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ON THE DIFFERENTIABILITY OF THE NORM IN TRACE CLASSES S_p

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This text provides an outline of the proof of the differentiability of the norm in the trace classes S_p.

Let H be a real Hilbert space. By K(H) we denote the space of all compact operators from H to H endowed with the operator norm $\| \ \|$.

If $A \in K(H)$, then A^* denotes the adjoint of A. We define the sequence $\{S_n(A)\}_{n=1}^{\infty}$ of s-numbers of the operator A by

$$S_n(A) = \lambda_n \qquad n = 1,2,\dots$$

where $\lambda_1 \geq \lambda_2 \geq \dots$ is the decreasing sequence of non-zero eigenvalues of the operator $(A^*A)^{1/2}$, each repeated the number of times equal to its multiplicity.

Let $1 \leq p \leq \infty$. We put

$$S_p = \{A \in K(H) : \|A\|_p = (\sum_{n=1}^{\infty} s_n^p(A))^{1/p} < \infty\}$$

It is well known that S_p is a Banach space under the norm $\| \ \|_p$ and that

$$\|A\|_p = (\operatorname{tr} (A^*A)^{p/2})^{1/p}$$

Let E et F be Banach spaces. For an arbitrary natural K, $\mathfrak{B}^K(E,F)$ denotes the Banach space of continuous K-linear operators $v : \underbrace{E \times \dots \times E}_{K \text{ times}} \to F$ equipped with the norm

$$\|v\| = \sup_{\|x_1\| = \dots = \|x_K\| = 1} \|v(x_1, \dots, x_K)\|$$

Let $\mathcal{O}$ be an open set in E. A mapping $f : \mathcal{O} \to F$ is said to be differentiable at $x \in \mathcal{O}$ if there exists a linear operator $f'(x) \in \mathfrak{B}^1(E,F)$ such that $\lim_{h \to 0} \|f(x+h) - f(x) - f'(x)h\| \ . \ \|h\|^{-1} = 0$.

This $f'(x)$, which is unique, is called the derivative of f at x. The higher-order derivatives $f^{(K)} : \mathcal{O} \to \mathfrak{B}^K(E,F)$ are defined in the usual manner by induction. It is well known that the mapping $f : \mathcal{O} \to F$ is n-times continuously differentiable (is class C_n, for short) if and only if for every $x \in \mathcal{O}$ there exist a convex neighbourhood $x \in U \subset \mathcal{O}$, mappings $L_K : U \to \mathfrak{B}^K(E,F)$ $(K = 1,2,\dots,n)$ and a function $R : U \times E \to F$ such that for every h with $x + H \in U$

$$f(x+h) = f(x) + L_1(x\,;\,h) + \dots + L_n(x\,;\,h) + R(x,h)$$

where $\lim\limits_{h \to 0} \|R(x\,;\,h)\| \,.\, \|h\|^{-n} = 0$, uniformly on U.

The differentiability of the norm in the space $L_p(\Omega,\mu)$ $(1 \leq p < \infty)$ was considered by Bonic and Frampton in [1]. This property can be formulated as follows :

<u>Theorem 1</u> : Let $1 < p < \infty$. Then

1°) p is an even integer then the norm in $L_p(\Omega,\mu)$ is class C_∞ away from zero ;

2°) if p is an odd integer, then the norm in $L_p(\Omega,\mu)$ is class C_{p-1} away from zero and is not class C_p ;

3°) if p is not an integer and $[p]$ denotes the integral part of p, then the norm in $L_p(\Omega,\mu)$ is class $C_{[p]}$ away from zero and is not class $C_{[p]+1}$;

4°) in the space c_o there exists an equivalent norm $|\,.\,|$ which is class C_∞ away from zero.

Part 4°) of this theorem has been observed by Kuiper (see [1]). For our considerations we need only the information that this smooth norm in c_o locally depends only on (the absolute values of) a finite number of coordinates (away from zero).

In the case of the trace classes S_p we have exactly the same result as in the case of L_p, but the proofs are a good deal more complicated.

