

## Relative fundamental groups and rational points

CHRISTOPHER LAZDA (\*)

ABSTRACT - In this paper we define a relative rigid fundamental group, which associates to a section of a smooth and proper morphism  $f : X \rightarrow S$  in characteristic  $p$ , a Hopf algebra in the ind-category of overconvergent  $F$ -isocrystals on  $S$ . We prove a base change property, which says that the fibres of this object are the Hopf algebras of the rigid fundamental groups of the fibres of  $f$ . We explain how to use this theory to define period maps as Kim does for varieties over number fields, and show in certain cases that the targets of these maps can be interpreted as varieties.

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(\*) Indirizzo dell'A.: Department of Mathematics, Room 6.13 Huxley Building, 180 Queen's Gate, Imperial College, London, SW7 2AZ UK.

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E-mail: c.lazda10@imperial.ac.uk

## Introduction

Let  $K$  be a number field and let  $C/K$  be a smooth, projective curve of genus  $g > 1$ , with Jacobian  $J$ . Then a famous theorem of Faltings states that the set  $C(K)$  of  $K$ -rational points on  $C$  is finite. The group  $J(K)$  is finitely generated, and under the assumption that its rank is strictly less than  $g$ , Chabauty in [14] was able to prove this theorem using elementary methods as follows. Let  $v$  be a place of  $K$ , of good reduction for  $C$ , and denote by  $C_v, J_v$  the base change to  $K_v$ . Then Chabauty defines a homomorphism

$$(1) \quad \log : J(K_v) \rightarrow H^0\left(J_v, \Omega_{J_v/K_v}^1\right)$$

and shows that there exists a non zero linear functional on  $H^0(J_v, \Omega_{J_v/K_v}^1)$  which vanishes on the image of  $J(K)$ . He then proves that pulling this back to  $J(K_v)$  gives an analytic function on  $J(K_v)$ , which is not identically zero on  $C(K_v)$ , and which vanishes on  $J(K)$ . Hence  $C(K) \subset C(K_v) \cap J(K)$  must be finite as it is contained in the zero set of a non-zero analytic function on  $C(K_v)$ .

In [29], Kim describes what he calls a ‘non-abelian lift’ of this method. Fix a point  $p \in C(K)$ . By considering the Tannakian category of integrable connections on  $C_v$ , one can define a ‘de Rham fundamental group’  $U^{\text{dR}} = \pi_1^{\text{dR}}(C_v, p)$ , which is a pro-unipotent group scheme over  $K_v$ , as well as, for any other  $x \in C(K_v)$ , path torsors  $P^{\text{dR}}(x) = \pi_1^{\text{dR}}(C_v, x, p)$  which are right torsors under  $U^{\text{dR}}$ . These group schemes and torsors come with extra structure, namely that of a Hodge filtration and, by comparison with the crystalline fundamental group of the reduction of  $C_v$ , a Frobenius action. He then shows that such torsors are classified by  $U^{\text{dR}}/F^0$ , and hence one can define ‘period maps’

$$(2) \quad j_n : C(K_v) \rightarrow U_n^{\text{dR}}/F^0$$

where  $U_n^{\text{dR}}$  is the  $n$ th level nilpotent quotient of  $U^{\text{dR}}$ . If  $n = 2$  then  $j_n$  is just the composition of the above log map with the inclusion  $C(K_v) \rightarrow J(K_v)$ . By analysing the image of this map, he is able to prove finiteness of  $C(K)$  under certain conditions, namely if the dimension of  $U_n^{\text{dR}}/F^0$  is greater than the dimension of the target of a global period map defined using the category of lisse étale sheaves on  $C$ . Moreover, when  $n = 2$ , this condition on dimensions is essentially Chabauty’s condition that  $\text{rank}_{\mathbb{Z}} J(K) < \text{genus}(C)$  (modulo the Tate-Shafarevich conjecture).

Our interest lies in trying to develop a function field analogue of these ideas. The analogy between function fields in one variable over finite fields and number fields has been a fruitful one throughout modern number theory, and indeed the analogue of Mordell’s conjecture was first proven for function fields by Grauert. In this paper we discuss the problem of defining a good analogue of the global period map. This is defined in [29] using the Tannakian category of lisse  $\mathbb{Q}_p$  sheaves on  $X$ , and this approach will not work in the function field setting. Neither  $p$ -adic nor  $\ell$ -adic étale cohomology will give satisfactory answers, the first because, for example,

the resulting fundamental group will be moduli dependent, i.e. will not be locally constant in families (see for example [34]), and the second because the  $\ell$ -adic topology on the resulting target spaces for period maps will not be compatible with the  $p$ -adic topology on the source varieties. Instead we will work with the category of overconvergent  $F$ -isocrystals.

Let  $K$  be a finite extension of  $\mathbb{F}_p(t)$ , and let  $k$  be the field of constants of  $K$ , i.e. the algebraic closure of  $\mathbb{F}_p$  inside  $K$ . Let  $\bar{S}$  be the unique smooth projective, geometrically irreducible curve over  $k$  whose function field is  $K$ . If  $C/K$  is a smooth, projective, geometrically integral curve then one can choose a regular model for  $C$ . This is a regular, proper surface  $\bar{X}/k$ , equipped with a flat, proper morphism  $f : \bar{X} \rightarrow \bar{S}$  whose generic fibre is  $C/K$ . Let  $S \subset \bar{S}$  be the smooth locus of  $f$ , and denote by  $f$  also the pullback  $f : X \rightarrow S$ . The idea is to construct, for any section  $p$  of  $f$ , a ‘non-abelian isocrystal’ on  $S$  whose fibre at any closed point  $s$  ‘is’ the rigid fundamental group  $\pi_1^{\text{rig}}(X_s, p_s)$ . The idea behind how to construct such an object is very simple.

Suppose that  $f : X \rightarrow S$  is a Serre fibration of topological spaces, with connected base and fibres. If  $p$  is a section, then for any  $s \in S$  the homomorphism  $\pi_1(X, p(s)) \rightarrow \pi_1(S, s)$  is surjective, and  $\pi_1(S, s)$  acts on the kernel via conjugation. This corresponds to a locally constant sheaf of groups on  $S$ , and the fibre over any point  $s \in S$  is just the fundamental group of the fibre  $X_s$ . This approach makes sense for any fundamental group defined algebraically as the Tannaka dual of a category of ‘locally constant’ coefficients. So if  $f : X \rightarrow S$  is a morphism of smooth varieties with section  $p$ , then  $f_* : \pi_1^{\mathcal{C}_X}(X, x) \rightarrow \pi_1^{\mathcal{C}_S}(S, s)$  is surjective, and  $\pi_1^{\mathcal{C}_S}(S, s)$  acts on the kernel. Here  $\mathcal{C}_{(-)}$  is any appropriate category of coefficients, for example vector bundles with integrable connection, unipotent isocrystals etc., and e.g.  $\pi_1^{\mathcal{C}_X}(X, x)$  is the Tannaka dual of this category with respect to the fibre functor  $x^*$ . This gives the kernel of  $f_*$  the structure of an ‘affine group scheme over  $\mathcal{C}_S$ ’, and it makes sense to ask what the fibre is over any closed point  $s \in S$ . The main theorem of the first section is the following.

**THEOREM.** *Suppose that  $f : X \rightarrow S$  is a smooth morphism of smooth varieties over an algebraically closed field  $k$  of characteristic zero. Assume that both  $S$  and the fibres of  $f$  are connected, and that  $X$  is the complement of a relative normal crossings divisor in a smooth and proper  $S$ -scheme  $\bar{X}$ . Let  $\mathcal{C}_S$  be the category of vector bundles with a regular integrable connection on  $S$ , and let  $\mathcal{C}_X$  be the category of vector bundles with a regular integrable connection on  $X$  which are iterated extensions of those of the form  $f^* \mathcal{E}$ , with  $\mathcal{E} \in \mathcal{C}_S$ . Then the fibre of the corresponding affine group scheme over  $\mathcal{C}_S$  at  $s \in S$  is the de Rham fundamental group  $\pi_1^{\text{dR}}(X_s, p_s)$  of the fibre.*

In Section 2 we discuss path torsors in the relative setting. We show in particular that for any other section  $q$  of  $f$  one can define an affine scheme  $\pi_1^{\text{dR}}(X/S, q, p)$  over  $\mathcal{C}_S$  which is a right torsor under the relative de Rham fundamental group

$\pi_1^{\text{dR}}(X/S, p)$ . The upshot of this is that we obtain

$$(3) \quad j_n : X(S) \rightarrow H^1(S, \pi_1^{\text{dR}}(X/S, p)_n)$$

which are a coarse characteristic zero function field analogue of Kim's global period maps. Of course, if we were really interested in the characteristic zero picture, we would want to define Hodge structures on these objects, and thus obtain finer period maps. However, our main interest lies in the positive characteristic case, and so we don't pursue these questions.

In Section 3 we define the relative rigid fundamental group in positive characteristic, mimicking the definition in characteristic zero. Instead of the category of vector bundles with regular integrable connections, we consider the category of overconvergent  $F$ -isocrystals (throughout Section 3 we will be over a finite field, and Frobenius will always mean the *linear* Frobenius). We then proceed to use Caro's theory of cohomological operations for arithmetic  $\mathcal{D}$ -modules in order to prove the analogue of the above theorem in positive characteristic.

The upshot of this is that for a smooth and proper map  $f : X \rightarrow S$  with geometrically connected fibres and smooth, geometrically connected base over a finite field  $k$ , and a section  $p$  of  $f$ , we can define an affine group scheme  $\pi_1^{\text{rig}}(X/S, p)$  over the category of overconvergent  $F$ -isocrystals on  $S$ , which we call the relative fundamental group at  $p$ . The fibre of this over any point  $s \in S$  is just the unipotent rigid fundamental group of the fibre  $X_s$  of  $f$  over  $s$ . As in the zero characteristic case, the general Tannakian formalism gives us path torsors  $\pi_1^{\text{rig}}(X/S, p, q)$  for any other  $q \in X(S)$ , and hence we can define a period map

$$(4) \quad X(S) \rightarrow H_{F, \text{rig}}^1(S, \pi_1^{\text{rig}}(X/S, p))$$

where the RHS is a classifying set of  $F$ -torsors under  $\pi_1^{\text{rig}}(X/S, p)$ , as well as finite level versions given by pushing out along the quotient map  $\pi_1^{\text{rig}}(X/S, p) \rightarrow \pi_1^{\text{rig}}(X/S, p)_n$ .

Finally, we study the targets of these period maps, and show that after replacing  $H_{F, \text{rig}}^1(S, \pi_1^{\text{rig}}(X/S, p))$ , the set classifying  $F$ -torsors, by the Frobenius invariant part of the set classifying torsors without  $F$ -structure,  $H_{\text{rig}}^1(S, \pi_1^{\text{rig}}(X/S, p))^{\phi=\text{id}}$ , then under very restrictive hypotheses on the morphism  $f : X \rightarrow S$ , we obtain the structure of an algebraic variety. The argument here is just a translation of the original argument of Kim into our context, and what for us are restrictive hypotheses are automatically satisfied in his case.

