

A class of Caffarelli-Kohn-Nirenberg type inequalities on the H-type group¹

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ABSTRACT - This work is devoted to establish a class of Caffarelli-Kohn-Nirenberg type inequalities on the H-type group. Inspired by the idea of Chern J.L. and Lin C.S., a function transformation is introduced. Combining some elementary inequalities and some accurate estimates, we establish a class of weighted Hardy-Sobolev type inequalities and then obtain our main result, namely Caffarelli-Kohn-Nirenberg type inequalities on the H-type group.

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1. Introduction

In [4], Caffarelli, Kohn, and Nirenberg established the following first order interpolation inequalities with weights, known as Caffarelli-Kohn-Nirenberg (CKN) inequalities.

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Theorem A *Let $p, q, r, \alpha, \beta, \gamma, \sigma$ and a satisfy*

$$(1.1) \quad \begin{cases} p, q \geq 1, r > 0, 0 \leq a \leq 1 \\ \frac{1}{p} + \frac{\alpha}{n}, \frac{1}{q} + \frac{\beta}{n}, \frac{1}{r} + \frac{\gamma}{n} > 0, \end{cases}$$

where $\gamma = a\sigma + (1 - a)\beta$. Then there exists a positive constant C such that the following inequality holds for all $u \in C_0^\infty(\mathbb{R}^n)$

$$(1.2) \quad \| |x|^\gamma u \|_{L^r} \leq C \| |x|^\alpha |\nabla u|^a \|_{L^p}^a \| |x|^\beta u \|_{L^q}^{1-a}$$

if and only if the following relations hold:

$$(1.3) \quad \frac{1}{r} + \frac{\gamma}{n} = a \left(\frac{1}{p} + \frac{\alpha - 1}{n} \right) + (1 - a) \left(\frac{1}{q} + \frac{\beta}{n} \right)$$

(this is dimensional balance),

$$0 \leq \alpha - \sigma \quad \text{if } a > 0,$$

$$\alpha - \sigma \leq 1 \quad \text{if } a > 0 \text{ and } \frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}.$$

Furthermore, on any compact set in parameter space in which (1.1), (1.3) and $0 \leq \alpha - \sigma \leq 1$ hold, the constant C is bounded.

It is well known that CKN inequalities include some important integral inequalities. For instance, the case $a = 1, \gamma = \alpha - 1$ and the case $a = 1, \alpha = \gamma = 0$ are corresponding to Hardy inequality[9] and Sobolev inequality, respectively. While the case $a = 1$ is Hardy-Sobolev inequality[1]. So, CKN inequalities can be expected to apply more extensively to related PDEs.

In 1986, **Theorem A** was extended to the higher order case by Lin C.S. (see [21]). In [13], Gutierrez and Wheeden gave a class of weighted local Sobolev interpolation inequalities and derived global Laudau inequalities. For applications, many authors have discussed the best constants and extremal functions of CKN inequalities (see e.g. [5],[20]), which play important roles in many problems, such as eigenvalue problems, existence problems of equation with singular weights, regularity problems, etc. (see [1], [3], [4], [5], [6], [7], [9], [12], [13], [20], [21] and references therein).

H-type group G was introduced by Kaplan in [18], which is a class of Carnot group. If the dimension of the center of Lie algebra of an H-type group is trivial, then the H-type group is isomorphic to Heisenberg group. Analysis on the H-type group is motivated by its role as an important

model in the general theory of vector fields satisfying Hörmander’s condition (see [2], [11], [10], [14], [15], [17], [18], [24]).

Some important integral inequalities have been obtained and applied to related problems. For example, Hardy-Sobolev type inequality has been obtained by the second author and Niu ([15]) On the H-type group. Therefore, we desire to know whether CKN inequality in **Theorem A** is true on the H-type group. This paper will give a positive answer for the case $1 < p < Q$, where Q is homogeneous dimension of H-type group.

To state our results, we need to describe some preliminaries for H-type group G (more details can be found in [10], [11], [18] and references therein).

Consider a Carnot group G of step 2, with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$. We assume that \mathfrak{g} is equipped with a scalar product $\langle \cdot, \cdot \rangle$, with respect to which the V_j ’s are mutually orthogonal. We use the exponential mapping $\exp : \mathfrak{g} \rightarrow G$ to define analytic maps $\zeta_i : G \rightarrow V_i, i = 1, 2$, through the equation $g = \exp(\zeta_1(g) + \zeta_2(g))$. Here, $\zeta(g) = \zeta_1(g) + \zeta_2(g)$ is such that $g = \exp(\zeta(g))$. With $m = \dim(V_1)$ and $V_1 = \text{span}\{X_1, \dots, X_m\}$ (X_1, \dots, X_m are orthonormal), the coordinates of the projection ζ_1 in the basis X_1, \dots, X_m are denoted by $x_1 = x_1(g), \dots, x_m = x_m(g)$; that is,

$$x_j(g) = \langle \zeta(g), X_j \rangle, \quad j = 1, \dots, m,$$

and we set $x = x(g) = (x_1(g), \dots, x_m(g)) \in R^m$. Similarly, for a fixed orthonormal basis Y_1, \dots, Y_k of V_2 , we define the exponential coordinates in the second layer V_2 of a point $g \in G$ with

$$y_i(g) = \langle \zeta(g), Y_i \rangle, \quad i = 1, \dots, k$$

and $y = (y_1, \dots, y_k) \in R^k$.

