REND. SEM. MAT. UNIV. PADOVA, Vol. 134 (2015)
DOI 10.4171/RSMUP/134-5

The Cyclic and Epicyclic Sites

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ABSTRACT - We determine the points of the epicyclic topos which plays a key role in the geometric encoding of cyclic homology and the lambda operations. We show that the category of points of the epicyclic topos is equivalent to projective geometry in characteristic one over algebraic extensions of the infinite semifield of "max-plus integers" $\mathbb{Z}_{\max}$. An object of this category is a pair (E, K) of a semimodule E over an algebraic extension K of $\mathbb{Z}_{\max}$. The morphisms are projective classes of semilinear maps between semimodules. The epicyclic topos sits over the arithmetic topos $\widehat{\mathbb{N}^\times}$ of [6] and the fibers of the associated geometric morphism correspond to the cyclic site. In two appendices we review the role of the cyclic and epicyclic toposes as the geometric structures supporting cyclic homology and the lambda operations.

MATHEMATICS SUBJECT CLASSIFICATION (2010). 18B25; 20L05, 19D55.

KEYWORDS. Grothendieck topos, Cyclic category, Groupoids, Characteristic one, Projective geometry.

1. Introduction

The theory of topoi of Grothendieck provides the best geometric framework to understand cyclic homology and the λ-operations using the topos associated to the cyclic category [3] and its epicyclic refinement [5]. Given a topos $\mathcal{T}$ a basic question is to determine the category of points of $\mathcal{T}$, *i.e.* of geometric morphisms from the topos of sets to $\mathcal{T}$. In this paper we show how to describe the category of points of the epicyclic topos in terms of projective geometry in characteristic 1. Given a small category $\mathcal{C}$, we denote by $\hat{\mathcal{C}}$ the topos of contravariant functors from $\mathcal{C}$ to the category $\mathfrak{Sets}$ of sets. The epicyclic topos $(\tilde{\Lambda}^{\rm op})^\wedge$ is obtained by taking the *opposite* of the epicyclic category $\tilde{\Lambda}$. This

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(***) C. Consani would like to thank the Collège de France for some financial support.

choice is dictated by the following natural construction. A commutative ring R determines a *covariant* functor $(-)^\natural : \mathfrak{Fin} \longrightarrow \mathfrak{Ab}$ from the category of finite sets to that of abelian groups. This functor assigns to a finite set J the tensor power $R^{\otimes J} = \bigotimes_{j\in J} R$. As explained in geometric terms here below, there is also a natural *covariant* functor $\widetilde{\Lambda} \longrightarrow \mathfrak{Fin}$. The composite of these two functors $\widetilde{\Lambda} \longrightarrow \mathfrak{Ab}$ provides, for any commutative ring R, a *covariant* functor $R^\natural$ from the epicyclic category to the category of abelian groups. In geometric terms $R^\natural$ is a sheaf of abelian groups over the topos $(\widetilde{\Lambda}^{\rm op})^\wedge$. Both the cyclic homology of R and its λ-operations are completely encoded by the associated sheaf $R^\natural$. In [5], we provided a conceptual understanding of the epicyclic category as projective geometry over the semifield $\mathbb{F} := \mathbb{Z}_{\max}$ of the tropical integers. In these terms the functor $(-)^\natural$ considered above assigns to a projective space the underlying finite set. This article pursues the relation between the epicyclic topos and (projective) geometry in characteristic 1 in more details. Our main result is the following (*cf.* Theorem 4.1)

THEOREM. *The category of points of the epicyclic topos $(\widetilde{\Lambda}^{\rm op})^\wedge$ is equivalent to the category $\mathcal{P}$ whose objects are pairs (K, E), where K is an algebraic extension of $\mathbb{F} = \mathbb{Z}_{\max}$ and E is an archimedean semimodule over K. The morphisms in $\mathcal{P}$ are projective classes of semilinear maps and injective semifield morphisms.*

It is important to realize the relevance of the language of Grothendieck topoi to interpret, for instance, the action of the edgewise subdivision on the points of the simplicial topos $\hat{\Delta}$. The preliminary Section 2 is dedicated to this description. It is well known (*cf.* [10]) that the points of $\hat{\Delta}$ correspond to intervals, *i.e.* totally ordered sets I with a smallest element b and a largest element $t \neq b$. For each integer $k > 0$, the edgewise subdivision Sd_k defines an endofunctor of the simplicial category Δ and one obtains in this way an action of the monoïd $\mathbb{N}^\times$ by geometric morphisms on the topos $\hat{\Delta}$. We show that the action of the edgewise subdivision on the points of $\hat{\Delta}$ is given by the operation of concatenation of k copies of the interval I: the intermediate top point t_j of the copy I_j is identified with the bottom point b_{j+1} of the subsequent copy I_{j+1}. Then, we form the small category $\Delta^{\rm op} \ltimes \mathbb{N}^\times$ crossed product of $\Delta^{\rm op}$ by the transposed action Sd^* of $\mathbb{N}^\times$ (*i.e.* $\mathrm{Sd}_k(f)^* = \mathrm{Sd}_k^*(f^*)$, where $f \mapsto f^*$ is the anti-isomorphism $\Delta \longrightarrow \Delta^{\rm op}$).