Theorem 2 : Let $1 < p < \infty$. Then

1°) if p is en even integer then the norm in S_p is class C_∞ away from zero ;

2°) if p is an odd integer then the norm in S_p is class C_{p-1} away from zero and is not class C_p ;

3°) if p is not an integer then the norm in S_p is class $C_{[p]}$ away from zero and is not class $C_{[p]+1}$;

4°) in K(H) there exists an equivalent norm $|||\cdot|||$ which is class C_∞ away from zero.

We begin with some general considerations on orthogonal projections on finite-dimensional subspaces spanned by eigenvectors of a compact operator. In the book by Gohberg and Krein [2] one can find the following useful lemma :

Lemma 3 : Let $X \neq 0$ be a compact operator acting in a complex Hilbert space with eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ and eigenvectors $\{x_n\}_{n=1}^{\infty}$. Let D be a circle

$D = \{z \in \mathbb{C} \ : \ |z - z_o| < r\}$ where $|z_o| > r$, and Γ be its boundary.

$\Gamma = \{z \in \mathbb{C} \ : \ |z - z_o| = r\}$ with positive orientation. Assume that $\lambda_m \in D$ for $m \in \mathfrak{M}$, $\lambda_K \notin D$ for $K \notin \mathfrak{M}$ and $\lambda_n \notin \Gamma$ for $n = 1,2,\dots$. Then the integral

$$- \frac{1}{2\pi i} \int_\Gamma (X - \lambda I)^{-1} \, d\lambda$$

is the orthogonal projection onto the subspace $E_U = \text{span}(x_m)_{m \in \mathfrak{M}}$.

Now let $\mathfrak{M}$ be a finite set of natural numbers. Let $\mathcal{O}_{\mathfrak{M}} \subset K(H)$ be the set of all compact operators A such that $s_m(A) \neq 0$ for $m \in \mathfrak{M}$. It follows from the continuity of s-numbers that $\mathcal{O}_{\mathfrak{M}}$ is open. Let $P_A^{\mathfrak{M}}$ denote the orthogonal projection on the finite dimensional subspace spanned by the eigenvectors of A^*A corresponding to the s-numbers $s_m(A)$, $m \in \mathfrak{M}$. The crucial proposition can be formulated as follows :

<u>Proposition 4</u> : The mapping $P_A^{\mathfrak{M}} : \mathcal{O}_{\mathfrak{M}} \to K(H)$ is class C_∞ .

<u>Proof</u> : Let $A_o \neq 0$ be a compact operator $A_o \in \mathcal{O}_{\mathfrak{M}}$. We shall prove that the mapping $P_A^{\mathfrak{M}}$ is infinitely many times differentiable at A_o. For this let us pick a positive number $\varepsilon > 0$ and a complex number $z_o \in \mathbb{C}$ such that $|s_m^2(A_o)-z_o| < \varepsilon$ for $m \in \mathfrak{M}$ and $|s_K^2(A_o)-z_o| > \varepsilon$ for $K \notin \mathfrak{M}$. From the continuity of s-numbers it follows that there is a $\delta > 0$ such that if B is an arbitrary compact operator with $\|B\| < \delta$ then we have also

$$| \; s_m^2 \; (A_o+B) \; - \; z_o \; | < \varepsilon \qquad \text{for } m \in \mathfrak{M}$$

$$| \; s_K^2 \; (A_o+B) \; - \; z_o \; | > \varepsilon \qquad \text{for } K \notin \mathfrak{M}$$

Put $\Gamma = \{z \in \mathbb{C} \; : \; |z - z_o| = \varepsilon\}$. By Lemma 3 for every compact operator B with $\|B\| < \delta$ the orthogonal projection $P_{A_o+B}^{\mathfrak{M}}$, considered as an operator acting in associated complex Hilbert space, can be represented in the form

$$P_{A_o+B}^{\mathfrak{M}} = - \frac{1}{2\pi i} \int_\Gamma (\; (A_o^* + B^*)(A_o+B) - \lambda I)^{-1} \, d\lambda$$

where $(A_o^* + B^*)(A_o+B)$ is meant as the operator acting in the complex Hilbert space.

At first we shall show that the operator $((A_o^* + B^*)(A_o+B) - \lambda I)^{-1}$ has an expansion in a Taylor's series at A_o, next we obtain the required result integrating this expansion over Γ.