For each $v \in V_1$, consider the orthogonal decomposition

$$V_1 = K_v \oplus R_v,$$

where $K_v = \ker(ad_v : V_1 \rightarrow V_2) = \{v' \in V_1 : [v, v'] = 0\}$. We say that the Lie algebra \mathfrak{g} is of H-type if the mapping $ad_v : R_v \rightarrow V_2$ is a surjective isometry for every unit vector $v \in V_1$ and the corresponding simple connected group G is named H-type group.

A family of dilations is defined by

$$\delta_\lambda(x, y) = (\lambda x, \lambda^2 y), \quad \text{for any } \lambda > 0, (x, y) \in G.$$

The homogeneous dimension of G with respect to dilations is $Q = m + 2k$.

Recently, using Hardy-Sobolev type inequality and interpolation technique in [17], a class of CKN type inequalities on the H-type group has been obtained as follows:

Theorem B Set $p, q, r, \beta, \gamma, \sigma$ and a such that

$$(1.4) \quad \begin{cases} 1 < p < Q, & q, r \geq 1, & 0 \leq a \leq 1, \\ m > \gamma r > -Q, & m > \beta q > -Q, \end{cases}$$

where

$$(1.5) \quad \gamma = a\sigma + (1 - a)\beta.$$

For any $u \in C_0^\infty(G)$, there exists a positive constant C such that

$$(1.6) \quad \begin{aligned} & \left(\int_G \frac{|x|^{-\gamma r}}{\rho^{-\gamma r}} \rho^{\gamma r} |u|^r \, dx dy \right)^{\frac{1}{r}} \\ & \leq C \left(\int_G |Xu|^p \, dx dy \right)^{\frac{a}{p}} \left(\int_G \frac{|x|^{-\beta q}}{\rho^{-\beta q}} \rho^{\beta q} |u|^q \, dx dy \right)^{\frac{1-a}{q}} \end{aligned}$$

holds if and only if the following relations hold:

$$(1.7) \quad \frac{1}{r} + \frac{\gamma}{Q} = a \left(\frac{1}{p} - \frac{1}{Q} \right) + (1 - a) \left(\frac{1}{q} + \frac{\beta}{Q} \right)$$

((1.7) is dimensional balance),

$$(1.8) \quad 0 \geq \sigma, \quad \text{if } a > 0,$$

$$(1.9) \quad \sigma \geq -1, \quad \text{if } a > 0 \text{ and } \frac{1}{p} - \frac{1}{Q} = \frac{1}{r} + \frac{\gamma}{Q},$$

where $Xu = (X_1u, \dots, X_mu)$ is the generalized horizontal gradient, $\rho = (|x(g)|^4 + |16|y(g)|^2)^{\frac{1}{4}}$ is the norm function and $dx dy$ is the Haar measure on G .

Obviously, **Theorem B** had initially investigated that the CKN inequality holds in the specific model of vector fields satisfying Hörmander’s condition. However, it is not completely analogous to **Theorem A**.

In this paper, we will establish the following generalizing CKN type inequalities on the H-type group G .

THEOREM 1.1. Set $p, q, r, \alpha, \beta, \gamma, \sigma$ and a such that

$$(1.10) \quad \begin{cases} 1 < p < Q, & q \geq 1, & r > 0, & 0 \leq a \leq 1, \\ \gamma r + Q > 0, & \alpha p + Q > 0, & \beta q + Q > 0, \\ m + (\alpha - \gamma)r > 0, & m + (\alpha - \beta)q > 0, \end{cases}$$

where

$$(1.11) \quad \gamma = a\sigma + (1 - a)\beta.$$

For any $u \in C_0^\infty(G)$, there exists a positive constant C such that

$$(1.12) \quad \left(\int_G \left(\frac{|x|}{\rho} \right)^{(\alpha-\gamma)r} \rho^{\gamma r} |u|^r \, dx dy \right)^{\frac{1}{r}} \leq C \left(\int_G |\rho^\alpha Xu|^p \, dx dy \right)^{\frac{a}{p}} \left(\int_G \left(\frac{|x|}{\rho} \right)^{(\alpha-\beta)q} \rho^{\beta q} |u|^q \, dx dy \right)^{\frac{1-a}{q}}$$

holds if and only if the following relations hold:

$$(1.13) \quad \frac{1}{r} + \frac{\gamma}{Q} = a \left(\frac{1}{p} + \frac{\alpha - 1}{Q} \right) + (1 - a) \left(\frac{1}{q} + \frac{\beta}{Q} \right)$$

(the dimensional balance condition),

$$(1.14) \quad \alpha - \sigma \geq 0, \quad \text{if } a > 0,$$

$$(1.15) \quad \alpha - \sigma \leq 1, \quad \text{if } a > 0 \text{ and } \frac{1}{p} + \frac{\alpha - 1}{Q} = \frac{1}{r} + \frac{\gamma}{Q}.$$