Section 3 gives a description of the epicyclic category in terms of oriented groupoïds. The ambiguïty in the choice of a representative of a projective class of semilinear maps in the category $\mathcal{P}$ of Theorem 4.1 is inconvenient when working, for example, with colimits. In Section 3.3 we provide a description of the cyclic and the epicyclic categories in terms of a category $\mathfrak{g}$ of *oriented groupoïds* whose morphisms are no longer given by equivalence classes. There are by now a number of equivalent descriptions of the cyclic and epicyclic categories, ranging from the most concrete *i.e.* given in terms of generators and relations, to the most conceptual as in [5]. The description of these categories in terms of oriented groupoïds turns out to be very useful to determine the points of the epicyclic topos by considering filtering colimits, in the category $\mathfrak{g}$, of the special points provided by the Yoneda embedding of the categories. It is in fact well known that any point of a topos of the form $\hat{\mathcal{C}}$ is obtained as a filtering colimit, in the category of flat functors $\mathcal{C} \longrightarrow \mathfrak{Sets}$, of these special points. On the other hand, there is no guarantee "a priori" that this colimit process yields the same result as the colimit taken in the category $\mathfrak{g}$. This matter is solved in two steps and in concrete terms in Section 4. In Proposition 4.3 we show how to associate to a pair (K, E) as in the above Theorem a point of $(\tilde{\Lambda}^{\rm op})^\wedge$. Conversely, in §§4.2-4.4 we explain a geometric procedure that allows one to reconstruct the structure of an oriented groupoïd from the flat functor naturally associated to a point of $(\tilde{\Lambda}^{\rm op})^\wedge$.

In §4.8 we explore the relations of $(\tilde{\Lambda}^{\rm op})^\wedge$ with the arithmetic site $\widehat{\mathbb{N}^\times}$, as recently defined in [6]. Let $\mathbb{N}^\times$ be the small category with a single object $\bullet$ and whose endomorphisms $\mathrm{End}(\bullet) = \mathbb{N}^\times$ form the multiplicative semigroup $\mathbb{N}^\times$ of positive integers. One has a canonical functor $\mathrm{Mod} : \tilde{\Lambda}^{\rm op} \longrightarrow \mathbb{N}^\times$ which is trivial on the objects and associates to a semilinear map of semimodules over $\mathbb{F} = \mathbb{Z}_{\max}$ the corresponding injective endomorphism $\mathrm{Fr}_n \in \mathrm{End}(\mathbb{F})$ (*cf.* [5] for details). This functor induces a geometric morphism of topoi $\mathrm{Mod} : (\tilde{\Lambda}^{\rm op})^\wedge \longrightarrow \widehat{\mathbb{N}^\times}$. The subcategory of $\tilde{\Lambda}^{\rm op}$ which is the kernel of this morphism is the cyclic category Λ ($\Lambda \simeq \Lambda^{\rm op}$).

In Appendix A we view the λ-operations as elements Λ_n^k of the convolution ring $\mathbb{Z}[\Delta^{\rm op} \ltimes \mathbb{N}^\times]$ with integral coefficients. We review their geometric meaning and the geometric proof of their commutation (*cf.* [9]) with the Hochschild boundary operator.

Appendix B is dedicated to the description of the cyclic homology of cyclic modules (*cf.* [3]) and its extension to epicyclic modules [8]. We stress the nuance between $\tilde{\Lambda}$ and $\tilde{\Lambda}^{\rm op}$ in a hopefully clear form. An *epicyclic* module E is a *covariant* functor $\tilde{\Lambda} \longrightarrow \mathfrak{Ab}$. These modules correspond to sheaves of abelian groups on the topos $(\tilde{\Lambda}^{\rm op})^\wedge$. At this point the nuance between the epicyclic category and its dual plays an important role since unlike the cyclic category the epicyclic category is not anti-isomorphic to itself. As explained earlier on in this introduction, a commutative ring R gives rise naturally to an epicyclic module $R^\natural$ and it is well known (*cf.* [8]) that the λ-operations on cyclic homology of R are obtained directly through the associated epicyclic module. We provide a simple and conceptual proof of the commutation of the λ-operations with the B operator of cyclic theory. Finally, we point out that the extended framework of epicyclic modules involves many more modules than those arising by composition, as explained earlier, from a covariant functor $\mathfrak{Fin} \longrightarrow \mathfrak{Ab}$. In fact, these particular (epicyclic) modules have *integral* weights and the λ-operations decompose their cyclic homology as direct sums of modules on which Λ_n^k acts by an integral power of k. This integrality property no longer holds for general epicyclic modules as can be easily checked by applying a twisting argument.

2. The action of the edgewise subdivision on points of $\hat{\Delta}$

We recall that the simplicial category Δ is the small category with objects the totally ordered sets $[n] := \{0, \ldots, n\}$, for each integer $n \geq 0$, and morphisms non-decreasing maps.

In this section we study, using the formalism of topoi, the edgewise subdivision functors $\mathrm{Sd}_k : \Delta \longrightarrow \Delta$, for $k \in \mathbb{N}^\times$ and their action on the points of the simplicial topos $\hat{\Delta}$.