We are still a long way away from getting a version of Kim's methods to work for function fields. There is still the question of how to define the analogue of the local period maps, and also to show that the domains of the period maps have the structure of varieties. Even then, it is very unclear what the correct analogue of the local integration theory will be in positive characteristic. There is still a very large amount of work to be done if such a project is to be completed.

## 1. Relative de Rham fundamental groups

Let  $f : X \rightarrow S$  be a smooth morphism of smooth complex varieties, and suppose that  $f$  admits a good compactification, that is, there exists  $\overline{X}$  smooth and proper over  $S$ , an open immersion  $X \hookrightarrow \overline{X}$  over  $S$ , such that  $D = \overline{X} \setminus X$  is a relative normal crossings divisor in  $\overline{X}$ . Let  $p \in X(S)$  be a section. For every closed point  $s \in S$  with fibre  $X_s$ , one can consider the topological fundamental group  $G_s := \pi_1(X_s^{\text{an}}, p(s))$ , and as  $s$  varies, these fit together to give a locally constant sheaf  $\pi_1(X/S, p)$  on  $S^{\text{an}}$ . Let

$$(5) \quad \hat{U}(\text{Lie } G_s) := \varprojlim \mathbb{C}[G_s]/\alpha^n$$

denote the completed enveloping algebra of the Malcev Lie algebra of  $G_s$ , where  $\alpha \subset \mathbb{C}[G_s]$  is the augmentation ideal. According to Proposition 4.2 of [25], as  $s$  varies, these fit together to give a pro-local system on  $S^{\text{an}}$ , i.e. a pro-object  $\hat{U}_p^{\text{top}}$  in the category of locally constant sheaves of finite dimensional  $\mathbb{C}$ -vector spaces on  $S^{\text{an}}$ . (Their theorem is a lot stronger than this, but this is all we need for now). According to Théorème 5.9 in Chapter II of [18], the pro-vector bundle with integrable connection  $\hat{U}_p^{\text{top}} \otimes_{\mathbb{C}} \mathcal{O}_{S^{\text{an}}}$  has a canonical algebraic structure. Thus given a smooth morphism  $f : X \rightarrow S$  as above, with section  $p$ , one can construct a pro-vector bundle with connection  $\hat{U}_p$  on  $S$ , whose fibre at any closed point  $s \in S$  is the completed enveloping algebra of the Malcev Lie algebra of  $\pi_1(X_s^{\text{an}}, p(s))$ .

Denoting by  $\mathfrak{g}_s$  the Malcev Lie algebra of  $\pi_1(X_s^{\text{an}}, p(s))$ ,  $\hat{U}(\mathfrak{g}_s) = (\hat{U}_p)_s$  can be constructed algebraically, as  $\mathfrak{g}_s$  is equal to  $\text{Lie } \pi_1^{\text{dR}}(X_s, p_s)$ , the Lie algebra of the Tannaka dual of the category of unipotent vector bundles with integrable connection on  $X_s$ . This suggests the question of whether or not there is an algebraic construction of  $\hat{U}_p$ ?

We will not directly answer this question - instead we will construct the Lie algebra associated to  $\hat{U}_p$  - this is a pro-system  $\hat{\mathcal{L}}_p$  of Lie algebras with connection on  $S$ . The way we will do so is very simple, and is closely related to ideas used in [35] to study relatively unipotent mixed motivic sheaves.

**DEFINITION 1.1.** To save ourselves saying the same thing over and over again, we make the following definition. A ‘good’ morphism is a smooth morphism  $f : X \rightarrow S$  of smooth varieties over a field  $k$ , with geometrically connected fibres and base, such that  $X$  is the complement of a relative normal crossings divisor in a smooth, proper  $S$ -scheme  $\overline{X}$ . Throughout this section we will assume that the ground field  $k$  is algebraically closed of characteristic 0.

We will assume that the reader is familiar with Tannakian categories, a good introductory reference is [31]. If  $\mathcal{T}$  is a Tannakian category over a field  $k$ , and  $\omega$  is a fibre functor on  $\mathcal{T}$ , in the sense of §1.9 of [20], we will denote the group scheme representing tensor automorphisms of  $\omega$  by  $G(\mathcal{T}, \omega)$ . We will also use the rudiments of algebraic geometry in Tannakian categories, as explained in §5 of [19] - in particular we will talk about affine (group) schemes over Tannakian categories. We

will denote the fundamental group of a Tannakian category by  $\pi(\mathcal{T})$ , this is an affine group scheme over  $\mathcal{T}$  which satisfies  $\omega(\pi(\mathcal{T})) = G(\mathcal{T}, \omega)$  for every fibre functor  $\omega$  (see for example 6.1 of [19]). If  $\mathcal{T}$  is a Tannakian category over  $k$ , and  $k'/k$  is a finite extension, then we will denote the category of  $k'$ -modules in  $\mathcal{T}$  by either  $\mathcal{T} \otimes_k k'$ , or  $\mathcal{T}_{k'}$ .

We will also assume familiarity with the theory of integrable connections and regular holonomic  $\mathcal{D}$ -modules on  $k$ -varieties, and will generally refer to [18] and [23] for details. We say that a regular integrable connection on  $X$  is unipotent if it is a successive extension of the trivial connection, and these form a Tannakian subcategory  $\mathcal{NIC}(X) \subset \text{IC}(X)$  of the Tannakian category of regular integrable connections.

DEFINITION 1.2. For  $X/k$  smooth and connected, the algebraic and de Rham fundamental groups of  $X$  at a closed point  $x \in X$  are defined by

$$(6) \quad \pi_1^{\text{alg}}(X, x) := x^*(\pi(\text{IC}(X))) = G(\text{IC}(X), x^*)$$

$$(7) \quad \pi_1^{\text{dR}}(X, x) := x^*(\pi(\mathcal{NIC}(X))) = G(\mathcal{NIC}(X), x^*).$$

REMARK 1.3. It follows from the Riemann-Hilbert correspondence that if  $k = \mathbb{C}$ , then these affine group schemes are the pro-algebraic and pro-unipotent completions of  $\pi_1(X^{\text{an}}, x)$  respectively.

If  $f : X \rightarrow Y$  is a morphism of smooth  $k$ -varieties, then we can form the pullback of vector bundles with integrable connection on  $Y$ , which preserves regularity and is the usual pull-back on the underlying  $\mathcal{O}_Y$ -module. This induces a homomorphism  $f_* : \pi_1^{\#}(X, x) \rightarrow \pi_1^{\#}(Y, f(x))$  for  $\# = \text{dR}, \text{alg}$ .

### 1.1 – The relative fundamental group and its pro-nilpotent Lie algebra

Let  $f : X \rightarrow S$  be a ‘good’ morphism. A regular integrable connection  $E$  on  $X$  is said to be relatively unipotent if there exists a filtration by horizontal sub-bundles, whose graded objects are all in the essential image of  $f^* : \text{IC}(S) \rightarrow \text{IC}(X)$ . We will denote the full subcategory of relatively unipotent objects in  $\text{IC}(X)$  by  $\mathcal{N}_f\text{IC}(X)$ , which is a Tannakian subcategory. Suppose that  $p \in X(S)$  is a section of  $f$ . We have functors of Tannakian categories

$$(8) \quad \mathcal{N}_f\text{IC}(X) \begin{array}{c} \xrightarrow{p^*} \\ \xleftarrow{f_*} \end{array} \text{IC}(S)$$

and hence, after choosing a point  $s \in S(k)$ , homomorphisms

$$(9) \quad G(\mathcal{N}_f\text{IC}(X), p(s)^*) \begin{array}{c} \xrightarrow{f_*} \\ \xleftarrow{p_*} \end{array} G(\text{IC}(S), s^*)$$

between their Tannaka duals. Let  $K_s$  denote the kernel of  $f_*$ . Then the splitting  $p_*$  induces an action of  $\pi_1^{\text{alg}}(S, s) = G(\text{IC}(S), s^*)$  on  $K_s$  via conjugation. This corresponds to an affine group scheme over  $\text{IC}(S)$ .

LEMMA 1.4. *This affine group scheme over  $\text{IC}(S)$  is independent of  $s$ .*

PROOF. Thanks to [19], §6.4,  $f_*$ ,  $p_*$  above come from homomorphisms

$$(10) \quad p^*(\pi(\mathcal{N}_f\text{IC}(X))) \begin{array}{c} \xrightarrow{f_*} \\ \xleftarrow{p_*} \end{array} \pi(\text{IC}(S))$$

of affine group schemes over  $\text{IC}(S)$ . If we let  $\mathcal{K}$  denote the kernel of  $f_*$ , then  $K_s = s^*(\mathcal{K})$ , and this induces the action of  $\pi_1^{\text{alg}}(S, s)$  on  $K_s$ .  $\square$

DEFINITION 1.5. The relative de Rham fundamental group  $\pi_1^{\text{dR}}(X/S, p)$  of  $X/S$  at  $p$  is defined to be the affine group scheme  $\mathcal{K}$  over  $\text{IC}(S)$ .

Let  $i_s : X_s \rightarrow X$  denote the inclusion of the fibre over  $s$ . Then there is a canonical functor  $i_s^* : \mathcal{N}_f\text{IC}(X) \rightarrow \mathcal{N}\text{IC}(X_s)$ . This induces a homomorphism  $\pi_1^{\text{dR}}(X_s, p_s) \rightarrow G(\mathcal{N}_f\text{IC}(X), p_s^*)$  which is easily seen to factor through the fibre  $\pi_1^{\text{dR}}(X/S, p)_s := s^*(\mathcal{K}) = K_s$  of  $\pi_1^{\text{dR}}(X/S, p)$  over  $s$ .