To prove **Theorem 1.1**, we employ the idea of [16], [17]. So we need firstly to establish a class of weighted Hardy-Sobolev type inequalities as follows:

THEOREM 1.2. *If $1 < p < Q$, $0 \leq s \leq p$, $\alpha > \frac{p - Q}{p}$, $p_*(s, p, Q) = \frac{p(Q - s)}{Q - p}$, then there exists a positive constant $C = C(s, p, \alpha, Q)$ such that for every $u \in D_\alpha^{1,p}(H_n)$,*

$$(1.16) \quad \int_G \frac{|x|^s}{\rho^s} \frac{|\rho^\alpha u|^{p_*(s,p,Q)}}{\rho^s} \, dx dy \leq C \left(\int_G |\rho^\alpha Xu|^p \, dx dy \right)^{\frac{Q-s}{Q-p}},$$

where $D_\alpha^{1,p}(G)$ is the closure of $C_0^\infty(G)$ with respect to the norm

$$\|u\|_\alpha^p = \int_G |\rho^\alpha Xu|^p \, dx dy.$$

REMARK 1.3. When $\alpha = 0$, **Theorem 1.2** and **Theorem 1.1** are Hardy-Sobolev type inequality (see [15]) and CKN type inequality (see [17]), re-

spectively. When $a = 1$, conditions of **Theorem 1.1** imply

$$0 \leq \alpha - \sigma = \alpha - \gamma \leq 1, \quad \frac{1}{r} + \frac{\sigma}{Q} = \frac{1}{p} + \frac{\alpha - 1}{Q},$$

and then $p \leq r \leq p^* = \frac{Qp}{Q-p}$. So, there exists $t \in [0, 1]$ such that

$$r = tp + (1-t)p^* = \frac{p(Q-tp)}{Q-p},$$

and $(\alpha - \sigma)r = tp$. Substituting r and $\alpha - \sigma$, (1.12) becomes the weighted Hardy-Sobolev type inequality (1.16).

As far as we know, on the H-type group, there are fewer inequalities, which are similar to (1.6) and (1.12). Main known results were derived by Lu in [22], [23] on stratified nilpotent Lie group. There, through approximating Sobolev functions by polynomials, Lu gave both local and global Sobolev interpolation inequalities (weighted and nonweighted version) of any higher orders for the Folland-Stein Sobolev spaces on stratified nilpotent Lie group and on Boman chain domains. But, these results are different from the results of **Theorem B** and **Theorem 1.1**.

The paper is organized as follows. In Section 2, we introduce some basic facts for the H-type group. Section 3 is devoted to the proof of **Theorem 1.2**. In Section 4, we complete the proof of our main result, namely **Theorem 1.1**, by using some accurate estimates and weighted Hardy-Sobolev type inequality.

2. Some known results

In [10], [11], Garofalo and Vassilev pointed out that

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{i=1}^k \langle [\xi, X_j], Y_i \rangle \frac{\partial}{\partial y_i} \\ &= \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{i=1}^k \langle [\xi_1, X_j], Y_i \rangle \frac{\partial}{\partial y_i}, \quad j = 1, \dots, m, \end{aligned}$$

where $\xi = \xi_1 + \xi_2 \in \mathfrak{g} = V_1 \oplus V_2$, $x = (x_1, \dots, x_m) \in R^m$, $y = (y_1, \dots, y_k) \in R^k$. Denote by

$$\begin{aligned} Xu &= (X_1u, X_2u, \dots, X_mu), \\ \operatorname{div}_X(u_1, u_2, \dots, u_m) &= \sum_{i=1}^m X_i u_i, \end{aligned}$$

and

$$Lu = \sum_{i=1}^m X_i^2 u$$

the horizontal gradient, horizontal divergence and sub-Laplacian, respectively.

The norm function on G has the form

$$\rho(g) = d(g, 0) = (|x(g)|^4 + 16|y(g)|^2)^{\frac{1}{4}},$$

where 0 is the unit element of G . It is easy to verify that (see [10], [11])

$$\begin{aligned} |X(|x(g)|^2)|^2 &= 4|x(g)|^2, & |X(|y(g)|^2)|^2 &= |x(g)|^2|y(g)|^2, \\ \langle X(|x(g)|^2), X(|y(g)|^2) \rangle &= 0, & |X\rho|^2 &= \frac{|x(g)|^2}{\rho^2}, \\ L(|x(g)|^2) &= 2m, & L(|y(g)|^2) &= \frac{k}{2}|x(g)|^2, \\ L\rho &= (Q - 1)\frac{|x(g)|^2}{\rho^3}, & \langle X(|x(g)|), X\rho \rangle &= \frac{|x(g)|^3}{\rho^3}. \end{aligned}$$

Denote by $B_R(\xi) \equiv B(\xi, R) = \{\eta \in G | d(\xi, \eta) = \rho(\eta^{-1}\xi) < R\}$ the ball centered at ξ with radius R and let $(x^*, y^*) = \left(\frac{x}{\rho}, \frac{y}{\rho^2}\right)$ be a point on the unit sphere $S = \{g \in G : \rho(g) = 1\}$.