2.1 – *The edgewise subdivision functors* Sd_k

Let F be a finite, totally ordered set and $k \in \mathbb{N}^\times$ a positive integer. We define the set

$$\mathrm{Sd}_k(F) := \{0, \ldots, k-1\} \times F \tag{1}$$

to be the cartesian product of the finite ordered set $\{0, \ldots, k-1\}$ with F, endowed with the lexicographic ordering. For $f \in \mathrm{Hom}_\Delta(F, F')$ a non-decreasing map (of finite, totally ordered sets), we let

$$\mathrm{Sd}_k(f) := \mathrm{Id} \times f : \mathrm{Sd}_k(F) \to \mathrm{Sd}_k(F') \tag{2}$$

Proposition 2.1. *For each* $k \in \mathbb{N}^\times$, (1) *and* (2) *define an endofunctor* $\mathrm{Sd}_k : \Delta \longrightarrow \Delta$. *They fulfill the property*

$$\mathrm{Sd}_{kk'} = \mathrm{Sd}_k \circ \mathrm{Sd}_{k'}, \quad \forall k, k' \in \mathbb{N}^\times.$$

Proof. The totally ordered sets $\mathrm{Sd}_k([n])$ and $[k(n+1)-1]$ have the same cardinality and are canonically isomorphic. The unique increasing bijection $\mathrm{Sd}_k([n]) \to [k(n+1)-1]$ is given by

$$(a, i) \mapsto i + a(n+1), \quad \forall a \in \{0, \ldots, k-1\}, \ i \in \{0, \ldots, n\}.$$

Let $f \in \mathrm{Hom}_\Delta([n], [m])$ then by definition $\mathrm{Sd}_k(f) \in \mathrm{Hom}_\Delta(\mathrm{Sd}_k([n]), \mathrm{Sd}_k([m]))$ is given by

$$\mathrm{Sd}_k(f)(i + a(n+1)) = f(i) + a(m+1), \quad \forall i, a, \ 0 \leq i \leq n, \ 0 \leq a \leq k-1. \tag{3}$$

One checks directly that $\mathrm{Sd}_{kk'} = \mathrm{Sd}_k \circ \mathrm{Sd}_{k'}$. □

We transfer the functors Sd_k to the opposite category Δ^{op} of finite intervals. Recall that by definition, an interval I is a totally ordered set with a smallest element b and a largest element $t \neq b$. The morphisms between intervals are the non-decreasing maps respecting b and t, *i.e.* $f : I \to J$, $f(b_I) = b_J$, $f(t_I) = t_J$.

For all $n \geq 0$ we denote by $n^* := \{0, \ldots, n+1\}$. The interval n^* parametrizes the hereditary subsets of $[n]$: indeed, to $j \in n^*$ corresponds $[j, n] := \{x \in [n] \mid x \geq j\}$, the latter set is empty for $j = n+1$. The duality between Δ and Δ^{op} is then provided by the contravariant functor $\Delta \xrightarrow{\sim} \Delta^{\mathrm{op}}$, $[n] \mapsto n^*$, which acts on morphisms as follows

(4) $\mathrm{Hom}_\Delta([n],[m]) \ni f \to f^* \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(m^*, n^*),\ f^{-1}([j,m]) = [f^*(j), n], \quad \forall j \in m^*.$

Let I be an interval, and $k \in \mathbb{N}^\times$, then one lets $\mathrm{Sd}_k^*(I)$ to be the quotient of the totally ordered set$\{0,\dots,k-1\} \times I = \mathrm{Sd}_k(I)$ (with lexicographic ordering) by the equivalence relation $(j, t_I) \sim (j+1, b_I)$ for $j \in \{0,\dots,k-2\}$. This defines an endofunctor Sd_k^* of the category of intervals whose action on morphisms sends $f : I \to J$ to $\mathrm{Sd}_k^*(f) = \mathrm{Id} \times f$. By restriction to finite intervals one obtains an endofunctor $\mathrm{Sd}_k^* : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$.

In particular, the interval $\mathrm{Sd}_k^*(n^*)$ has $k(n+2) - (k-1) = k(n+1) + 1$ elements and one obtains a canonical identification of $\mathrm{Sd}_k^*(n^*)$ with the hereditary subsets of $\mathrm{Sd}_k([n])$ as follows

$$\{0,\dots,k-1\} \times n^* \ni (b,j) \mapsto \{(a,i) \in \{0,\dots,k-1\} \times [n] \mid a > b \text{ or } a = b \ \&\ j \geq i\}.$$

Note that the right hand side of the above formula depends only upon the class of $(b,j) \in \mathrm{Sd}_k^*(n^*)$.

Lemma 2.2. *For* $f \in \mathrm{Hom}_\Delta([n],[m])$, *one has* $(\mathrm{Sd}_k(f))^* = \mathrm{Sd}_k^*(f^*)$.

Proof. The morphism $(\mathrm{Sd}_k(f))^*$ is defined by the equivalence

$$\mathrm{Sd}_k(f)(x) \geq y \iff (\mathrm{Sd}_k(f))^*(y) \leq x$$

Let $x = i + a(n+1)$, $y = j + b(m+1)$ with $0 \leq i \leq n$, $0 \leq j \leq m$, $0 \leq a \leq k-1$, $0 \leq b \leq k-1$. Then by (3) one has $\mathrm{Sd}_k(f)(x) = f(i) + a(m+1)$, thus the condition $\mathrm{Sd}_k(f)(x) \geq y$ determines $(\mathrm{Sd}_k(f))^*(y)$ as follows

$$f(i) + a(m+1) \geq j + b(m+1) \iff a > b \text{ or } a = b \text{ and } f(i) \geq j$$
$$\iff i + a(n+1) \geq f^*(j) + b(n+1) = \mathrm{Sd}_k^*(f^*)(y).$$

This provides the required equality $(\mathrm{Sd}_k(f))^* = \mathrm{Sd}_k^*(f^*)$. □

Theorem 2.3. *The action* $\widehat{\mathrm{Sd}_k}$ *of the geometric morphism* Sd_k *(*$k \in \mathbb{N}^\times$*) on the points of the topos* $\hat{\Delta}$ *is described by the endofunctor* Sd_k^* *on the category of intervals.*