THEOREM 1.6. *Suppose that  $k = \mathbb{C}$ . Then  $\phi : \pi_1^{\text{dR}}(X_s, p_s) \rightarrow \pi_1^{\text{dR}}(X/S, p)_s$  is an isomorphism.*

PROOF. The point  $s$  gives us fibre functors  $p_s^*$  on  $\mathcal{N}\text{IC}(X_s)$ ,  $p(s)^*$  on  $\mathcal{N}_f\text{IC}(X)$  and  $s^*$  on  $\text{IC}(S)$ . Write

$$(11) \quad \mathcal{K} = G(\mathcal{N}\text{IC}(X_s), p_s^*), \quad \mathcal{G} = G(\mathcal{N}_f\text{IC}(X), p(s)^*), \quad \mathcal{H} = G(\text{IC}(S), s^*)$$

and also let

$$(12) \quad K = \pi_1(X_s^{\text{an}}, p(s)), \quad G = \pi_1(X^{\text{an}}, p(s)), \quad H = \pi_1(S^{\text{an}}, s)$$

be the topological fundamental groups of  $X_s, X, S$  respectively. Then  $\mathcal{K} = K^{\text{un}}$ , the pro-unipotent completion of  $K$ , and  $\mathcal{H} = H^{\text{alg}}$ , the pro-algebraic completion of  $H$ . We need to show that the sequence of affine group schemes

$$(13) \quad 1 \rightarrow \mathcal{K} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 1$$

is exact, and we will use the equivalences of categories

$$(14) \quad \text{IC}(X) \simeq \text{Rep}_{\mathbb{C}}(\pi_1(X^{\text{an}}, p(s))), \quad \text{IC}(S) \simeq \text{Rep}_{\mathbb{C}}(\pi_1(S^{\text{an}}, s))$$

$$(15) \quad \text{IC}(X_s) \simeq \text{Rep}_{\mathbb{C}}(\pi_1(X_s^{\text{an}}, p(s))).$$

By Proposition 1.3 in Chapter I of [35],  $\ker(\mathcal{G} \rightarrow \mathcal{H})$  is pro-unipotent. Hence according to Proposition 1.4 of *loc. cit.*, in order to show that  $\phi$  is an isomorphism, we must show the following.

- If  $E \in \mathcal{N}_f\text{IC}(X)$  is such that  $i_s^*(E)$  is trivial, then  $E \cong f^*(F)$  for some  $F$  in  $\text{IC}(S)$ .
- Let  $E \in \mathcal{N}_f\text{IC}(X)$ , and let  $F_0 \subset i_s^*(E)$  denote the largest trivial sub-object. Then there exists  $E_0 \subset E$  such that  $F_0 = i_s^*(E_0)$ .
- There is a pro-action of  $\mathcal{G}$  on  $\hat{\mathcal{U}}(\text{Lie } \mathcal{K})$  such that the corresponding action of  $\text{Lie } \mathcal{G}$  extends the left multiplication by  $\text{Lie } \mathcal{K}$ .

The first is straightforward. Since  $f$  is topologically a fibration with section  $p$ , we have a split exact sequence

$$(16) \quad 1 \rightarrow K \rightarrow G \hookrightarrow H \rightarrow 1$$

and a representation  $V$  of  $G$  such that  $K$  acts trivially. We must show that  $V$  is the pullback of an  $H$ -representation - this is obvious! The second is no harder, we must show that if  $V$  is a  $G$ -representation, then  $V^K$  is a sub- $G$ -module of  $V$ . But since  $K$  is normal in  $G$ , this is clear. For the third, note that  $\hat{\mathcal{U}}(\text{Lie } \mathcal{K}) = \hat{\mathcal{U}}(\text{Lie } K) = \varprojlim \mathbb{C}[K]/\alpha^n$ , where  $\alpha$  is the augmentation ideal of  $\mathbb{C}[K]$ . Let  $H$  act on  $\mathbb{C}[K]/\alpha^n$  by conjugation and  $K$  by left multiplication. I claim that  $\mathbb{C}[K]/\alpha^n$  is finite dimensional, and unipotent as a  $K$ -representation.

Indeed, there are extensions of  $K$ -representations

$$(17) \quad 0 \rightarrow \alpha^n/\alpha^{n+1} \rightarrow \mathbb{C}[K]/\alpha^{n+1} \rightarrow \mathbb{C}[K]/\alpha^n \rightarrow 0$$

and hence, since the action of  $K$  on  $\alpha^n/\alpha^{n+1}$  is trivial, it follows by induction that each  $\mathbb{C}[K]/\alpha^n$  is unipotent. There are also surjections

$$(18) \quad (\alpha/\alpha^2)^{\otimes n} \twoheadrightarrow \alpha^n/\alpha^{n+1}$$

for each  $n$ , and hence by induction, to show finite dimensionality it suffices to show that  $\alpha/\alpha^2$  is finite dimensional. But  $\alpha/\alpha^2 \cong K^{\text{ab}} \otimes_{\mathbb{Z}} \mathbb{C}$  is finite dimensional, as  $K$  is finitely generated.

Now, since  $\mathbb{C}[K]/\alpha^n$  is unipotent as a  $K$ -representation, it is relatively unipotent as a  $G = K \times H$ -representation, hence  $\mathbb{C}[K]/\alpha^n$  is naturally an object in  $\text{Rep}_{\mathbb{C}}(\mathcal{G})$ . Thus there is a pro-action of  $\mathcal{G}$  on  $\hat{\mathcal{U}}(\text{Lie } \mathcal{K})$ , and the action extends left multiplication by  $\text{Lie } \mathcal{K}$  as required.  $\square$

**REMARK 1.7.** The co-ordinate algebra of  $\pi_1^{\text{dR}}(X/S, p)$  is an ind-object in the category of regular integrable connections on  $S$ . Hence we may view  $\pi_1^{\text{dR}}(X/S, p)$  as an affine group scheme over  $S$  in the usual sense, together with a regular integrable connection on the associated  $\mathcal{O}_S$ -Hopf algebra.

If  $g : T \rightarrow S$  is any morphism of smooth varieties over  $k$ , then there is a homomorphism of fundamental groups

$$(19) \quad \pi_1^{\text{dR}}(X_T/T, p_T) \rightarrow \pi_1^{\text{dR}}(X/S, p) \times_S T := g^*(\pi_1^{\text{dR}}(X/S, p))$$



which corresponds to a horizontal morphism

$$(20) \quad \mathcal{O}_{\pi_1^{\mathrm{dR}}(X/S,p)} \otimes_{\mathcal{O}_S} \mathcal{O}_T \rightarrow \mathcal{O}_{\pi_1^{\mathrm{dR}}(X_T/T,p_T)}.$$

PROPOSITION 1.8. *If  $k = \mathbb{C}$  then this is an isomorphism.*

PROOF. We know by the previous theorem that this induces an isomorphism on fibres over any point  $t \in T(\mathbb{C})$ . Hence by rigidity, it is an isomorphism.  $\square$

Write  $G = \pi_1^{\mathrm{dR}}(X/S, p)$  and let  $G_n$  denote the quotient of  $G$  by the  $n$ th term in its lower central series. Let  $A_n$  denote the Hopf algebra of  $G_n$ , and  $I_n \subset A_n$  the augmentation ideal.  $L_n := \mathcal{H}om_{\mathcal{O}_S}(I_n/I_n^2, \mathcal{O}_S)$  is the Lie algebra of  $G_n$ . This is a coherent, nilpotent Lie algebra with connection, i.e the bracket  $[\cdot, \cdot] : L_n \otimes L_n \rightarrow L_n$  is horizontal. There are natural morphisms  $L_{n+1} \rightarrow L_n$ , which form a pro-system of nilpotent Lie algebras with connection  $\hat{L}_p$ , whose universal enveloping algebra is the object  $\hat{U}_p$  considered in the introduction to this section.

## 1.2 – An algebraic proof of Theorem 1.6

Although we have a candidate for the relative fundamental group of a ‘good’ morphism  $f : X \rightarrow S$  at a section  $p$ , so far we have only proved it is a good candidate when the ground field is the complex numbers. One might hope to be able to reduce to the case  $k = \mathbb{C}$  via base change and finiteness arguments, but this approach will not work in a straightforward manner. Also, such an argument will not easily adapt to the case of positive characteristic, as in general one will not be able to lift a smooth proper family, even locally on the base. In this section, we will give a more algebraic proof, that will more easily adapt to positive characteristic. Recall that we have an affine group scheme  $\pi_1^{\mathrm{dR}}(X/S, p)$  over  $\mathrm{IC}(S)$ , and a comparison morphism

$$(21) \quad \phi : \pi_1^{\mathrm{dR}}(X_s, p_s) \rightarrow \pi_1^{\mathrm{dR}}(X/S, p)_s$$

for any point  $s \in S$ . We will prove that this map is an isomorphism.

It follows from Proposition 1.4 in Chapter I of [35] and Appendix A of [22] that we need to prove the following:

- (Injectivity) Every  $E \in \mathcal{N}\mathrm{IC}(X_s)$  is a quotient of  $i_s^*(F)$  for some  $F \in \mathcal{N}_f\mathrm{IC}(X)$ .
- (Surjectivity I) Suppose that  $E \in \mathcal{N}_f\mathrm{IC}(X)$  is such that  $i_s^*(E)$  is trivial. Then there exists  $F \in \mathrm{IC}(S)$  such that  $E \cong f^*(F)$ .
- (Surjectivity II) Let  $E \in \mathcal{N}_f\mathrm{IC}(X)$ , and let  $F_0 \subset i_s^*(E)$  denote the largest trivial sub-object. Then there exists  $E_0 \subset E$  such that  $F_0 = i_s^*(E_0)$ .

To do so, we will need to use the language of algebraic  $\mathcal{D}$ -modules. We define the functor

$$f_*^{\mathrm{dR}} : \mathcal{N}_f\mathrm{IC}(X) \rightarrow \mathrm{IC}(S)$$

by  $f_*^{\mathrm{dR}}(E) = \mathcal{H}^{-d}(f_+E)$  where  $f_+$  is the usual push-forward for regular holonomic complexes of  $\mathcal{D}$ -modules,  $d$  is the relative dimension of  $f : X \rightarrow S$ , and we are considering a regular integrable connection on  $X$  as a  $\mathcal{D}_X$ -module in the usual way.

LEMMA 1.9. *The functor  $f_*^{\mathrm{dR}}$  lands in the category of regular integrable connections, and is a right adjoint to  $f^*$ .*

PROOF. The content of the first claim is in the coherence of direct images in de Rham cohomology, using the comparison result 1.4 of [21], and the fact that a regular holonomic  $\mathcal{D}_X$ -module is a vector bundle iff it is coherent as an  $\mathcal{O}_X$ -module.

To see this coherence, we first use adjointness of  $f_+$  and  $f^+$ , together with the facts that  $f^+\mathcal{O}_S = \mathcal{O}_X[-d]$  and  $f_+\mathcal{O}_X$  is concentrated in degrees  $\geq -d$ , to get canonical adjunction morphism  $f_*^{\mathrm{dR}}(\mathcal{O}_X) \rightarrow \mathcal{O}_S$  of regular holonomic  $\mathcal{D}_X$ -modules. I claim that this is an isomorphism. Indeed, by 1.4 of [21] we have  $f_*^{\mathrm{dR}}(\mathcal{O}_X) \cong \mathbf{R}^0 f_* (\mathcal{O}_{X/S}^*)$ , and since  $f$  admits a good compactification  $\bar{f} : \bar{X} \rightarrow S$  the latter is isomorphic to  $\mathbf{R}^0 \bar{f}_* (\mathcal{O}_{\bar{X}/S}^*(\log D))$  where  $D = \bar{X} \setminus X$ . Since  $\bar{f}$  is proper with connected fibres, we know that  $\bar{f}_*(\mathcal{O}_{\bar{X}}) \cong \mathcal{O}_S$ , and since  $\mathbf{R}^0 \bar{f}_* (\mathcal{O}_{\bar{X}/S}^*(\log D))$  is a sub- $\mathcal{O}_S$ -module of  $\bar{f}_*(\mathcal{O}_{\bar{X}})$ , the claim follows.