The polar coordinates formula (see [2], [8], [14]) is stated as follows: There is a unique Radon measure $d\mu$ on S such that for all $f \in L^1(G)$,

$$(2.1) \quad \int_G f(x, y) \, dx dy = \int_0^\infty \int_S f \circ \delta_\rho(x^*, y^*) \rho^{Q-1} \, d\mu d\rho.$$

Moreover, authors of [2], [14] gave the explicit polar coordinates transformation, Jacobian and the following criteria for the integrability of $|x|^p \rho^q$.

- 1) If $p + m > 0$ and $p + q + Q > 0$, then $\int_{B_1} |x|^p \rho^q \, dx dy < +\infty$;
- 2) If $p + m > 0$ and $p + q + Q < 0$, then $\int_{G \setminus B_1} |x|^p \rho^q \, dx dy < +\infty$.

Setting

$$A = \left(\int_G |\rho^2 X u|^p \, dx dy \right)^{\frac{1}{p}}, \quad B = \left(\int_G \frac{|x|^{(\alpha-\beta)q}}{\rho^{(\alpha-\beta)q}} \rho^{\beta q} |u|^q \, dx dy \right)^{\frac{1}{q}},$$

(1.12) can be rewritten as

$$\left(\int_G \frac{|x|^{(\alpha-\gamma)r}}{\rho^{(\alpha-\gamma)r}} \rho^{\gamma r} |u|^r \, dx dy \right)^{\frac{1}{r}} \leq CA^a B^{1-a}.$$

Rescaling u such that $A^a B^{1-a} = 1$, our goal becomes to investigate the following inequality

$$\left(\int_G \frac{|x|^{(\alpha-\gamma)r}}{\rho^{(\alpha-\gamma)r}} \rho^{\gamma r} |u|^r \, dx dy \right)^{\frac{1}{r}} \leq C.$$

In the sequel, let $\Phi(\xi)(0 \leq \Phi \leq 1)$ be a fixed C_0^∞ function on G satisfying

$$(2.2) \quad \Phi = \begin{cases} 1, & \text{if } \rho < \frac{1}{2}, \\ 0, & \text{if } \rho > 1. \end{cases}$$

3. Weighted Hardy-Sobolev type inequality

In this section, we devote to establish a weighted Hardy-Sobolev type inequality. To this end, we introduce the following known facts firstly.

PROPOSITION 3.1 [Theorem 4.2, [17]]. *Let $p \geq 1, \alpha \in R$ and $\frac{1}{p} + \frac{\alpha - 1}{Q} > 0$. For any $f \in C_0^\infty(G)$, there exists a positive constant $C = \left(\frac{p}{Q - p + \alpha p}\right)^p$ such that*

$$(3.1) \quad \int_G \rho^{2p} \frac{|x|^p}{\rho^p} \frac{|f|^p}{\rho^p} \, dx dy \leq C \int_G \rho^{2p} |Xf|^p \, dx dy.$$

PROPOSITION 3.2 [see [3]]. *For all $\xi_1, \xi_2 \in R^n$, the following inequalities hold:*

1). *If $p \leq 2$,*

$$|\xi_1 + \xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2} \langle \xi_1, \xi_2 \rangle \leq C(p) |\xi_2|^p,$$

$$|\xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2} \langle \xi_1, \xi_2 - \xi_1 \rangle \geq C(p) \frac{|\xi_2 - \xi_1|^p}{(|\xi_1| + |\xi_2|)^{2-p}};$$

2). If $p > 2$,

$$|\xi_1 + \xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2}\langle \xi_1, \xi_2 \rangle \leq \frac{p(p-1)}{2}(|\xi_1| + |\xi_2|)^{p-2}|\xi_2|^2,$$

$$|\xi_2|^p - |\xi_1|^p - p|\xi_1|^{p-2}\langle \xi_1, \xi_2 - \xi_1 \rangle \geq C(p)|\xi_2 - \xi_1|^p,$$

where $\langle \xi_1, \xi_2 \rangle$ represents the common inner product.

PROPOSITION 3.3 (Lemma 2.1, [19]). For any $\xi, \eta \in R^n$ and $\lambda > 0$,

$$|\xi|^{\lambda+1} + \lambda|\eta|^{\lambda+1} - (\lambda + 1)|\eta|^{\lambda-1}\langle \xi, \eta \rangle \geq 0.$$

Moreover, the equality holds if and only if $\xi = \eta$.