Proof. One can prove this theorem using the fact that any point of $\hat{\Delta}$ is obtained as a filtering colimit of the points associated to the Yoneda embedding of Δ^{op} in the category of points of $\hat{\Delta}$. One shows that on such points the action $\widehat{\mathrm{Sd}_k}$ coincides with the functor $\mathrm{Sd}_k^* : \Delta^{\mathrm{op}} \longrightarrow \Delta^{\mathrm{op}}$. We shall nevertheless find it more instructive to give, in §2.3, a concrete direct proof of the equality between the following two flat functors $\Delta \longrightarrow \mathfrak{Sets}$ associated to an interval I

(5) $F_1([n]) = \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(I)), F_2([n]) =$

$$= \left(\coprod_{m \geq 0} (\mathrm{Hom}(m^*, I) \times \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(m^*))) \right) / \sim$$

here F_2 is the inverse image functor of the point p_I of $\hat{\Delta}$ applied to the contravariant functor $Y : \Delta \longrightarrow \mathfrak{Sets}$, $Y = X \circ \mathrm{Sd}_k$, where $X = h_{[n]}$ is the Yoneda embedding, so that

$$Y([m]) = X(\mathrm{Sd}_k([m])) = \mathrm{Hom}_\Delta(\mathrm{Sd}_k([m]), [n]) = \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(m^*)). \tag{6}$$

Thus F_2 corresponds to the point $\widehat{\mathrm{Sd}_k}(p_I)$ and the equality between F_1 and F_2 (*cf.* Lemma 2.7) yields the result. □

Corollary 2.4. *The point of the simplicial topos $\hat{\Delta}$ associated to the interval $[0,1] \subset \mathbb{R}$ is a fixed point for the action of $\mathbb{N}^\times$ on $\hat{\Delta}$.*

Proof. The statement follows using the affine isomorphism

$$\mathrm{Sd}_k^*([0,1]) = \{0, \dots, k-1\} \times [0,1] \to [0,1], \quad (a, x) \mapsto \frac{a}{k} + \frac{x}{k}. \tag{7}$$

2.2 – *Canonical decomposition of* $\varphi \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(I))$

Let I be an interval, $I/\!\sim$ be the quotient of I by the identification $b \sim t$. Consider the map $\pi : \mathrm{Sd}_k^*(I) \to I/\!\sim$, $(j, x) \mapsto x$. For $\varphi \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(I))$, we define the rank of φ as the cardinality of the set $Z = I^o \cap \mathrm{Range}(\pi \circ \varphi)$, where $I^o = I \setminus \{b, t\}$.

Proposition 2.5. *Let $\varphi \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(I))$ and r its rank. Then, one has a unique decomposition*

$$\varphi = \mathrm{Sd}_k^*(\alpha) \circ \beta, \quad \beta \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(r^*)), \ \alpha \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(r^*, I).$$

Moreover, the morphism $\alpha \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(r^*, I)$ is the unique increasing injection $\{1, \dots, r\} \hookrightarrow I^o$ which admits Z as range. The composite $\pi_r \circ \beta$ is surjective, where $\pi_r : \mathrm{Sd}_k^*(r^*) \to r^*/\!\sim$ is the canonical surjection.

Proof. Let $\alpha \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(r^*, I)$ be the map whose restriction to $\{1, \dots, r\}$ is the unique increasing injection into I^o which admits Z as range. Recall that an element $x \in \mathrm{Sd}_k^*(I)$ is given by a pair $x = (j, y) \in \{0, \dots, k-1\} \times I = \mathrm{Sd}_k(I)$ with the identifications $(j, t) \sim (j+1, b)$ for $j \in \{0, \dots, k-2\}$. Similarly an element $z \in \mathrm{Sd}_k^*(r^*)$ is given by a pair $z = (i, u) \in \{0, \dots, k-1\} \times r^* = \mathrm{Sd}_k(r^*)$ with the identifications $(j, r+1) \sim (j+1, 0)$ for $j \in \{0, \dots, k-2\}$. Let $s \in n^* = \{0, \dots, n+1\}$, then $\varphi(s) \in \mathrm{Sd}_k^*(I)$ is given by a pair $\varphi(s) = (j, y) \in \{0, \dots, k-1\} \times I = \mathrm{Sd}_k(I)$ unique up to the above identifications. If $y \in \{b, t\}$, one defines $\beta(s) = (j, 0) \in \mathrm{Sd}_k^*(r^*)$ if $y = b$, and $\beta(s) := (j, r+1) \in \mathrm{Sd}_k^*(r^*)$ if $y = t$. This definition is compatible with the identifications. Let us now assume that $y \notin \{b, t\}$. Then $y \in Z$ and there exists a unique element $v \in \{1, \dots, r\}$ such that $y = \alpha(v)$. One then defines $\beta(s) := (j, v) \in$

$\mathrm{Sd}_k^*(r^*)$. The map $\beta : n^* \to \mathrm{Sd}_k^*(r^*)$ so defined is non-decreasing, *i.e.* for $s < s'$ one has $\beta(s') \geq \beta(s)$ since the inequality $\varphi(s') \geq \varphi(s)$ shows that either $j' > j$ in which case $(j', v') \geq (j, v)$ is automatic, or $j = j'$ and in that case $y' > y$ which shows that $v' \geq v$. Moreover since $\mathrm{Sd}_k^*(\alpha) = \mathrm{Id} \times \alpha$ one has $\varphi = \mathrm{Sd}_k^*(\alpha) \circ \beta$.