Observe that for all operators X and Y (in real or complex Hilbert space), if X is invertible and $\|Y\| \; \|X^{-1}\| < 1$, then

$$(X+Y)^{-1} = X^{-1}[I + \sum_{\nu=1}^{\infty} (-Y \, X^{-1})^\nu]$$

Indeed, our assumption on Y implies that the series on the right-hand side is absolutely convergent and we can verify this equality by multiplying it by $(X+Y)$.

Now substitute in the above formula $X = A_o^*A_o - \lambda I$, $Y = A_o^*B + B^*A_o + B^*B$. Since for every $\lambda \in \Gamma$ the operator $(A_o^*A_o - \lambda I)$ is invertible, we get

$$((A_o^* + B^*)(A_o+B) - \lambda I)^{-1} = (A_o^*A_o - \lambda I)^{-1}[I + \sum_{\nu=1}^{\infty} (-(A_o^*B + B^*A_o + B^*B)(A_o^*A_o - \lambda I)^{-1})^{\nu}]$$

for all B such that $\|A_o^*B + B^*A_o + B^*B\| \max_{\lambda \in \Gamma} \|A_o^*A_o - \lambda I)^{-1}\| < 1$, i.e. for all B with $\|B\| < \delta'$. Rearranging the terms according to the powers of B and of B^* we can obtain the desired expansion in a Taylor's series. Finally, by integration this expansion over Γ we get the 'real" Taylor's formula for $P^{m}_{A_o+B}$, as an operator acting in a real Hilbert space, with a good estimate for the remainder. This proves that the mapping $P^{m}_{A} : K(H) \to K(H)$ is infinitely many times differentiable at A_o.

The easiest way to see the idea of the proof of Theorem 2 is to consider the case 4°). Therefore we begin with it

Case 4°) : we define a new norm on K(H) as follows

$$|||A||| = |\{s_n(A)\}|$$

where $\{s_n(A)\}_{n=1}^{\infty}$ is the sequence of s-numbers of the operator A and $|.|$ is the Kuiper's norm from Theorem 1, 4°). This norm is obviously equivalent to the usual operator norm in K(H). We shall show that this norm is infinitely many times differentiable away from zero. For this we take any compact operator $A_o \neq 0$.
As it was observed, the norm $|.|$ locally depends on a finite number of coordinates away from zero. Hence there exist a natural number N, a convex neighbourhood V of A_o and mappings $L_K : V \to \mathfrak{B}^K(c_o,F)$ $(K = 1,2,\dots)$ such that for every $A \in V$ the K-linear form $L_K(A)$ depends only on the first N coordinates and that for every compact operator B, with $A+B \in V$ and every natural μ we

have

$$|\{s_n(A+B)\}| = |\{s_n(A)\}| + \sum_{K=1}^{\mu} L_K(A; \{s_n(A+B)-s_n(A)\}) + R(A; \{s_n(A+B)-s_n(A)\}),$$

where $R : V \times c_o \to \mathbb{R}$ is the real function satisfying $\lim_{x \to 0} R(A; \{x_n\}).\|\{x_n\}\| = 0$, uniformly on V.

Now let us take any $A \in V$. To simplify the notation let us assume that only one of the s-numbers of A has multiplicity greater than 1 and that we have $s_1(A) = \dots = s_m(A) > s_{m+1}(A) > \dots$. It follows from the general form of continuous K-linear symmetric forms on c_o and the assumption on the multiplicity of s-numbers of A, that

$$|\{s_n(A+B)\}| = |\{s_n(A)\}| + \sum_{K=1}^{\mu} \sum_{\alpha}^{*} a_\alpha(A)[\sum_{\pi}^{**}(s_1(A+B)-s_1(A))^{\alpha_{\pi(1)}} \dots\dots\dots$$

$$\dots\dots(s_m(A+B)-s_m(A))^{\alpha_{\pi(m)}}] . (s_{m+1}(A+B)-s_{m+1}(A))^{\alpha_{m+1}} \dots\dots$$

$$\dots\dots(s_N(A+B)-s_N(A))^{\alpha_N} + R(A; \{s_n(A+B)-s_n(A)\}),$$

where $\sum_{\alpha}^{*}$ is extended over all sequences $(\alpha_i)_{i=1}^{N}$ of non-negative integers with $\alpha_1 \geq \dots \geq \alpha_m$ and $\sum_{1}^{\mu} \alpha_i = K$, $\sum_{\pi}^{**}$ is extended over all permutations π of the set $\{1,2,\dots,n\}$.