Hence in particular  $f_*^{\mathrm{dR}}(\mathcal{O}_X)$  is coherent, and via the projection formula, so is  $f_*^{\mathrm{dR}}(f^*F)$  for any  $F \in \mathrm{IC}(S)$ . Hence using exact sequences in cohomology and induction on unipotency degree,  $f_*^{\mathrm{dR}}E$  is coherent whenever  $E$  is relatively unipotent.

To prove the second claim, we just use that  $f^+$  is adjoint to  $f_+$ ,  $f^+ = f^*[-d]$  on the subcategory of regular integrable connections, and  $f_+E$  is concentrated in degrees  $\geq -d$  whenever  $E$  is a regular integrable connection.  $\square$

REMARK 1.10. Although the proposition is stated in [21] for  $k = \mathbb{C}$ , the same proof works for any algebraically closed field of characteristic zero.

Thus we get a canonical morphism  $\varepsilon_E : f^* f_*^{\mathrm{dR}} E \rightarrow E$  which is the counit of the adjunction between  $f^*$  and  $f_*^{\mathrm{dR}}$ .

EXAMPLE 1.11. Suppose that  $S = \mathrm{Spec}(k)$ . Then

$$(22) \quad f_*^{\mathrm{dR}} E = H_{\mathrm{dR}}^0(X, E) = \mathrm{Hom}_{\mathcal{N}\mathrm{IC}(X)}(\mathcal{O}_X, E)$$

and the adjunction becomes the identification

$$(23) \quad \mathrm{Hom}_{\mathcal{N}\mathrm{IC}(X)}(V \otimes_k \mathcal{O}_X, E) = \mathrm{Hom}_{\mathrm{Vec}_k}(V, \mathrm{Hom}_{\mathcal{N}\mathrm{IC}(X)}(\mathcal{O}_X, E)).$$

Since  $f_*^{\mathrm{dR}}$  takes objects in  $\mathcal{N}_f \mathrm{IC}(X)$  to objects in  $\mathrm{IC}(S)$ , it commutes with base change and there is an isomorphism of functors

$$(24) \quad H_{\mathrm{dR}}^0(X_s, -) \circ j_s^* \cong s^* \circ f_*^{\mathrm{dR}} : \mathcal{N}_f \mathrm{IC}(X) \rightarrow \mathrm{Vec}_k.$$

To see this, first note that by using the five lemma, the projection formula and

induction on the relative unipotence degree, it suffices to prove that  $f_*^{\mathrm{dR}}(\mathcal{O}_X)$  and  $\mathbf{R}_{\mathrm{dR}}^1 f_* (\mathcal{O}_X)$  commute with base change, where we have written  $\mathbf{R}_{\mathrm{dR}}^1 f_* (\mathcal{O}_X) = \mathcal{H}^{-d+1}(f_+ \mathcal{O}_X)$ . Again, using 1.4 of [21] this boils down to proving that  $\mathbf{R}^i f_* (\Omega_{X/S}^*)$  commute with base change for  $i = 0, 1$ . Since we know that they are coherent (and hence locally free), and that  $f : X \rightarrow S$  admits a good compactification  $\bar{f} : \bar{X} \rightarrow S$ , this follows from Corollary 8.6 of [27] together with the standard isomorphism  $\mathbf{R}^i f_* (\Omega_{X/S}^*) \cong \mathbf{R}^i \bar{f}_* (\Omega_{\bar{X}/S}^*(\log D))$  where  $D = \bar{X} \setminus X$ .

**PROPOSITION 1.12.** *Suppose that  $i_s^* E$  is trivial. Then the counit  $\varepsilon_E : f^* f_*^{\mathrm{dR}} E \rightarrow E$  is an isomorphism.*

**PROOF.** Pulling back  $\varepsilon_E$  by  $i_s^*$ , and using base change, we get a morphism

$$(25) \quad \mathcal{O}_{X_s} \otimes_k H_{\mathrm{dR}}^0(X_s, i_s^* E) \rightarrow i_s^* E$$

which by the explicit description of 1.11 is seen to be an isomorphism (as  $i_s^* E$  is trivial). Hence by rigidity,  $\varepsilon_E$  must be an isomorphism.  $\square$

**PROPOSITION 1.13.** *Let  $E \in \mathcal{N}_f \mathrm{IC}(X)$ , and let  $F_0 \subset i_s^*(E)$  denote the largest trivial sub-object. Then there exists  $E_0 \subset E$  such that  $F_0 = i_s^*(E_0)$ .*

**PROOF.** Let  $F = i_s^*(E)$ , then we have  $H_{\mathrm{dR}}^0(X_s, F) = \mathrm{Hom}_{\mathrm{IC}(X_s)}(\mathcal{O}_{X_s}, F)$ . I first claim that the natural map  $\mathcal{O}_{X_s} \otimes_k H_{\mathrm{dR}}^0(X_s, F) \rightarrow F$  is injective and identifies  $F_0$  with  $\mathcal{O}_{X_s} \otimes_k H_{\mathrm{dR}}^0(X_s, F)$ . But since the category  $\mathcal{N} \mathrm{IC}(X_s)$  is Tannakian, this is equivalent to claiming that for an affine group scheme  $G$  over  $k$ , and  $V$  a representation of  $G$ , the natural map of  $G$  representations  $H^0(G, V) \rightarrow V$  is injective, and identifies  $H^0(G, V)$  with the largest trivial sub- $G$ -module of  $V$ . This is obvious.

To complete the proof, set  $E_0 = f^* f_*^{\mathrm{dR}}(E)$ , then by the base change results we know that  $i_s^*(E_0) \cong F_0$ , and that the natural map  $E_0 \rightarrow E$  restricts to the inclusion  $F_0 \rightarrow F$  on the fibre  $X_s$ .

**COROLLARY 1.14.** *The map  $\pi_1^{\mathrm{dR}}(X_s, p_s) \rightarrow \pi_1^{\mathrm{dR}}(X/S, p)_s$  is a surjection.*

We now turn to the proof of injectivity of the comparison map, borrowing heavily from ideas used in Section 2.1 of [24]. We define objects  $U_n$  of  $\mathcal{N} \mathrm{IC}(X_s)$ , the category of unipotent integrable connections on  $X_s$  inductively as follows.  $U_1$  will just be  $\mathcal{O}_{X_s}$ , and  $U_{n+1}$  will be the extension of  $U_n$  by  $\mathcal{O}_{X_s} \otimes_k H_{\mathrm{dR}}^1(X_s, U_n^\vee)^\vee$  corresponding to the identity under the isomorphisms

$$(26) \quad \begin{aligned} \mathrm{Ext}_{\mathrm{IC}(X_s)}(U_n, \mathcal{O}_{X_s} \otimes_k H_{\mathrm{dR}}^1(X_s, U_n^\vee)^\vee) &\cong H_{\mathrm{dR}}^1(X_s, U_n^\vee \otimes_k H_{\mathrm{dR}}^1(X_s, U_n^\vee)^\vee) \\ &\cong H_{\mathrm{dR}}^1(X_s, U_n^\vee) \otimes_k H_{\mathrm{dR}}^1(X_s, U_n^\vee)^\vee \\ &\cong \mathrm{End}_k(H_{\mathrm{dR}}^1(X_s, U_n^\vee)). \end{aligned}$$

If we look at the long exact sequence in de Rham cohomology associated to the short exact sequence  $0 \rightarrow U_n^\vee \rightarrow U_{n+1}^\vee \rightarrow H_{\text{dR}}^1(X_s, U_n^\vee) \otimes_k \mathcal{O}_{X_s} \rightarrow 0$  we get

$$(27) \quad 0 \rightarrow H_{\text{dR}}^0(X_s, U_n^\vee) \rightarrow H_{\text{dR}}^0(X_s, U_{n+1}^\vee) \rightarrow H_{\text{dR}}^1(X_s, U_n^\vee) \\ \xrightarrow{\delta} H_{\text{dR}}^1(X_s, U_n^\vee) \rightarrow H_{\text{dR}}^1(X_s, U_{n+1}^\vee).$$

LEMMA 1.15. *The connecting homomorphism  $\delta$  is the identity.*

PROOF. By dualising, the extension

$$(28) \quad 0 \rightarrow U_n^\vee \rightarrow U_{n+1}^\vee \rightarrow \mathcal{O}_{X_s} \otimes_k H_{\text{dR}}^1(X_s, U_n^\vee) \rightarrow 0$$

corresponds to the identity under the isomorphism

$$(29) \quad \text{Ext}_{\text{IC}(X_s)}(\mathcal{O}_{X_s} \otimes_k H_{\text{dR}}^1(X_s, U_n^\vee), U_n^\vee) \cong \text{End}_k(H_{\text{dR}}^1(X_s, U_n^\vee)).$$

Now the lemma follows from the fact that for an extension  $0 \rightarrow E \rightarrow F \rightarrow \mathcal{O}_{X_s} \otimes_k V \rightarrow 0$  of a trivial bundle by  $E$ , the class of the extension under the isomorphism

$$(30) \quad \text{Ext}_{\text{IC}(X_s)}(\mathcal{O}_{X_s} \otimes_k V, E) \cong V^\vee \otimes H_{\text{dR}}^1(X_s, E) \cong \text{Hom}_k(V, H_{\text{dR}}^1(X_s, E))$$

is just the connecting homomorphism for the long exact sequence

$$(31) \quad 0 \rightarrow H_{\text{dR}}^0(X_s, E) \rightarrow H_{\text{dR}}^0(X_s, F) \rightarrow V \rightarrow H_{\text{dR}}^1(X_s, E).$$

□

In particular, any extension of  $U_n$  by a trivial bundle  $V \otimes_k \mathcal{O}_{X_s}$  is split after pulling back to  $U_{n+1}$ , and  $H_{\text{dR}}^0(X_s, U_{n+1}^\vee) \cong H_{\text{dR}}^0(X_s, U_n^\vee)$ . It then follows by induction that  $H_{\text{dR}}^0(X_s, U_n^\vee) \cong H_{\text{dR}}^0(X_s, \mathcal{O}_{X_s}) \cong k$  for all  $n$ .

DEFINITION 1.16. We define the unipotent class of an object  $E \in \mathcal{N}\text{IC}(X_s)$  inductively as follows. If  $E$  is trivial, then we say  $E$  has unipotent class 1. If there exists an extension

$$(32) \quad 0 \rightarrow V \otimes_k \mathcal{O}_{X_s} \rightarrow E \rightarrow E' \rightarrow 0$$

with  $E'$  of unipotent class  $\leq m - 1$ , then we say that  $E$  has unipotent class  $\leq m$ .

Now let  $x = p(s)$ ,  $u_1 = 1 \in (U_1)_x \cong k$ , and choose a compatible system of elements  $u_n \in (U_n)_x$  mapping to  $u_1$ .