We prove the uniqueness of this decomposition. Since $\varphi \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(I))$ preserves the base points, Range $(\pi \circ \varphi)$ contains the base point and its cardinality is $r + 1$. Thus the map $\alpha \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(r^*, I)$ is the unique map whose restriction to $\{1, \ldots, r\}$ is the increasing injection to I^o and which admits $Z = I^o \cap$ Range $(\pi \circ \varphi)$ as range. Moreover α is injective and so is $\mathrm{Sd}_k^*(\alpha)$. Thus the map $\beta \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(r^*))$ is uniquely determined by the equality $\varphi = \mathrm{Sd}_k^*(\alpha) \circ \beta$. Finally $\pi_r \circ \beta : n^* \to r^*/\sim$ is surjective since otherwise the range of $\mathrm{Sd}_k^*(\alpha) \circ \beta$ would be strictly smaller than the range of φ. $\square$

COROLLARY 2.6. *For any interval I the map*

$$\mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, I) \times \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(n^*)) \to \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(I)), (\alpha, \beta) \mapsto \mathrm{Sd}_k^*(\alpha) \circ \beta$$

is surjective.

2.3 – *Explicit description of the isomorphism $F_1 \simeq F_2$*

Let $F_j : \Delta \longrightarrow \mathfrak{Sets}$ be the flat functors defined in (5). By definition

$$F_2([n]) = \left(\coprod_{m \geq 0} (\mathrm{Hom}(m^*, I) \times \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(m^*))) \right) /\sim \tag{8}$$

where the equivalence relation is generated by

$$(\alpha \circ f, \beta) \sim (\alpha, \mathrm{Sd}_k^*(f) \circ \beta)$$

for $f \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(m^*, r^*)$, $\beta \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(m^*))$, $\alpha \in \mathrm{Hom}(r^*, I)$.

LEMMA 2.7. *The map*

$$\Phi : F_2([n]) \to F_1([n]) = \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(I)), \quad (\alpha, \beta) \mapsto \mathrm{Sd}_k^*(\alpha) \circ \beta$$

is a bijection of sets.

PROOF. The map Φ is well defined since $\Phi(\alpha \circ f, \beta) = \Phi(\alpha, \mathrm{Sd}_k^*(f) \circ \beta)$. Corollary 2.6 shows that Φ is surjective. To show the injectivity it is enough to prove that for any $(\alpha, \beta) \in \mathrm{Hom}(m^*, I) \times \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(m^*))$ one has $(\alpha, \beta) \sim (\alpha_c, \beta_c)$ where

$$\varphi = \mathrm{Sd}_k^*(\alpha_c) \circ \beta_c, \;\; \beta_c \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(r^*)), \;\; \alpha_c \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(r^*, I)$$

is the canonical decomposition of $\varphi = \mathrm{Sd}_k^*(\alpha) \circ \beta$.

One has the canonical decomposition $\beta = \mathrm{Sd}_k^*(\alpha_0) \circ \beta_0$ with $\beta_0 \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, \mathrm{Sd}_k^*(\ell^*))$, $\mathrm{Id}_\ell \circ \beta_0$ surjective. Thus $(\alpha, \beta) \sim (\alpha \circ \alpha_0, \beta_0)$. Since $\mathrm{Id}_\ell \circ \beta_0$ is surjective, Range $\alpha \circ \alpha_0 \subset$

Range $\mathrm{Id} \circ \varphi =$ Range α_c, and thus $\alpha \circ \alpha_0 = \alpha_c \circ \rho$

$$(\alpha, \beta) \sim (\alpha \circ \alpha_0, \beta_0) = (\alpha_c \circ \rho, \beta_0) \sim (\alpha_c, \mathrm{Sd}_k^*(\rho) \circ \beta_0) = (\alpha_c, \beta_c).$$

□

2.4 – *The small category* $\Delta^{\mathrm{op}} \ltimes \mathbb{N}^\times$

We denote by $\Delta^{\mathrm{op}} \ltimes \mathbb{N}^\times$ the small category semi-direct product of Δ^{op} by the action of $\mathbb{N}^\times$ implemented by the endofunctors Sd_k^*, for $k \in \mathbb{N}^\times$. It has the same objects as Δ^{op} while one adjoins to the collection of morphisms of Δ^{op} the new morphisms $\pi_n^k : \mathrm{Sd}_k^*(n^*) = (k(n+1)-1)^* \to n^*$ such that

$$\pi_n^k \circ \pi_{k(n+1)-1}^\ell = \pi_n^{k\ell} \in \mathrm{Hom}_{\Delta^{\mathrm{op}} \ltimes \mathbb{N}^\times}((k\ell(n+1)-1)^*, n^*) \tag{9}$$

where π_n^k implements the endofunctor Sd_k^*, *i.e.*

$$\alpha \circ \pi_n^k = \pi_m^k \circ \mathrm{Sd}_k^*(\alpha), \quad \forall \alpha \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(n^*, m^*). \tag{10}$$

Using this set-up one checks that any morphism ϕ in $\Delta^{\mathrm{op}} \ltimes \mathbb{N}^\times$ is uniquely of the form $\phi = \pi_n^k \circ \alpha$ with α a morphism in Δ^{op}. Any such ϕ composes as follows

$$(\pi_m^k \circ \beta) \circ (\pi_n^\ell \circ \alpha) = \pi_m^{k\ell} \circ (\mathrm{Sd}_\ell^*(\beta) \circ \alpha) \tag{11}$$

where $\alpha \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}(r^*, (\ell(n+1)-1)^*)$ and

$$\mathrm{Sd}_\ell^*(\beta) \in \mathrm{Hom}_{\Delta^{\mathrm{op}}}((\ell(n+1)-1)^*, (k\ell(m+1)-1)^*)$$

so that $\mathrm{Sd}_\ell^*(\beta) \circ \alpha$ makes sense and belongs to $\mathrm{Hom}_{\Delta^{\mathrm{op}}}(r^*, (k\ell(m+1)-1)^*)$. Using Proposition 2.1 one checks that, if one takes (11) as a definition, the product is associative.