The above formula can be rewritten in the form

$$|||A+B||| = |||A||| + \sum_{K=1}^{\mu} \sum_{\alpha}^{*} b_\alpha(A) [\sum_{\pi}^{**} s_1(A+B)^{\alpha_{\pi(1)}} \dots s_m(A+B)^{\alpha_{\pi(m)}}]$$

$$s_{m+1}(A+B)^{\alpha_{m+1}} \dots s_N(A+B)^{\alpha_N} + R'(A; B)$$

where $R' : V \times K(H) \to \mathbb{R}$ is a mapping satisfying $\lim_{B \to 0} R'(A;B).\|B\|^{-\mu} = 0$, uniformly on V.

The case where there are more s-numbers of multiplicity greater than 1 can be handled analogously.

Thus the complete the proof it is enough to show the following fact :

<u>Lemma 5</u> : Let $A_o \neq 0$ be a compact operator and $s_{i+1}(A_o)$ be an s-number of multiplicity m, i.e. $s_{i+1}(A_o)=\dots=s_{i+m}(A_o) > s_{i+m+1}(A_o)$. Then for every sequence $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m$ the mapping

$$\varphi(C) = \sum_{\pi}^{**} s_{i+1}(C)^{\alpha_{\pi(1)}} \dots s_{i+m}(C)^{\alpha_{\pi(m)}}$$

is infinitely many times differentiable at A_o.

<u>Proof of Lemma</u> Let us take some sequence $\alpha_1 \geq \dots \geq \alpha_m$ and define the function $\overline{\varphi} : \mathbb{R}^m \to \mathbb{R}$ by

$$\overline{\varphi}(x_1 \dots x_m) = \sum_{\pi}^{**} x_1^{\alpha_{\pi(1)}} \dots x_m^{\alpha_{\pi(m)}}$$

furthermore for every natural $\nu = 1,2,\dots$ and every natural $j = 1,\dots,m$ let us define the functions g_ν , h_j : $\mathbb{R}^m \to \mathbb{R}$ by

$$g_\nu(x_1 \dots x_m) = \sum_{n=1}^{m} x_n^{2\nu}$$

$$h_j(x_1 \dots x_m) = \sum_{1 \leq n_1 < \dots < n_j \leq m} x_{n_1} \dots x_{n_j}$$

It is easy to show that if $x^o = (x_1^o \dots x_m^o) \in \mathbb{R}^m$ satisfies $x_n^o \neq 0$ for n=1...m, then every function h_j can be expressed as an infinitely many times differentiable function of the g_ν (ν=1...m) in some neighbourhood of $(g_1(x_o)..g_m(x_o)) \in \mathbb{R}^m$ Moreover the function $\overline{\varphi}$ can be expressed as an infinitely many times differentiable function of g_ν (ν=1,2,...) and h_j (j=1,...,m) in some neighbourhood of $(h_1(x^o),\dots,h_m(x^o),g_1(x^o)\dots)$. Thus, $\overline{\varphi}$ is an infinitely many times differentiable function of g_ν (ν=1,2...) in some neighbourhood of $(g_1(x^o),\dots)$.

This implies that to complete the proof of the differentiability of φ at A_o it is sufficient to show that the mapping

$$\tilde{g}_{\nu}(C) = \sum_{n=i+1}^{i+m} s_m^{2\nu}(C)$$

for every natural ν is class C_∞ at A_o. But this follows immediately from Proposition 4. This completes the proof of the case 4°).

Case 1°) is obvious.

The proof of 2°) and 3°) starts with showing that the mapping $\|\ \|^p$ is class C_q for $q = p-1$ or $q = [p]$ respectively. It is done using the formula mentioned in the proof of Proposition 4. We need the exact form of the Taylor's series for $P_A^{\mathfrak{m}}$ since the norm $\|\ \|$ in ℓ_p does not have the "localization property" of the Kuiper's norm $|\cdot|$ that we have used before. The corresponding computations and estimates are therefore more complicated, thus we omit them.
The fact that the norm $\|\ \|$ is not of class C_q is obvious because the space S_p contains a subspace isometric to ℓ_p.

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