PROPOSITION 1.17. *Let  $F \in \mathcal{N}\text{IC}(X_s)$  be an object of unipotent class  $\leq m$ . Then for all  $n \geq m$  and any  $f \in F_x$  there exists a morphism  $\alpha : U_n \rightarrow F$  such that  $\alpha_x(u_n) = f$ .*

PROOF. We copy the proof of Proposition 2.1.6 of [24] and use induction on  $m$ . The case  $m = 1$  is straightforward. For the inductive step, let  $F$  be of unipotent

class  $m$ , and choose an exact sequence

$$(33) \quad 0 \rightarrow E \xrightarrow{\psi} E \xrightarrow{\phi} G \rightarrow 0$$

with  $E$  trivial and  $G$  of unipotent class  $< m$ . By induction there exists a morphism  $\beta : U_{n-1} \rightarrow G$  such that  $\phi_x(f) = \beta_x(u_{n-1})$ . Pulling back the extension (33) first by the morphism  $\beta$  and then by the natural surjection  $U_n \rightarrow U_{n-1}$  gives an extension of  $U_n$  by  $E$ , which must split, as observed above.

$$(34) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & F'' & \xrightarrow{\quad} & U_n & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & F' & \longrightarrow & U_{n-1} & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \xrightarrow{\psi} & F & \xrightarrow{\phi} & G & \longrightarrow & 0 \end{array}$$

Let  $\gamma : U_n \rightarrow F$  denote the induced morphism, then  $\phi_x(\gamma_x(u_n) - f) = 0$ . Hence there exists some  $e \in E_x$  such that  $\psi_x(e) = \gamma_x(u_n) - f$ . Again by induction we can choose  $\gamma' : U_n \rightarrow E$  with  $\gamma'_x(u_n) = e$ . Finally let  $\alpha = \gamma - \psi \circ \gamma'$ , it is easily seen that  $\alpha_x(u_n) = f$ .  $\square$

**COROLLARY 1.18.** *Every  $E$  in  $\mathcal{NIC}(X_s)$  is a quotient of  $U_m^{\oplus N}$  for some  $m, N \in \mathbb{N}$ .*

**PROOF.** Suppose that  $E$  is of unipotent class  $\leq m$ . Let  $e_1, \dots, e_N$  be a basis for  $E_x$ . Then there is a morphism  $\alpha : U_m^{\oplus N} \rightarrow E$  with every  $e_i$  in the image of the induced map on fibres. Thus  $\alpha_x$  is surjective, and hence so is  $\alpha$ .  $\square$

We now try to inductively define relatively nilpotent integrable connections  $W_n$  on  $X$  which restrict to the  $U_n$  on fibres. Define higher direct images in de Rham cohomology by  $\mathbf{R}_{\mathrm{dR}}^i f_*(E) = \mathcal{H}^{i-d}(f_+ E)$ , and begin the induction with  $W_1 = \mathcal{O}_X$ . As part of the induction we will assume that  $\mathbf{R}_{\mathrm{dR}}^0 f_*(W_n^\vee) \cong \mathbf{R}_{\mathrm{dR}}^0 f_*(\mathcal{O}_X) = \mathcal{O}_S$ , that  $\mathbf{R}_{\mathrm{dR}}^1 f_*(W^\vee)$  and  $\mathbf{R}_{\mathrm{dR}}^1 f_*(W)$  are both coherent, i.e. regular integrable connections, and that there exists a morphism  $p^* W_n^\vee \rightarrow \mathcal{O}_S$  such that the composite  $\mathcal{O}_S \cong f_* W_n^\vee \cong p^* f^* f_* W_n^\vee \rightarrow p^* W_n^\vee \rightarrow \mathcal{O}_S$  is an isomorphism. We will define  $W_{n+1}$  to be an extension of  $W_n$  by the sheaf  $f^* \mathbf{R}_{\mathrm{dR}}^1 f_*(W_n^\vee)^\vee$ , and thus consider the extension group

$$(35) \quad \mathrm{Ext}_{\mathrm{IC}(X)} \left( W_n, f^* \mathbf{R}_{\mathrm{dR}}^1 f_*(W_n^\vee)^\vee \right) \cong H_{\mathrm{dR}}^1 \left( X, W_n^\vee \otimes_{\mathcal{O}_X} f^* \mathbf{R}_{\mathrm{dR}}^1 f_*(W_n^\vee)^\vee \right).$$

The Leray spectral sequence, together with the induction hypothesis and the projection formula, gives us the 5-term exact sequence

$$\begin{aligned}
(36) \quad 0 &\rightarrow H_{\mathrm{dR}}^1\left(S, \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) \rightarrow \mathrm{Ext}_{\mathrm{IC}(X)}\left(W_n, f^* \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) \\
&\rightarrow \mathrm{End}_{\mathrm{IC}(S)}\left(\mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) \rightarrow H_{\mathrm{dR}}^2\left(S, \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) \\
&\rightarrow H_{\mathrm{dR}}^2(X, W_n^\vee \otimes_{\mathcal{O}_X} (\mathbf{R}_{\mathrm{dR}}^1 f_* W_n^\vee)^\vee).
\end{aligned}$$

Now, the projection  $p^* W_n^\vee \rightarrow \mathcal{O}_S$  induces a map

$$(37) \quad H_{\mathrm{dR}}^i\left(S, p^* W_n^\vee \otimes_{\mathcal{O}_X} (\mathbf{R}_{\mathrm{dR}}^1 f_* W_n^\vee)^\vee\right) \rightarrow H_{\mathrm{dR}}^i\left(S, \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right)$$

such that the composite (dotted) arrow

$$\begin{array}{ccc}
H_{\mathrm{dR}}^i\left(S, \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) & \longrightarrow & H_{\mathrm{dR}}^i\left(X, W_n^\vee \otimes_{\mathcal{O}_X} (\mathbf{R}_{\mathrm{dR}}^1 f_* W_n^\vee)^\vee\right) \\
\downarrow & & \downarrow \\
H_{\mathrm{dR}}^i\left(S, \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) & \longleftarrow & H_{\mathrm{dR}}^i\left(S, p^* W_n^\vee \otimes_{\mathcal{O}_X} (\mathbf{R}_{\mathrm{dR}}^1 f_* W_n^\vee)^\vee\right)
\end{array}
\quad (38)$$

is an isomorphism, since it can be identified with the map induced by the composite arrow  $\mathcal{O}_S \cong f_* W_n^\vee \cong p^* f^* f_* W_n^\vee \rightarrow p^* W_n^\vee \rightarrow \mathcal{O}_S$ . Hence both the maps

$$\begin{aligned}
(39) \quad H_{\mathrm{dR}}^1\left(S, \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) &\rightarrow \mathrm{Ext}_{\mathrm{IC}(X)}\left(W_n, f^* \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) \\
H_{\mathrm{dR}}^2\left(S, \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) &\rightarrow H_{\mathrm{dR}}^2(X, W_n^\vee \otimes_{\mathcal{O}_X} (\mathbf{R}_{\mathrm{dR}}^1 f_* W_n^\vee)^\vee)
\end{aligned}$$

appearing in the 5-term exact sequence split. So there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}_{\mathrm{IC}(X)}\left(W_n, f^* \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) & \longrightarrow & \mathrm{Ext}_{\mathrm{IC}(X_s)}\left(U_n, \mathcal{O}_{X_s} \otimes_k H_{\mathrm{dR}}^1(X_s, U_n^\vee)^\vee\right) \\
\downarrow & & \parallel \\
\mathrm{End}_{\mathrm{IC}(S)}\left(\mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) & \longrightarrow & \mathrm{End}_k\left(H_{\mathrm{dR}}^1(X_s, U_n^\vee)^\vee\right)
\end{array}
\quad (40)$$

where the horizontal arrows are just restrictions to fibres, and the left hand vertical arrow is surjective. The identity morphism in  $\mathrm{End}_k(H_{\mathrm{dR}}^1(X_s, U_n^\vee)^\vee)$  clearly lifts to  $\mathrm{End}_{\mathrm{IC}(S)}(\mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee)$ , and there exists a unique element of the extension group  $\mathrm{Ext}_{\mathrm{IC}(X)}(W_n, f^* \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee)$  lifting the identity in  $\mathrm{End}_{\mathrm{IC}(S)}(\mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee)$ , and which maps to zero under the above splitting

$$(41) \quad \mathrm{Ext}_{\mathrm{IC}(X)}\left(W_n, f^* \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right) \rightarrow H_{\mathrm{dR}}^1\left(S, \mathbf{R}_{\mathrm{dR}}^1 f_* (W_n^\vee)^\vee\right).$$

Let  $W_{n+1}$  be the corresponding extension.

**PROPOSITION 1.19.** *Every object of  $\mathcal{N}\mathrm{IC}(X_s)$  is a quotient of  $i_s^* E$  for some  $E \in \mathcal{N}_f \mathrm{IC}(X)$ .*

PROOF. To finish the induction step, we must show that

$$(42) \quad \mathbf{R}_{\mathrm{dR}}^0 f_*(W_{n+1}^\vee) \cong \mathbf{R}_{\mathrm{dR}}^0 f_*(W_n^\vee),$$

that  $\mathbf{R}_{\mathrm{dR}}^1 f_*(W_{n+1}^\vee)$  and  $\mathbf{R}_{\mathrm{dR}}^1 f_*(W_n^\vee)$  are coherent, and that there exists a morphism  $p^*W_{n+1}^\vee \rightarrow \mathcal{O}_S$  as in the induction hypothesis. For the first claim, if we look at the long exact sequence of relative de Rham cohomology

$$(43) \quad 0 \rightarrow \mathbf{R}_{\mathrm{dR}}^0 f_*(W_n^\vee) \rightarrow \mathbf{R}_{\mathrm{dR}}^0 f_*(W_{n+1}^\vee) \rightarrow \dots$$

we simply note that the given map restricts to an isomorphism on the fibre over  $s$ , and is hence an isomorphism. For the second, we simply use the long exact sequence in cohomology and the inductive hypothesis for  $\mathbf{R}_{\mathrm{dR}}^1 f_*(W_n)$  and  $\mathbf{R}_{\mathrm{dR}}^1 f_*(W_n^\vee)$ . For the third, note that it follows from the construction of  $W_{n+1}$  that the exact sequence

$$(44) \quad 0 \rightarrow p^*W_n^\vee \rightarrow p^*W_{n+1}^\vee \rightarrow (\mathbf{R}_{\mathrm{dR}}^1 f_* W_n^\vee)^\vee \rightarrow 0$$

splits when we push out via the map  $p^*W_n^\vee \rightarrow \mathcal{O}_S$ . This splitting induces a map  $p^*W_{n+1}^\vee \rightarrow \mathcal{O}_S$  such that the diagram

$$(45) \quad \begin{array}{ccc} p^*W_n^\vee & \longrightarrow & p^*W_{n+1}^\vee \\ & \searrow & \downarrow \\ & & \mathcal{O}_S \end{array}$$

commutes. Now the fact that the diagram

$$(46) \quad \begin{array}{ccccccc} & & f_*W_{n+1}^\vee & \longrightarrow & p^*f^*f_*W_{n+1}^\vee & \longrightarrow & p^*W_{n+1}^\vee \\ & \nearrow & \uparrow & & \uparrow & & \uparrow \\ \mathcal{O}_S & \longrightarrow & f_*W_n^\vee & \longrightarrow & p^*f^*f_*W_n^\vee & \longrightarrow & p^*W_n^\vee \longrightarrow \mathcal{O}_S \end{array}$$

commutes implies that the composite along the top row is an isomorphism, finishing the proof.  $\square$

**COROLLARY 1.20.** *Let  $f : X \rightarrow S$  be a ‘good’ morphism over an algebraically closed field of characteristic zero, and  $p$  a section of  $f$ . Then the natural ‘base change’ map  $\pi_1^{\mathrm{dR}}(X_s, p_s) \rightarrow \pi_1^{\mathrm{dR}}(X/S, p)_s$  is an isomorphism.*

**REMARK 1.21.** It is possible to define a relative fundamental group when  $k$  is not necessarily algebraically closed (but still of characteristic 0) using identical methods. One can then show that the corresponding ‘base change’ question can be deduced from what we have proved in the algebraically closed case. Since this argument is rather fiddly, and not necessary in the context of this paper, we have omitted it.