Let $\mathfrak{Fin}_*$ be the category of finite pointed sets and let $\mathcal{F}$ be the functor which associates to an interval I the pointed set $I_* = I/\sim$ with base point the class of $b \sim t$. To any morphism of intervals $f : I \to J$ corresponds the quotient map f_* which preserves the base point. By restricting $\mathcal{F}$ to Δ^{op} one gets a covariant functor $\mathcal{F} : \Delta^{\mathrm{op}} \longrightarrow \mathfrak{Fin}_*$. The following Proposition shows that $\mathcal{F}$ can be extended to $\Delta^{\mathrm{op}} \ltimes \mathbb{N}^\times$.

Proposition 2.8. *For any* $n \geq 0, k \in \mathbb{N}^\times$, *let* $(\pi_n^k)_* : \mathcal{F}(\mathrm{Sd}_k^*(n^*)) \to \mathcal{F}(n^*)$ *be given by the residue modulo* $n+1$. *Then the extension of the functor* $\mathcal{F}$ *on morphisms given by*

$$\phi = \pi_n^k \circ \alpha \mapsto \phi_* := (\pi_n^k)_* \circ \alpha_*$$

determines a functor $\mathcal{F} : \Delta^{\mathrm{op}} \ltimes \mathbb{N}^\times \longrightarrow \mathfrak{Fin}_*$.

Proof. One checks directly that the definition of $(\pi_n^k)_*$ is compatible with the rules (9) and (10) so that the required functoriality follows. □

3. The epicyclic category and the oriented groupoïds

3.1 – *Generalities on groupoïds*

A groupoïd G is a small category where the morphisms are invertible. Given a subset $X \subset G$ of a groupoïd, we set $X^{-1} := \{\gamma^{-1} \mid \gamma \in X\}$. Let $G^{(0)}$ be the set of objects of G and denote by $r, s : G \to G^{(0)}$ the range and the source maps respectively. We view $G^{(0)}$ as the subset of units of G. The following definition is a direct generalization to groupoïds of the notion of right ordered group (*cf.* [7])

Definition 3.1. *An oriented groupoïd (G, G_+) is a groupoïd G endowed with a subcategory $G_+ \subset G$, such that the following relations hold*

$$G_+ \cap G_+^{-1} = G^{(0)}, \qquad G_+ \cup G_+^{-1} = G. \tag{12}$$

Let (G, G_+) be an oriented groupoïd and let $x \in G^{(0)}$. The set $G_x := \{\gamma \in G \mid s(\gamma) = x\}$ is endowed with the total order defined by

$$\gamma \leq \gamma' \iff \gamma' \circ \gamma^{-1} \in G_+. \tag{13}$$

This order is right invariant by construction: *i.e.* for any $\beta \in G$, with $r(\beta) = x$, one has

$$\gamma \leq \gamma' \iff \gamma \circ \beta \leq \gamma' \circ \beta.$$

In the following subsections we describe two constructions of oriented groupoïds associated to a group action.

3.1.1 – $G = X \ltimes H$.

Let H be a group acting on a set X. Then the semi-direct product $G := X \ltimes H$ is a groupoïd with source, range and composition law defined respectively as follows

$$s(x, h) := x, \quad r(x, h) := hx, \quad (x, h) \circ (y, k) := (y, hk).$$

(As in any groupoïd the composition $\gamma \circ \gamma'$ is only defined when $s(\gamma) = r(\gamma')$ which holds here if and only if $x = ky$). One has a canonical homomorphism of groupoïds $\rho : G \to H$, $\rho(x, h) = h$.

Lemma 3.2. *Let (H, H_+) be a right ordered group. Assume that H acts on a set X. Then the semi-direct product $G = X \ltimes H$ with $G_+ := \rho^{-1}(H_+)$ is an oriented groupoïd .*

Proof. By definition, the subset $H_+ \subset H$ of the group H is stable under product and fulfills the equalities: $H_+ \cap H_+^{-1} = \{1\}$, $\quad H_+ \cup H_+^{-1} = H$. This implies (12) using $\rho^{-1}(\{1\}) = G^{(0)}$. □

Let, in particular, $(H, H_+) = (\mathbb{Z}, \mathbb{Z}_+)$ act by translation on the set $X = \mathbb{Z}/(m+1)\mathbb{Z}$ of integers modulo $m+1$. Then one obtains the oriented groupoïd

$$\mathfrak{g}(m) := (\mathbb{Z}/(m+1)\mathbb{Z}) \ltimes \mathbb{Z}. \tag{14}$$

The oriented groupoïds $\mathfrak{g}(m)$ will play a crucial role in this article.

3.1.2 – $G = (X \times X)/H$.