Proof of Theorem 1.2: The condition $\alpha > \frac{p-Q}{p}$ deduces that

$$\alpha p_*(s, p, Q) - s + Q > 0, \quad p\alpha + Q > 0,$$

which ensure that the right and left integral of (1.16) are well defined. For any $u \in C_0^\infty(G)$, take $w = \rho^zu$ and then

$$\begin{aligned} \int_G |Xw|^p \, dx dy &= \int_G |\rho^z Xu + \alpha \rho^{z-1} u X\rho|^p \, dx dy \\ &\leq \int_G \left(\rho^\alpha |Xu| + |\alpha| \rho^{z-1} |u| \frac{|x|}{\rho} \right)^p \, dx dy \\ &\leq C \int_G \left(\rho^{p\alpha} |Xu|^p + |\alpha|^p \rho^{z-1} |u|^p \left(\frac{|x|}{\rho} \right)^p \right) \, dx dy. \end{aligned}$$

From (3.1), we see that $\int_G |Xw|^p \, dx dy < +\infty$, i.e., $w \in D_0^{1,p}(G)$. A straightforward computation shows that

$$(3.2) \quad \int_G \frac{|x|^s}{\rho^s} \frac{|\rho^z u|^{p_*(s,p,Q)}}{\rho^s} \, dx dy = \int_G \frac{|x|^s}{\rho^s} \frac{|w|^{p_*(s,p,Q)}}{\rho^s} \, dx dy$$

and

$$(3.3) \quad \int_G |\rho^z Xu|^p \, dx dy = \int_G |Xw - \alpha \rho^{-1} w X\rho|^p \, dx dy.$$

According to the **Proposition 3.2**, we will discuss under the case $p \leq 2$ and $p > 2$, respectively.

3.1 – Case $p \leq 2$

By the first case of **Proposition 3.2**, it holds

$$\begin{aligned}
 & C(p) \int_G |Xw - \alpha\rho^{-1}wX\rho|^p \, dx dy \\
 & \geq \int_G \left\{ |Xw|^p - |\alpha|^p \frac{|w|^p}{\rho^p} |X\rho|^p \right. \\
 & \quad \left. - p\alpha|\alpha|^{p-2} \frac{w|w|^{p-2}}{\rho^{p-1}} |X\rho|^{p-2} \langle X\rho, Xw - \alpha\rho^{-1}wX\rho \rangle \right\} dx dy \\
 & = \int_G \left\{ |Xw|^p + (p-1)|\alpha|^p \frac{|x|^p}{\rho^p} \frac{|w|^p}{\rho^p} \right. \\
 & \quad \left. - p\alpha|\alpha|^{p-2} \frac{w|w|^{p-2}}{\rho^{p-1}} |X\rho|^{p-2} \langle X\rho, Xw \rangle \right\} dx dy.
 \end{aligned}$$

Since

$$\begin{aligned}
 & p\alpha|\alpha|^{p-2} \int_G \frac{w|w|^{p-2}}{\rho^{p-1}} |X\rho|^{p-2} \langle X\rho, Xw \rangle \, dx dy \\
 & = \alpha|\alpha|^{p-2} \int_G \rho^{1-p} |X\rho|^{p-2} \langle X\rho, X|w|^p \rangle \, dx dy \\
 & = -\alpha|\alpha|^{p-2} \int_G |w|^p \operatorname{div}_X(\rho^{1-p} |X\rho|^{p-2} X\rho) \, dx dy \\
 & = -\alpha|\alpha|^{p-2} (Q-p) \int_G \frac{|x|^p}{\rho^p} \frac{|w|^p}{\rho^p} \, dx dy,
 \end{aligned}$$

it follows

$$\begin{aligned}
 (3.4) \quad & C(p) \int_G |Xw - \alpha\rho^{-1}wX\rho|^p \, dx dy \geq \int_G |Xw|^p \, dx dy \\
 & \quad + \alpha|\alpha|^{p-2} (Q-p + \alpha(p-1)) \int_G \frac{|x|^p}{\rho^p} \frac{|w|^p}{|x|^p} \, dx dy.
 \end{aligned}$$

Because of $\alpha > \frac{p-Q}{p}$, it knows that $Q-p + \alpha(p-1) > 0$. If $\alpha < 0$, com-

binning the Hardy type inequality (3.1), we can deduce from (3.3) and (3.4) that

$$(3.5) \quad C(p) \int_G |\rho^\alpha Xw|^p \, dx dy \geq C_1 \int_G |Xw|^p \, dx dy,$$

where

$$\begin{aligned} C_1 &= 1 + \alpha|\alpha|^{p-2}(Q - p + \alpha(p - 1)) \left(\frac{p}{Q - p}\right)^p \\ &= \left(\frac{p}{Q - p}\right)^p \left\{ \left(\frac{Q - p}{p}\right)^p + \alpha|\alpha|^{p-2}(Q - p) + |\alpha|^p(p - 1) \right\}. \end{aligned}$$

Choosing $\xi = \frac{p - Q}{p}, \eta = \alpha$ in **Proposition 3.3**, we have $C_1 > 0$. If $\alpha \geq 0$, (3.5) holds obviously with $C_1 = 1$. Then, (1.16) can be obtained by Hardy-Sobolev type inequality (see [15], namely the case $\alpha = 0$ in (1.16)).