## 2. Path torsors, non-abelian crystals and period maps

If  $\mathcal{T}$  is a Tannakian category over an arbitrary field  $k$ , and  $\omega_i$  are fibre functors on  $\mathcal{T}$ ,  $i = 1, 2$ , with values in the category of quasi-coherent sheaves on some  $k$ -scheme  $S$ , then the functor of isomorphisms  $\omega_1 \rightarrow \omega_2$  is representable by an affine  $S$ -scheme, which is a  $(G(\mathcal{T}, \omega_1), G(\mathcal{T}, \omega_2))$ -bitorsor. This allows us to define path torsors under the algebraic and de Rham fundamental groups. In this section, we show how to do this in the relative case.

### 2.1 – Torsors in Tannakian categories

Let  $\mathcal{C}$  be a Tannakian category over a field  $k$ . A Tannakian  $\mathcal{C}$ -category is a Tannakian category  $\mathcal{D}$  together with an exact,  $k$ -linear tensor functor  $t : \mathcal{C} \rightarrow \mathcal{D}$ . We say it is neutral over  $\mathcal{C}$  if there exists an exact, faithful  $k$ -linear tensor functor  $\omega : \mathcal{D} \rightarrow \mathcal{C}$  such that  $\omega \circ t \cong \text{id}$ . Such functors will be called fibre functors. If such a functor  $\omega$  is fixed, we say  $\mathcal{D}$  is neutralised. Thanks to §6.4 of [19], we have a homomorphism

$$(47) \quad t^* : \pi(\mathcal{D}) \rightarrow t(\pi(\mathcal{C}))$$

of affine group schemes over  $\mathcal{D}$ . Hence applying  $\omega$  gives us a homomorphism

$$(48) \quad \omega(t^*) : \omega(\pi(\mathcal{D})) \rightarrow \pi(\mathcal{C})$$

of affine group schemes over  $\mathcal{C}$ . We define  $G(\mathcal{D}, \omega) := \ker \omega(t^*)$ .

For an affine group scheme  $G$  over  $\mathcal{C}$ , let  $\mathcal{O}_G$  be its Hopf algebra, a representation of  $G$  is then defined to be an  $\mathcal{O}_G$ -comodule. That is an object  $V \in \mathcal{C}$  together with a map  $\delta : V \rightarrow \mathcal{O}_G \otimes V$  satisfying the usual axioms.

**DEFINITION 2.1.** A torsor under  $G$  is a non-empty affine scheme  $\text{Sp}(\mathcal{O}_P)$  over  $\mathcal{C}$ , together with a  $\mathcal{O}_G$ -comodule structure on  $\mathcal{O}_P$ , such that the induced map  $\mathcal{O}_P \otimes \mathcal{O}_P \rightarrow \mathcal{O}_P \otimes \mathcal{O}_G$  is an isomorphism.

**EXAMPLE 2.2.** Suppose that  $\mathcal{C} = \text{Rep}_k(H)$ , for some affine group scheme  $H$  over  $k$ . Then an affine group scheme  $G$  over  $\mathcal{C}$  ‘is’ just an affine group scheme  $G_0$  over  $k$  together with an action of  $H$ . A representation of  $G$  ‘is’ then just an  $H$ -equivariant representation of  $G_0$ , or in other words, a representation of the semi-direct product  $G_0 \rtimes H$ .

Representations have another interpretation. Suppose that  $V$  is an  $\mathcal{O}_G$ -comodule, and let  $R$  be a  $\mathcal{C}$ -algebra. A point  $g \in G(R)$  is then a morphism  $\mathcal{O}_G \rightarrow R$  of  $\mathcal{C}$ -algebras, and hence for any such  $g$  we get a morphism

$$(49) \quad V \rightarrow V \otimes R$$



which extends linearly to a morphism

$$(50) \quad V \otimes R \rightarrow V \otimes R.$$

This is an isomorphism, with inverse given by the map induced by  $g^{-1}$ . Hence we get an  $R$ -linear action of  $G(R)$  on  $V \otimes R$ , for all  $\mathcal{C}$ -algebras  $R$ . The same proof as in the absolute case (Proposition 2.2 of [31]) shows that a representation of  $G$  (defined in terms of comodules) is equivalent to an  $R$ -linear action of  $G(R)$  on  $V \otimes R$ , for all  $R$ .

For  $G$  an affine group scheme over  $\mathcal{C}$ , let  $\text{Rep}_{\mathcal{C}}(G)$  denote its category of representations, which is a Tannakian category over  $k$ . There are canonical functors

$$(51) \quad \mathcal{C} \begin{array}{c} \xrightarrow{t} \\ \xleftarrow{\omega} \end{array} \text{Rep}_{\mathcal{C}}(G)$$

given by ‘trivial representation’ and ‘forget the representation’. This makes  $\text{Rep}_{\mathcal{C}}(G)$  neutral over  $\mathcal{C}$ . There is a natural homomorphism  $G \rightarrow \omega(\pi(\text{Rep}_{\mathcal{C}}(G)))$  which comes from the fact that by definition,  $G$  acts on  $\omega(V)$  for all  $V \in \text{Rep}_{\mathcal{C}}(G)$ . Since this action is trivial on everything of the form  $t(W)$ ,  $W \in \mathcal{C}$ , again by definition, this homomorphism factors to give a homomorphism

$$(52) \quad G \rightarrow G(\text{Rep}_{\mathcal{C}}(G), \omega).$$

Conversely, if  $\mathcal{D}$  is neutral over  $\mathcal{C}$ , with fibre functor  $\omega$ , then the action of  $\omega(\pi(\mathcal{D}))$  on  $\omega(V)$ , for all  $V \in \mathcal{D}$ , induces an action of  $G(\mathcal{D}, \omega)$  on  $\omega(V)$ , and hence a functor

$$(53) \quad \mathcal{D} \rightarrow \text{Rep}_{\mathcal{C}}(G(\mathcal{D}, \omega)).$$

PROPOSITION 2.3. *In the above situation, the homomorphism*

$$(54) \quad G \rightarrow G(\text{Rep}_{\mathcal{C}}(G), \omega)$$

*is an isomorphism, and the functor*

$$(55) \quad \mathcal{D} \rightarrow \text{Rep}_{\mathcal{C}}(G(\mathcal{D}, \omega))$$

*is an equivalence of categories.*

PROOF. Suppose first that  $\mathcal{C} \cong \text{Rep}_k(H)$  is neutral. In the first case, we can identify  $G$  with an affine group scheme  $G_0$  over  $k$  together with an action of  $H$ , and the category  $\text{Rep}_{\mathcal{C}}(G)$  with the category of representations of the semi-direct product  $G_0 \rtimes H$ . The functor  $\omega : \text{Rep}_{\mathcal{C}}(G) \rightarrow \mathcal{C}$  can be identified with the forgetful functor from  $G_0 \rtimes H$ -representations to  $H$ -representations, and the morphism

$$(56) \quad \omega(\pi(\text{Rep}_{\mathcal{C}}(G))) \rightarrow \pi(\mathcal{C})$$

with the natural map

$$(57) \quad G_0 \rtimes H \rightarrow H$$

of affine group schemes with  $H$ -action. Thus the kernel of this map is identified with

$G_0$  together with its given  $H$ -action. In other words,

$$(58) \quad G \rightarrow G(\mathrm{Rep}_{\mathcal{C}}(G), \omega)$$

is an isomorphism.

In the second case,  $\mathcal{D}$  is also neutral, and corresponds to representations of some affine group scheme  $G$ . The functors  $t, \omega$  give a surjection  $G \rightarrow H$  and a splitting  $H \rightarrow G$  which induces an action of  $H$  on  $G_0 := \ker(G \rightarrow H)$  such that  $G \cong G_0 \rtimes H$ . Then  $G(\mathcal{D}, \omega)$  is identified with  $G_0$  together with its  $H$ -action, and  $\mathcal{D} \rightarrow \mathrm{Rep}_{\mathcal{C}}(G(\mathcal{D}, \omega))$  with the natural functor from  $G = G_0 \rtimes H$ -representations to  $H$ -equivariant  $G_0$ -representations. It is thus an equivalence.

If  $\mathcal{C}$  is not neutral, then we can choose a fibre functor with values in some  $k$ -scheme  $S$ , apply Théorème 1.12 of [20] and replace the affine group scheme  $H$  by a certain groupoid acting on a  $S$  (for more details see §3.3). The argument is then formally identical.  $\square$

REMARK 2.4. Our definition of the fundamental group  $\pi_1^{\mathrm{dR}}(X/S, p)$  is then just  $G(\mathcal{N}_f \mathrm{IC}(X), p^*)$ , as an affine group scheme over  $\mathrm{IC}(S)$ .