Let H be a group acting freely on a set X. Let $G(X,H) = (X \times X)/H$ be the quotient of $X \times X$ by the diagonal action of H

$$G(X,H) := (X \times X)/\sim \qquad (x,y) \sim (h(x), h(y)),\ \forall h \in H.$$

Let r and s be the two projections of $G(X,H)$ on $G^{(0)} := X/H$ defined by $r(x,y) = x$ and $s(x,y) = y$. Let $\gamma, \gamma' \in G(X,H)$ be such that $s(\gamma) = r(\gamma')$. Then, for $\gamma \sim (x,y)$ and $\gamma' \sim (x', y')$, there exists a unique $h \in H$ satisfying $x' = h(y)$: this because $s(\gamma) = r(\gamma')$ and H acts freely on X. Then, the pair $(h(x), y')$ defines an element of $G(X,H)$ independent of the choice of the pairs representing the elements γ and γ'. We denote by $\gamma \circ \gamma'$ the class of $(h(x), y')$ in $G(X,H)$. This construction defines a groupoïd law on $G(X,H) = (X \times X)/H$.

Lemma 3.3. *Let H be a group acting freely on a set X. Assume that X is totally ordered and that H acts by order automorphisms. Then $G(X,H) = (X \times X)/H$ is an oriented groupoïd with*

$$G_+(X,H) = \{(x,y) \in G(X,H) \mid x \geq y\}. \tag{15}$$

Proof. Since H acts by order automorphisms the condition $x \geq y$ is independent of the choice of a representative (x,y) of a given $\gamma \in G(X,H) = (X \times X)/H$. This condition defines a subcategory $G_+(X,H)$ of $G(X,H)$. The conditions (12) then follow since X is totally ordered. □

Lemma 3.4. *Let $X = \mathbb{Z}$ with the usual total order. Let $m \in \mathbb{N}$ and let the group $\mathbb{Z}$ act on X by $h(x) := x + (m+1)h$, $\forall x \in X, h \in \mathbb{Z}$. Then the oriented groupoïd $G = (X \times X)/\mathbb{Z}$ is canonically isomorphic to the oriented groupoïd $\mathfrak{g}(m)$ of* (14).

Proof. The associated oriented groupoïd (G, G_+) is by construction the quotient of $\mathbb{Z} \times \mathbb{Z}$ by the equivalence relation: $(x,y) \sim (x + \ell(m+1), y + \ell(m+1))$, $\forall \ell \in \mathbb{Z}$. Thus the following map defines a bijective homomorphism of groupoïds

$$\psi : G \to \mathfrak{g}(m) = (\mathbb{Z}/(m+1)\mathbb{Z}) \ltimes \mathbb{Z}, \qquad \psi(x,y) = (\pi(y), x - y)$$

where $\pi : \mathbb{Z} \to \mathbb{Z}/(m+1)\mathbb{Z}$ is the natural projection. One has by restriction $G_+ \xrightarrow[\sim]{\psi} \mathfrak{g}_+(m)$, since $x \geq y \iff x - y \geq 0$, so that ψ is in fact an isomorphism of oriented groupoïds. □

3.2 – *The oriented groupoïd associated to an archimedean set*

In this section we explain how to associate an oriented groupoïd to an archimedean set and describe the special properties of the oriented groupoïds thus obtained. We first recall from [4] the definition of an archimedean set.

DEFINITION 3.5. *An archimedean set is a pair (X, θ) of a non-empty, totally ordered set X and an order automorphism $\theta \in \operatorname{Aut} X$, such that $\theta(x) > x$, $\forall x \in X$. The automorphism θ is also required to fulfill the following archimedean property*

$$\forall x, y \in X,\ \exists n \in \mathbb{N} \text{ s.t. } \quad y \leq \theta^n(x).$$

Let (X, θ) be an archimedean set and let $G(X, \theta)$ be the oriented groupoïd associated by Lemma 3.3 to the action of $\mathbb{Z}$ on X by integral powers of θ. Thus

$$G(X, \theta) := (X \times X)/\sim, \quad (x, y) \sim (\theta^n(x), \theta^n(y)),\ \forall n \in \mathbb{Z}$$

and

$$G_+(X, \theta) := \{(x, y) \in G(X, \theta) \mid x \geq y\}. \tag{16}$$

Next proposition describes the properties of the pair $(G(X, \theta), G_+(X, \theta))$ so obtained.

PROPOSITION 3.6. *The oriented groupoïd $(G, G_+) = (G(X, \theta), G_+(X, \theta))$ fulfills the following conditions*

(1) $\forall x, y \in G^{(0)}$, $\exists \gamma \in G_+$ *s.t.* $s(\gamma) = y$, $r(\gamma) = x$.
(2) *For $x \in X$, the ordered groups $G_x^x := \{\gamma \mid s(\gamma) = r(\gamma) = x\}$ are isomorphic to $(\mathbb{Z}, \leq)$.*
(3) *Let $\gamma \in G$ with $s(\gamma) = y$ and $r(\gamma) = x$. Then the map: $G_y^y \ni \rho \mapsto \gamma \circ \rho \circ \gamma^{-1} \in G_x^x$ is an isomorphism of ordered groups.*

PROOF. Since θ is an order automorphism of X, the group $\mathbb{Z}$ acts by order automorphisms. We check the three conditions (i)-(iii).

(*i*) For $x, y \in X$, there exists $n \in \mathbb{N}$ such that $\theta^n(x) \geq y$. Then $\gamma \sim (\theta^n(x), y)$ belongs to G_+ and $s(\gamma) = y$, $r(\gamma) = x$.