3.2 - Case $p > 2$

Taking $\xi_1 = \alpha\rho^{-1}wX\rho, \xi_2 = Xw - \alpha\rho^{-1}wX\rho$ in **Proposition 3.2**,

$$\begin{aligned} & 2^{p-2} \frac{p(p-1)}{2} (|\alpha\rho^{-1}wX\rho| + |Xw|)^{p-2} |Xw - \alpha\rho^{-1}wX\rho|^2 \\ & \geq \frac{p(p-1)}{2} (2|\alpha\rho^{-1}wX\rho| + |Xw|)^{p-2} |Xw - \alpha\rho^{-1}wX\rho|^2 \\ & \geq \frac{p(p-1)}{2} (|\alpha\rho^{-1}wX\rho| + |Xw - \alpha\rho^{-1}wX\rho|)^{p-2} \\ & \quad \cdot |Xw - \alpha\rho^{-1}wX\rho|^2 \\ & \geq |Xw|^p - |\alpha\rho^{-1}wX\rho|^p \\ & \quad - p|\alpha\rho^{-1}wX\rho|^{p-2} \langle \alpha\rho^{-1}wX\rho, Xw - \alpha\rho^{-1}wX\rho \rangle \\ & = |Xw|^p + (p-1)|\alpha|^p \frac{|x|^p}{\rho^p} \frac{|w|^p}{\rho^p} \\ & \quad - p\alpha|\alpha|^{p-2} |X\rho|^{p-2} \langle \rho^{1-p}X\rho, w|w|^{p-2}Xw \rangle. \end{aligned}$$

Denote by $C_1 = 2^{p-2} \frac{p(p-1)}{2}$ and then

$$\begin{aligned}
& C_1 \int_G (|\alpha\rho^{-1}wX\rho| + |Xw|)^{p-2} |Xw - \alpha\rho^{-1}wX\rho|^2 \, dx dy \\
& \geq \int_G \left\{ |Xw|^p + (p-1)|\alpha|^p \frac{|x|^p}{\rho^p} \frac{|w|^p}{\rho^p} \right. \\
(3.6) \quad & \quad \left. - p\alpha|\alpha|^{p-2}|X\rho|^{p-2} \langle \rho^{1-p}X\rho, w|w|^{p-2}Xw \rangle \right\} \, dx dy \\
& = \int_G \left\{ |Xw|^p + \alpha|\alpha|^{p-2}(Q-p+\alpha(p-1)) \frac{|x|^p}{\rho^p} \frac{|w|^p}{\rho^p} \right\} \, dx dy \\
& \geq C_2 \int_G |Xw|^p \, dx dy,
\end{aligned}$$

where $C_2 > 0$ and the argument of the last inequality in (3.6) is similar to the case $p \leq 2$.

On the other hand, by Hölder inequality, Minkowski inequality and Hardy type inequality (3.1), it is obtained that

$$\begin{aligned}
& C_1 \int_G (|\alpha\rho^{-1}wX\rho| + |Xw|)^{p-2} |Xw - \alpha\rho^{-1}wX\rho|^2 \, dx dy \\
& \leq C_1 \left(\int_G |Xw - \alpha\rho^{-1}wX\rho|^p \, dx dy \right)^{\frac{2}{p}} \\
& \quad \cdot \left(\int_G (|\alpha\rho^{-1}wX\rho| + |Xw|)^p \, dx dy \right)^{\frac{p-2}{p}} \\
(3.7) \quad & \leq C_1 \left(\int_G |Xw - \alpha\rho^{-1}wX\rho|^p \, dx dy \right)^{\frac{2}{p}} \\
& \quad \cdot \left\{ \left(\int_G |\alpha\rho^{-1}wX\rho|^p \, dx dy \right)^{\frac{1}{p}} + \left(\int_G |Xw|^p \, dx dy \right)^{\frac{1}{p}} \right\}^{p-2} \\
& \leq C_1 \left(1 + |\alpha| \frac{p}{Q-p} \right)^{p-2} \left(\int_G |Xw|^p \, dx dy \right)^{\frac{p-2}{p}} \\
& \quad \cdot \left(\int_G |Xw - \alpha\rho^{-1}wX\rho|^p \, dx dy \right)^{\frac{2}{p}}.
\end{aligned}$$

Combining (3.6) and (3.7), there exists a constant $C_3 > 0$ such that

$$C_3 \int_G |Xw|^p \, dx dy \leq \int_G |Xw - \alpha \rho^{-1} w X\rho|^p \, dx dy.$$

From the Hardy-Sobolev type inequality on the H-type group, it arrives at (1.16).

4. CKN type inequalities with $1 < p < Q$

For the necessity of **Theorem 1.1**, since the detailed computations can be found in [17], [16], [4], we give only a simple sketch as follows: (1.13) and (1.14) can be obtained by scaling and translation, while (1.15) can be deduced by construction a class of specific test functions.

In the sequel, we mainly devote to the proof of the sufficiency. In addition, if $a = 0$, then (1.12) obviously holds; if $a = 1$, we complete the proof in **Remark 1.3**. So we need only to treat the case $0 < a < 1$.