In order to define torsors of isomorphisms in the relative setting, we must first recall Deligne's construction in the absolute case, which uses the notion of a coend. So suppose that we have categories  $\mathcal{X}$  and  $\mathcal{S}$ , and a functor  $F : \mathcal{X} \times \mathcal{X}^{\mathrm{op}} \rightarrow \mathcal{S}$ . The coend of  $F$  is the universal pair  $(\zeta, s)$  where  $s$  is an object of  $\mathcal{S}$  and  $\zeta : F \rightarrow s$  is a bi-natural transformation. Here  $s$  is the constant functor at  $s \in \mathrm{Ob}(\mathcal{S})$ , and by bi-natural we mean that it is natural in both variables. If such an object exists, we will denote it by

$$(59) \quad \int^{\mathcal{X}} F(x, x).$$

If  $\mathcal{S}$  is cocomplete then the coend always exists and is given concretely by the formula (see Chapter IX, Section 6 of [30])

$$(60) \quad \int^{\mathcal{X}} F(x, x) = \mathrm{colim} \left( \coprod_{f: x \rightarrow y \in \mathrm{Mor}(\mathcal{X})} F(x, y) \rightrightarrows \coprod_{x \in \mathrm{Ob}(\mathcal{X})} F(x, x) \right).$$

Suppose that  $\mathcal{C}$  is a Tannakian category, and let  $\omega_1, \omega_2 : \mathcal{C} \rightarrow \mathrm{QCoh}(S)$  be two fibre functors on  $\mathcal{C}$ . In [20], Deligne defines

$$(61) \quad L_S(\omega_1, \omega_2) = \int^{\mathcal{C}} \omega_1(V) \otimes \omega_2(V)^{\vee}$$

to be the coend of the bifunctor

$$(62) \quad \omega_1 \otimes \omega_2^{\vee} : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \rightarrow \mathrm{QCoh}(S),$$

and in §6 of *loc. cit.*, uses the tensor structure of  $\mathcal{C}$  to define a multiplication on  $L_S(\omega_1, \omega_2)$  which makes it into a quasi-coherent  $\mathcal{O}_S$ -algebra. He then proves that  $\text{Spec}(L_S(\omega_1, \omega_2))$  represents the functor of isomorphisms from  $\omega_1$  to  $\omega_2$ .

Now let  $\mathcal{C}$  be a Tannakian category, let  $\mathcal{D}$  be neutral over  $\mathcal{C}$ , and suppose that  $\omega_1, \omega_2 : \mathcal{D} \rightarrow \mathcal{C}$  are two fibre functors from  $\mathcal{D}$  to  $\mathcal{C}$ . Define the coend

$$(63) \quad L_{\mathcal{C}}(\omega_1, \omega_2) := \int^{\mathcal{D}} \omega_1(V) \otimes \omega_2(V)^{\vee} \in \text{Ind}(\mathcal{C}).$$

If  $\eta : \mathcal{C} \rightarrow \text{QCoh}(S)$  is a fibre functor, then  $\eta$  commutes with colimits, and hence  $\eta(L_{\mathcal{C}}(\omega_1, \omega_2)) = L_S(\eta\omega_1, \eta\omega_2)$ : this is a quasi-coherent  $\mathcal{O}_S$ -algebra, functorial in  $\eta$ . Since algebraic structures in Tannakian categories, such as commutative algebras, Hopf algebras, and so on, can be constructed ‘functorially in fibre functors’, (see for example §5.11 of [19]), it follows that there is a unique way of defining a  $\mathcal{C}$ -algebra structure on  $L_{\mathcal{C}}(\omega_1, \omega_2)$  lifting the  $\mathcal{O}_S$ -algebra structure on each  $\eta(L_{\mathcal{C}}(\omega_1, \omega_2))$ . Moreover, since  $\eta(\text{Sp}(L_{\mathcal{C}}(\omega_1, \omega_2)))$  is a  $(\eta\omega_1(\pi(\mathcal{D})), \eta\omega_2(\pi(\mathcal{D})))$ -bitorsor, functorially in  $\eta$ , the affine scheme

$$(64) \quad P_{\mathcal{C}}(\omega_1, \omega_2) := \text{Sp}(L_{\mathcal{C}}(\omega_1, \omega_2))$$

is a  $(\omega_1(\pi(\mathcal{D})), \omega_2(\pi(\mathcal{D})))$ -bitorsor over  $\mathcal{C}$ .

What we actually want, however, is a  $(G_{\mathcal{C}}(\mathcal{D}, \omega_2), G_{\mathcal{C}}(\mathcal{D}, \omega_2))$ -bitorsor. We get this as follows. Suppose that  $V \in \mathcal{D}$ , then by the definition of  $L_{\mathcal{C}}(\omega_1, \omega_2)$  we get a morphism

$$(65) \quad \omega_1(V) \otimes \omega_2(V)^{\vee} \rightarrow L_{\mathcal{C}}(\omega_1, \omega_2)$$

which corresponds to a morphism

$$(66) \quad \omega_1(V) \rightarrow \omega_2(V) \otimes L_{\mathcal{C}}(\omega_1, \omega_2).$$

Thus a morphism  $L_{\mathcal{C}}(\omega_1, \omega_2) \rightarrow R$  for some  $\mathcal{C}$ -algebra  $R$  induces an  $R$ -linear morphism

$$(67) \quad \omega_1(V) \otimes R \rightarrow \omega_2(V) \otimes R$$

which is in fact an isomorphism, since it is so after applying any fibre functor.

**DEFINITION 2.5.** Define  $P_{\text{triv}}(\omega_1, \omega_2)$  to be the sub-functor of  $P_{\mathcal{C}}(\omega_1, \omega_2)$  which takes  $R$  to the set of all morphisms  $L_{\mathcal{C}}(\omega_1, \omega_2) \rightarrow R$  such that for every  $V$  in the essential image of  $t : \mathcal{C} \rightarrow \mathcal{D}$ , the induced automorphism of  $R \otimes \omega_1(V) = R \otimes \omega_2(V)$  is the identity.

**PROPOSITION 2.6.** *The functor  $P_{\text{triv}}(\omega_1, \omega_2)$  is representable by an affine scheme over  $\mathcal{C}$ , and is a  $(G_{\mathcal{C}}(\mathcal{D}, \omega_1), G_{\mathcal{C}}(\mathcal{D}, \omega_2))$ -bitorsor in the category of affine schemes over  $\mathcal{C}$ .*

PROOF. First note that if  $V \in \text{Ob}(\mathcal{D})$ , then  $\omega_i(\pi(\mathcal{D}))$  acts on  $\omega_i(V)$ , and  $G(\mathcal{D}, \omega_i)$  is the largest subgroup of  $\omega_i(\pi(\mathcal{D}))$  whose action on  $\omega_i(V)$  is trivial for all  $V$  in the essential image of  $t$ .

Now, if  $p \in P_{\text{triv}}(\omega_1, \omega_2)(R)$  and  $g \in G_{\mathcal{C}}(\mathcal{D}, \omega_1)(R)$  then  $gp \in P_{\mathcal{C}}(\omega_1, \omega_2)(R)$  acts trivially on everything of the form  $t(W)$ , and hence lies in  $P_{\text{triv}}(\omega_1, \omega_2)(R)$ . Hence  $G(\mathcal{D}, \omega_1)$  acts on  $P_{\text{triv}}(\omega_1, \omega_2)$ . For  $p, p' \in P_{\text{triv}}(\omega_1, \omega_2)(R)$ ,  $p^{-1}p'$  is an automorphism of  $\omega_1(V) \otimes R$  which is trivial for all  $V$  in the essential image of  $t$ . Hence it must be an element of  $G(\mathcal{D}, \omega_1)(R) \subset \omega(\pi_1(\mathcal{D}))(R)$ . The same arguments work for  $G_{\mathcal{C}}(\mathcal{D}, \omega_2)$ .

Thus  $P_{\text{triv}}(\omega_1, \omega_2)$  is a bi-pseudo-torsor, and to complete the proof, we must show that  $P_{\text{triv}}(\omega_1, \omega_2)$  is represented by a non-empty affine scheme over  $\mathcal{C}$ . Now, consider the ind-object of  $\mathcal{C}$

$$(68) \quad L_{\mathcal{C}}(\text{id}, \text{id}) = \text{colim} \left( \coprod_{f: V \rightarrow W \in \text{Mor}(\mathcal{C})} V \otimes W^{\vee} \rightrightarrows \coprod_{V \in \text{Ob}(\mathcal{C})} V \otimes V^{\vee} \right).$$

For every fibre functor  $\eta : \mathcal{C} \rightarrow \text{QCoh}(S)$ , there is a natural Hopf algebra structure on  $\eta(L_{\mathcal{C}}(\text{id}, \text{id})) = L_S(\eta, \eta)$ , and the Spec of this Hopf algebra is the affine group scheme of tensor automorphisms of  $\eta$ . Thus as before,  $L_{\mathcal{C}}(\text{id}, \text{id})$  has a Hopf algebra structure, and its formal Spec satisfies the defining property of the fundamental groupoid  $\pi(\mathcal{C})$  of  $\mathcal{C}$ . Hence one can construct a morphism of affine  $\mathcal{C}$ -schemes

$$(69) \quad P_{\mathcal{C}}(\omega_1, \omega_2) \rightarrow \pi(\mathcal{C})$$

which is the formal Spec of the obvious morphism  $L_{\mathcal{C}}(\text{id}, \text{id}) \rightarrow L_{\mathcal{C}}(\omega_1, \omega_2)$ . Then  $P_{\text{triv}}(\omega_1, \omega_2)$  is the fibre of  $P_{\mathcal{C}}(\omega_1, \omega_2) \rightarrow \pi(\mathcal{C})$  over the identity section  $\text{Sp}(1) \rightarrow \pi(\mathcal{C})$ . Hence it is the formal Spec of the algebra  $L_{\text{triv}}(\omega_1, \omega_2)$  defined by the push-out diagram

$$(70) \quad \begin{array}{ccc} L_{\mathcal{C}}(\text{id}, \text{id}) & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ L_{\mathcal{C}}(\omega_1, \omega_2) & \longrightarrow & L_{\text{triv}}(\omega_1, \omega_2) \end{array}$$

and is thus representable by an affine  $\mathcal{C}$ -scheme.

To prove that  $P_{\text{triv}}(\omega_1, \omega_2) \neq \emptyset$ , it suffices to show that  $\eta(P_{\text{triv}}(\omega_1, \omega_2)) \neq \emptyset$  for any fibre functor  $\eta : \mathcal{C} \rightarrow \text{QCoh}(S)$ . For any  $f : T \rightarrow S$ ,  $\eta(P_{\text{triv}}(\omega_1, \omega_2))(T)$  is the subset of  $\text{Isom}^{\otimes}(f^* \circ \eta\omega_1, f^* \circ \eta\omega_2)$  which maps to the identity under the natural map

$$(71) \quad r : \text{Isom}^{\otimes}(f^* \circ \eta\omega_1, f^* \circ \eta\omega_2) \rightarrow \text{Isom}^{\otimes}(f^* \circ \eta\omega_1 t, f^* \circ \eta\omega_2 t) = \text{Aut}^{\otimes}(f^* \circ \eta).$$

Since  $\eta\omega_1$  and  $\eta\omega_2$  are fibre functors on  $\mathcal{D}$ , according to 1.13 of [20] the affine scheme  $\underline{\text{Isom}}^{\otimes}(\eta\omega_1, \eta\omega_2)$  of tensor isomorphisms from  $\eta\omega_1$  to  $\eta\omega_2$  is faithfully flat

over  $S$ , and in particular is non-empty. Hence there is certainly some  $S$ -scheme  $f : T \rightarrow S$  such that

$$(72) \quad \text{Isom}^{\otimes}(f^* \circ \eta\omega_1, f^* \circ \eta\omega_2)$$

is non-empty. Pick such a  $T$ , and pick some  $p \in \text{Isom}^{\otimes}(f^* \circ \eta\omega_1, f^* \circ \eta\omega_2)$ . Since the morphism  $\omega_1(\pi(\mathcal{D})) \rightarrow \pi(\mathcal{C})$  admits a section, the induced homomorphism

$$(73) \quad \text{Aut}^{\otimes}(f^* \circ \eta\omega_1) \rightarrow \text{Aut}^{\otimes}(f^* \circ \eta)$$

is surjective, and hence there exists some  $g \in \text{Aut}^{\otimes}(f^* \circ \eta\omega_1)$  mapping to  $r(p) \in \text{Aut}^{\otimes}(f^* \circ \eta)$ . Then  $p' := g^{-1}p$  is an element of the set  $\text{Isom}^{\otimes}(f^* \circ \eta\omega_1, f^* \circ \eta\omega_2)$  and  $r(p') = \text{id}$ , thus  $\eta(P_{\text{triv}}(\omega_1, \omega_2))(T) \neq \emptyset$ .