(*ii*) Let $x \in X$. The conditions $s(\gamma) = r(\gamma) = x$ imply that the class of $\gamma \in G(X, \theta)$ admits a unique representative of the form $(\theta^n(x), x)$. One easily checks that the map $G_x^x \to (\mathbb{Z}, \leq)$, $(\theta^n(x), x) \mapsto n$ is an isomorphism of ordered groups.

(*iii*) Let $x, y \in X$ with $\gamma \sim (x, y)$. Then for $\rho \sim (\theta^n(y), y) \in G_y^y$ one gets $\gamma \circ \rho \circ \gamma^{-1} \sim (\theta^n(x), x)$, thus the unique isomorphism with $(\mathbb{Z}, \leq)$ is preserved. □

Let (G, G_+) be an oriented groupoïd fulfilling the three conditions of Proposition 3.6. Let $x \in G^{(0)}$, consider the set $G_x = \{\gamma \in G \mid s(\gamma) = x\}$ with the total order defined by (13) and with the action of $\mathbb{Z}$ given, for $\gamma_x \in G_+$ the positive generator of G_x^x, by

$$\theta(\gamma) := \gamma \circ \gamma_x. \tag{17}$$

When one applies this construction to the case $(G, G_+) = (G(X,\theta), G(X,\theta)_+)$, for (X,θ) an archimedean set, with $x \in G^{(0)} = X/\theta$ one obtains, after choosing a lift $\tilde{x} \in X$ of x, an isomorphism

$$j_{\tilde{x}} : X \xrightarrow{\sim} G_x, \qquad j_{\tilde{x}}(z) = (z, x). \tag{18}$$

The following proposition shows that the two constructions $(X,\theta) \mapsto G(X,\theta)$ and $(G, G_+) \mapsto (G_x, \theta)$ are reciprocal.

Proposition 3.7. *Let (G, G_+) be an oriented groupoïd fulfilling the conditions of Proposition* 3.6 *and let $x \in G^{(0)}$. Consider the set $G_x := \{\gamma \in G \mid s(\gamma) = x\} = X$ endowed with the total order* (13) *and the action of $\mathbb{Z}$ on it given by* (17). *Then (X,θ) is an archimedean set and one has an isomorphism of groupoïds*

$$(G(X,\theta), G(X,\theta)_+) \cong (G, G_+).$$

Proof. The implication $\gamma \le \gamma' \Longrightarrow \theta(\gamma) \le \theta(\gamma')$ follows since right multiplication preserves the order. Moreover, the condition (iii) of Proposition 3.6 implies that $\theta(\gamma) > \gamma, \forall \gamma \in X = G_x$. Next we show that the archimedean property holds on (X,θ). Let $\gamma \le \gamma' \in G_x$, with $y = r(\gamma)$ and $y' = r(\gamma')$. By applying the condition (i) of Proposition 3.6, we choose $\delta \in G_+$ such that $s(\delta) = y'$ and $r(\delta) = y$. Then $\gamma'' = \delta \circ \gamma'$ fulfills $s(\gamma'') = s(\gamma)$ and $r(\gamma'') = r(\gamma)$ and thus there exists $n \in \mathbb{Z}$ such that $\gamma'' = \gamma \circ \gamma_x^n$. Moreover, one has $\gamma'' = \delta \circ \gamma' \ge \gamma'$. It follows that (X,θ) is an archimedean set. If one replaces $x \in G^{(0)}$ by $y \in G^{(0)}$, then the condition (i) implies that there exists $\alpha \in G_+$ with $s(\alpha) = y$, $r(\alpha) = x$. Then the map $G_x \to G_y$, $\gamma \mapsto \gamma \circ \alpha$ is an order isomorphism which satisfies

$$(\gamma \circ \gamma_x) \circ \alpha = (\gamma \circ \alpha) \circ (\alpha^{-1} \circ \gamma_x \circ \alpha).$$

Since condition (iii) implies $\alpha^{-1} \circ \gamma_x \circ \alpha = \gamma_y$, one obtains an isomorphism of the corresponding archimedean sets.

Finally, we compare the pair (G, G_+) with $(G(X,\theta), G_+(X,\theta))$. We define a map $f : G(X,\theta) \to G$ as follows: given a pair (γ, γ') of elements of $X = G_x$, one sets $f(\gamma, \gamma') := \gamma \circ \gamma'^{-1}$. One has

$$f(\theta(\gamma), \theta(\gamma')) = f(\gamma \circ \gamma_x, \gamma' \circ \gamma_x) = \gamma \circ \gamma'^{-1} = f(\gamma, \gamma').$$

To show that f is a groupoïd homomorphism it is enough to check that $f(\gamma, \gamma') \circ f(\gamma', \gamma'') = f(\gamma, \gamma'')$ and this can be easily verified. Next we prove that f is bijective. Let $\alpha \in G$. By applying condition (i) of Proposition 3.6, there exists $\gamma \in G$ such that $r(\gamma) = r(\alpha)$ and $s(\gamma) = x$. Let then $\gamma' = \alpha^{-1}\gamma$. Since $s(\gamma') = x$, both γ, γ' belong to $X = G_x$ and moreover $f(\gamma, \gamma') = \alpha$ showing that f is surjective. Let γ_j, γ_j' be elements of $X = G_x$ such that $f(\gamma_1, \gamma_1') = f(\gamma_2, \gamma_2')$. One then has $\gamma_1 \circ \gamma_1'^{-1} = \gamma_2 \circ \gamma_2'^{-1}$ and hence $\gamma_2^{-1} \circ \gamma_1 = \gamma_2'^{-1} \circ \gamma_2 = \gamma_x^n$ for some $n \in \mathbb{Z}$. It follows