4.1 – Sufficiency when $0 < a < 1$, $0 \leq \alpha - \sigma \leq 1$

For $p \leq \left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{Q}\right)^{-1} \leq p^*$, an argument similar to the one in the **Remark 1.3**, shows that there exists $t(0 \leq t \leq 1)$ such that $\left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{Q}\right)^{-1} = \frac{p(Q - tp)}{Q - p}$ and $(\alpha - \sigma)\left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{Q}\right)^{-1} = tp$. Hence,

$$\begin{aligned} & \int_G \left(\frac{|x|}{\rho}\right)^{(\alpha - \sigma)\left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{Q}\right)^{-1}} \frac{|\rho^\alpha u|^{\left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{Q}\right)^{-1}}}{\rho^{(\alpha - \sigma)\left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{Q}\right)^{-1}}} \, dx dy \\ & \leq C \left(\int_G |\rho^\alpha Xu|^p \, dx dy \right)^{\frac{Q - tp}{Q - p}}. \end{aligned}$$

By (1.11) and (1.13), $\frac{1}{r} = a\left(\frac{1}{p} + \frac{\alpha - \sigma - 1}{Q}\right) + \frac{1 - a}{q} \leq 1$. For $r > 1$, we have

$$\begin{aligned}
 & \int_G \left(\frac{|x|}{\rho}\right)^{(\alpha-\gamma)r} \rho^{2r} |u|^r dx dy \\
 & \leq \left(\int_G \left(\frac{|x|}{\rho}\right)^{(\alpha-\sigma)\left(\frac{1}{p} + \frac{2-\sigma-1}{Q}\right)^{-1}} \frac{|\rho^2 u|^{\left(\frac{1}{p} + \frac{2-\sigma-1}{Q}\right)^{-1}}}{\rho^{(\alpha-\sigma)\left(\frac{1}{p} + \frac{2-\sigma-1}{Q}\right)^{-1}}} dx dy \right)^{a\left(\frac{1}{p} + \frac{2-\sigma-1}{Q}\right)} \\
 & \quad \cdot \left(\int_G \left(\frac{|x|}{\rho}\right)^{(\alpha-\beta)q} \rho^{\beta q} |u|^q dx dy \right)^{\frac{1-a}{q}} \left(\int_G \left(\frac{|x|}{\rho}\right)^{(\alpha-\gamma)r} \rho^{2r} |u|^r dx dy \right)^{\frac{r-1}{r}} \\
 & \leq C \left(\int_G |\rho^2 Xu|^p dx dy \right)^{\frac{a}{p}} \\
 & \quad \cdot \left(\int_G \left(\frac{|x|}{\rho}\right)^{(\alpha-\beta)q} \rho^{\beta q} |u|^q dx dy \right)^{\frac{1-a}{q}} \left(\int_G \left(\frac{|x|}{\rho}\right)^{(\alpha-\gamma)r} \rho^{2r} |u|^r dx dy \right)^{\frac{r-1}{r}},
 \end{aligned}$$

and hence (1.12) holds. For $r = 1$, we argue with the same way.

4.2 – Sufficiency when $0 < a < 1$, $\alpha - \sigma > 1$

(1.15) tells us $a < 1, \frac{1}{p} + \frac{\alpha - 1}{Q} \neq \frac{1}{r} + \frac{\gamma}{Q}$. Using the discussion in the end of Section 2, we assume $A^a B^{1-a} = 1$ and want to prove that

$$(4.1) \quad \int_G \left(\frac{|x|}{\rho}\right)^{(\alpha-\gamma)r} \rho^{2r} |u|^r dx dy$$

is bounded. In Subsection 4.1, (1.12) has been checked for $\alpha - \sigma = 1$ and $\alpha - \sigma = 0$. Hence, we conclude that

$$(4.2) \quad \int_G \left(\frac{|x|}{\rho}\right)^{(\alpha-\delta)s} \rho^{\delta s} |u|^s dx dy \leq C, \quad \int_G \left(\frac{|x|}{\rho}\right)^{(\alpha-\varepsilon)t} \rho^{\varepsilon t} |u|^t dx dy \leq C,$$

where δ, s, ε and t satisfy

$$(4.3) \quad \begin{cases} \delta = b \cdot \alpha + (1 - b)\beta, \\ \frac{1}{s} = \frac{b}{p} + \frac{1 - b}{q} - \frac{b}{Q}, \\ \varepsilon = d \cdot (\alpha - 1) + (1 - d)\beta, \\ \frac{1}{t} = \frac{d}{p} + \frac{1 - d}{q} \end{cases}$$

for some choices of b and d , $0 \leq b, d \leq 1$, and then

$$(4.4) \quad \begin{cases} m + (\alpha - \delta)s > 0, & \delta s + Q > 0, \\ m + (\alpha - \varepsilon)t > 0, & \varepsilon t + Q > 0. \end{cases}$$