REMARK 2.7. We can rephrase this as follows. Consider the functors of  $\mathcal{C}$  algebras

$$\underline{\text{Isom}}^{\otimes}(\omega_1, \omega_2) : \mathcal{C}\text{-alg} \rightarrow (\text{Set})$$

$$(74) \quad R \mapsto \text{Isom}^{\otimes}(\omega_1(-) \otimes R, \omega_2(-) \otimes R);$$

$$\underline{\text{Aut}}^{\otimes}(\text{id}) : \mathcal{C}\text{-alg} \rightarrow (\text{Set})$$

$$(75) \quad R \mapsto \text{Aut}^{\otimes}((-) \otimes R);$$

as well as the sub-functor  $\underline{\text{Isom}}_{\mathcal{C}}^{\otimes}(\omega_1, \omega_2)$ , the ‘functor of  $\mathcal{C}$ -isomorphisms  $\omega_1 \rightarrow \omega_2$ ’, defined to be the fibre over the identity of the natural morphism

$$(76) \quad \underline{\text{Isom}}^{\otimes}(\omega_1, \omega_2) \rightarrow \underline{\text{Aut}}^{\otimes}(\text{id}).$$

Then the functor  $\underline{\text{Isom}}_{\mathcal{C}}^{\otimes}(\omega_1, \omega_2)$  is representable by the affine scheme  $P_{\text{triv}}(\omega_1, \omega_2)$  over  $\mathcal{C}$ , which is a  $(G_{\mathcal{C}}(\mathcal{D}, \omega_1), G_{\mathcal{C}}(\mathcal{D}, \omega_2))$  bitorsor.

## 2.2 – Path torsors under relative fundamental groups

Let  $k$  be an algebraically closed field of characteristic zero,  $S$  a connected, smooth variety over  $k$  and  $f : X \rightarrow S$  a ‘good’ morphism. Let  $p, x$  be sections of  $f$ . We can apply the above methods to obtain an affine scheme over  $\text{IC}(S)$ , the torsor of paths from  $x$  to  $p$ , which can be considered as an affine scheme  $P(x) = \pi_1^{\text{dR}}(X/S, x, p)$  over  $S$ , together with an integrable connection on  $\mathcal{O}_{P(x)}$  (as a quasi-coherent  $\mathcal{O}_S$ -algebra). This is naturally a left torsor under  $\pi_1^{\text{dR}}(X/S, x)$  and a right torsor under  $\pi_1^{\text{dR}}(X/S, p) =: G$ . Moreover, the action map  $P(x) \times G \rightarrow P(x)$  is compatible with the connections, in the sense that the associated comodule structure

$$(77) \quad \mathcal{O}_{P(x)} \rightarrow \mathcal{O}_{P(x)} \otimes_{\mathcal{O}_S} \mathcal{O}_G$$

is horizontal, the RHS being given the tensor product connection. If  $G_n$  is the

quotient of  $G$  by the  $n$ th term in its lower central series, we will denote the push-out torsor  $P(x) \times^G G_n$  by  $P(x)_n$ . As before, the action map  $P(x)_n \times G_n \rightarrow P(x)_n$  is compatible with the connections.

DEFINITION 2.8. A  $\nabla$ -torsor under  $G_n$  is a  $G_n$ -torsor  $P$  over  $S$  in the usual sense, together with a regular integrable connection on  $\mathcal{O}_P$ , such that the action map

$$(78) \quad \mathcal{O}_P \rightarrow \mathcal{O}_P \otimes \mathcal{O}_{G_n}$$

is horizontal. The set of isomorphism classes of  $\nabla$ -torsors is denoted  $H_{\nabla}^1(S, G_n)$ .

Thus we have ‘period maps’

$$(79) \quad X(S) \rightarrow H_{\nabla}^1(S, G_n)$$

which takes  $x \in X(S)$  to the path torsor  $P(x)_n$ .

REMARK 2.9. (1) This is not a good period map to study. For instance, if  $k = \mathbb{C}$ , then the relative fundamental group is not just an affine group scheme with connection. There are reasons to expect that one can put a ‘non-abelian’ variation of Hodge structure on this fundamental group. Similar considerations will apply to the path torsors, and the period maps should take these variations of Hodge structures into account.

- (2) We can use the pro-nilpotent Lie algebra of  $\pi_1^{\text{dR}}(X/S, \rho)$  and the Campbell-Hausdorff law to view  $\pi_1^{\text{dR}}(X/S, \rho)$  as a non-abelian sheaf of groups on the infinitesimal site of  $S/k$ . We can use this interpretation to give an alternative definition of the cohomology set  $H_{\nabla}^1(S, G_n)$ .
- (3) A natural question to ask is whether or not, as in the situation studied by Kim, the targets for the period maps have the structure of algebraic varieties. Since we are more interested in the positive characteristic case, we will not pursue this question here.

### 3. Crystalline fundamental groups of smooth families in char $p$

Our goal in this section is to define the fundamental group of a smooth family  $f : X \rightarrow S$  of varieties over a finite field. Many of our arguments are essentially the same as those we gave in Section 1.

We will assume that the reader is familiar with the theory of rigid cohomology and overconvergent ( $F$ -)isocrystals, a good reference is [2]. Assume that  $k$  is a finite field, of order  $q = p^a$  and characteristic  $p > 0$ . Frobenius will always refer to linear Frobenius. If  $U/K$  is a variety, the category of overconvergent ( $F$ -)isocrystals on  $U/K$  is denoted  $(F\text{-})\text{Isoc}^{\dagger}(U/K)$ . These are Tannakian categories over  $K$ .

We define  $\mathcal{N}\text{Isoc}^{\dagger}(U/K)$  to be the full subcategory of  $\text{Isoc}^{\dagger}(U/K)$  on objects admitting a filtration whose graded pieces are constant. Chiarellotto and Le Stum

in [16] define the rigid fundamental group  $\pi_1^{\text{rig}}(U, x)$  of  $U$  at a  $k$ -rational point  $x$  to be the Tannaka dual of  $\mathcal{N}\text{Isoc}^\dagger(U/K)$  with respect to the fibre functor  $x^*$ . This is a pro-unipotent group scheme over  $K$ .

Now suppose that  $g : X \rightarrow S$  is a ‘good’, proper morphism over  $k$ , and let  $p : S \rightarrow X$  be a section.

**DEFINITION 3.1.** We say that  $E \in F\text{-Isoc}^\dagger(X/K)$  is relatively unipotent if there is a filtration of  $E$ , whose graded pieces are all in the essential image of  $g^* : F\text{-Isoc}^\dagger(S/K) \rightarrow F\text{-Isoc}^\dagger(X/K)$ . The full subcategory of relatively unipotent overconvergent  $F$ -isocrystals is denoted  $\mathcal{N}_g F\text{-Isoc}^\dagger(X/K)$ .

The pair of functors

$$(80) \quad \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \xrightleftharpoons[g^*]{p^*} F\text{-Isoc}^\dagger(S/K)$$

makes  $\mathcal{N}_g F\text{-Isoc}^\dagger(X/K)$  neutral over  $F\text{-Isoc}^\dagger(S/K)$  in the sense of §2.1. Hence we get an affine group scheme  $G(\mathcal{N}_g F\text{-Isoc}^\dagger(X/K), p^*)$  in  $F\text{-Isoc}^\dagger(S/K)$ .

**DEFINITION 3.2.** We define the relative fundamental group to be the affine group scheme  $G(\mathcal{N}_g F\text{-Isoc}^\dagger(X/K), p^*)$  in  $F\text{-Isoc}^\dagger(S/K)$ .

For  $s \in S$  a closed point, let  $i_s : X_s \rightarrow X$  denote the inclusion of the fibre over  $s$  and let  $g_s : X_s \rightarrow \text{Spec } k(s)$  denote the structure morphism. Let  $K(s)$  denote the unique unramified extension of  $K$  with residue field  $k(s)$ . Let  $\mathcal{V}(s)$  denote the ring of integers of  $K(s)$ . In keeping with notation of previous sections, let  $\pi_1^{\text{rig}}(X/S, p)_s$  denote the affine group scheme  $s^*(\pi_1^{\text{rig}}(X/S, p))$  over  $K(s)$ . The pull-back functor

$$(81) \quad i_s^* : \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \rightarrow \mathcal{N}\text{Isoc}^\dagger(X_s/K(s))$$

induces a homomorphism

$$(82) \quad \phi : \pi_1^{\text{rig}}(X_s, p_s) \rightarrow \pi_1^{\text{rig}}(X/S, p)_s$$

of affine group schemes over  $K$ . We would like to show again that when  $S$  is an affine curve, this is an isomorphism. The question is whether or not the sequence of affine group schemes corresponding to the sequence of neutral Tannakian categories

$$(83) \quad \mathcal{N}\text{Isoc}(X_s/K(s)) \leftarrow \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s) \leftarrow F\text{-Isoc}^\dagger(S/K) \otimes_K K(s)$$

is exact. Thus, as before, this boils down to the following three questions.

- (1) If  $E \in \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s)$  is such that  $i_s^* E$  is constant, is  $E$  of the form  $g^* F$  for some  $F \in F\text{-Isoc}^\dagger(S/K) \otimes_K K(s)$ ?
- (2) If  $E \in \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s)$ , and  $F_0 \subset i_s^* E$  denotes the largest constant subobject, then does there exist  $E_0 \subset E$  such that  $F_0 = i_s^* E_0$ ?
- (3) Given  $E \in \text{Isoc}^\dagger(X_s/K(s))$ , does there exist  $F \in \mathcal{N}_g F\text{-Isoc}^\dagger(X/K) \otimes_K K(s)$  such that  $E$  is a quotient of  $i_s^* F$ ?



















































