^2$
M_{12}	χ_7	$2 \cdot 3^3$	Co_3	χ_6	$2^7 \cdot 7$
M_{22}	χ_2	$3 \cdot 7$	Fi_{22}	χ_{57}	$3^9 \cdot 7 \cdot 13$
M_{23}	χ_3	$3^2 \cdot 5$	HS	χ_7	$5^2 \cdot 7$
M_{24}	χ_7	$2^2 \cdot 3^2 \cdot 7$	Co_1	χ_2	$2^2 \cdot 3 \cdot 23$
McL	χ_{14}	$3^6 \cdot 7$	${}^2F_4(2)'$	χ_{20}	$2^6 \cdot 3^3$
He	χ_{15}	$2^7 \cdot 7^2$	Suz	χ_{43}	$2^{10} \cdot 3^5$
Co_2	χ_{22}	$3^6 \cdot 5^3$			

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