Obviously,

$$\frac{1}{t} + \frac{\varepsilon}{Q} = d \left(\frac{1}{p} + \frac{\alpha - 1}{Q} \right) + (1 - d) \left(\frac{1}{q} + \frac{\beta}{Q} \right),$$

$$\frac{1}{r} + \frac{\gamma}{Q} = a \left(\frac{1}{p} + \frac{\alpha - 1}{Q} \right) + (1 - a) \left(\frac{1}{q} + \frac{\beta}{Q} \right),$$

$$\frac{1}{s} + \frac{\delta}{Q} = b \left(\frac{1}{p} + \frac{\alpha - 1}{Q} \right) + (1 - b) \left(\frac{1}{q} + \frac{\beta}{Q} \right).$$

1) Case $\frac{1}{p} + \frac{\alpha - 1}{Q} < \frac{1}{q} + \frac{\beta}{Q}$. Take $b < a < d$ and then

$$(4.5) \quad \frac{1}{t} + \frac{\varepsilon}{Q} < \frac{1}{r} + \frac{\gamma}{Q} < \frac{1}{s} + \frac{\delta}{Q}.$$

A direct computation shows that

$$\frac{1}{r} - \frac{1}{s} = (a - b) \left(\frac{1}{p} - \frac{1}{Q} - \frac{1}{q} \right) + \frac{a(\alpha - \sigma)}{Q},$$

$$\frac{1}{r} - \frac{1}{t} = (a - d) \left(\frac{1}{p} - \frac{1}{q} \right) + \frac{a(\alpha - \sigma - 1)}{Q},$$

$$\begin{aligned} & \left(\frac{1}{r} + \frac{\alpha - \gamma}{m} \right) - \left(\frac{1}{t} + \frac{\alpha - \varepsilon}{m} \right) \\ &= (a - d) \left(\frac{1}{p} - \frac{1}{q} + \frac{1}{m}(\beta + 1 - \alpha) \right) + a(\alpha - \sigma - 1) \left(\frac{1}{Q} + \frac{1}{m} \right), \end{aligned}$$

$$\begin{aligned} & \left(\frac{1}{r} + \frac{\alpha - \gamma}{m} \right) - \left(\frac{1}{s} + \frac{\alpha - \delta}{m} \right) \\ &= (a - b) \left(\frac{1}{p} - \frac{1}{q} - \frac{1}{Q} + \frac{1}{m}(\beta - \alpha) \right) + a(\alpha - \sigma) \left(\frac{1}{Q} + \frac{1}{m} \right), \end{aligned}$$

Since $a > 0, \alpha - \sigma > 1$, we see

$$0 < \frac{a(\alpha - \sigma - 1)}{Q} < \frac{a(\alpha - \sigma)}{Q},$$

$$0 < a(\alpha - \sigma - 1) \left(\frac{1}{Q} + \frac{1}{m} \right) < a(\alpha - \sigma) \left(\frac{1}{Q} + \frac{1}{m} \right)$$

and for sufficiently small $|d - a|$ and $|b - a|$,

$$(4.6) \quad \begin{cases} \frac{1}{r} > \frac{1}{s}, & \frac{1}{r} > \frac{1}{t}, \\ \frac{1}{r} + \frac{\alpha - \gamma}{m} > \frac{1}{t} + \frac{\alpha - \varepsilon}{m}, \\ \frac{1}{r} + \frac{\alpha - \gamma}{m} > \frac{1}{s} + \frac{\alpha - \delta}{m}. \end{cases}$$

Combining (4.5) and (4.6), one gets

$$(4.7) \quad \begin{cases} m + (\varepsilon - \gamma) \frac{tr}{t - r} > 0, & (\gamma - \varepsilon) \frac{tr}{t - r} + Q > 0, \\ m + (\delta - \gamma) \frac{sr}{s - r} > 0, & (\gamma - \delta) \frac{sr}{s - r} + Q < 0. \end{cases}$$

By Hölder inequality it yields

$$(4.8) \quad \begin{aligned} & \left(\int_G \left(\frac{|x|}{\rho} \right)^{(\alpha-\gamma)r} \rho^{\gamma r} \Phi |u|^r dx dy \right)^{\frac{1}{r}} \\ & \leq C \left(\int_G \left(\frac{|x|}{\rho} \right)^{(\alpha-\varepsilon)t} \rho^{\varepsilon t} |u|^t dx dy \right)^{\frac{1}{t}} \end{aligned}$$

and

$$(4.9) \quad \begin{aligned} & \left(\int_G \left(\frac{|x|}{\rho} \right)^{(\alpha-\gamma)r} \rho^{\gamma r} (1 - \Phi) |u|^r dx dy \right)^{\frac{1}{r}} \\ & \leq C \left(\int_G \left(\frac{|x|}{\rho} \right)^{(\alpha-\delta)s} \rho^{\delta s} |u|^s dx dy \right)^{\frac{1}{s}}, \end{aligned}$$

where Φ is defined in (2.2). Combining (4.2), (4.8) and (4.9), we conclude that (4.1) holds.

2) Case $\frac{1}{p} + \frac{\lambda - 1}{Q} > \frac{1}{q} + \frac{\beta}{n}$. Take $d < a < b$ such that $|d - a|$ and $|b - a|$ are sufficiently small. Then (4.3)-(4.7) hold. This implies that (4.2), (4.8) and (4.9) are true. So (4.1) holds.

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