SÉMINAIRE D'ANALYSE FONCTIONNELLE
ÉCOLE POLYTECHNIQUE

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Séminaire d'analyse fonctionnelle (Polytechnique) (1978-1979), exp. nº 22, p. 1-10
<http://www.numdam.org/item?id=SAF_1978-1979____A19_0>

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S E M I N A I R E

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1978-1979

ON THE RELATION BETWEEN SEVERAL NOTIONS OF UNCONDITIONAL STRUCTURE - A COUNTEREXAMPLE

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Exposé No XXII

23 Mars 1979

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The purpose of this talk is to describe some results obtained by W. B. Johnson, J. Lindenstrauss and myself concerning the unconditional structure of a space built by Kalton and Peck in [3]. In particular this space does not have the G.L. Lust property in spite of the fact that it has an unconditional decomposition into two dimensional subspaces. It is easy to check that the last property implies the G. L. property: every absolutely summing operator from a space with a decomposition into two dimensional subspaces factors through ℓ_1 - this was observed by Yoav Benyamini. So this answers a question raised by Pisier [6].

The space Z_2 of Kalton and Peck is the space of all sequences $\{(a_n, b_n)\}_{n=1}^{\infty}$ of couples of real numbers such that

$$\|\{(a_n,b_n)\}_{n=1}^{\infty}\| = \Big(\sum_{n=1}^{\infty} b_n^2\Big)^{1/2} + \Big(\sum_{n=1}^{\infty}\Big(a_n - b_n \log\Big(|b_n|\Big(\sum_{n=1}^{\infty} b_n^2\Big)^{-1/2}\Big)\Big)^2\Big)^{1/2}$$

is finite. The expression above is not a norm but is equivalent to a norm, that is one of the main points in [3].

Define

$$e_n = \{(\delta_{n,i}, 0)\}_{i=1}^{\infty}, \quad f_n = \{(0, \delta_{n,i})\}_{i=1}^{\infty}, \; n = 1,2,\ldots$$

it is easily checked that the sequence

$$e_1, f_1, e_2, f_2, \dots$$

forms a basis for Z_2. It is also obvious that $Z_2 = \sum_{n=1}^{\infty} \oplus [e_n, f_n]$ and the sum is unconditional. $[e_n]_{n=1}^{\infty}$ is isomorphic to ℓ_2 and also $Z_2/[e_n]_{n=1}^{\infty}$ is, but Z_2 is not isomorphic to a Hilbert space since $[f_n]_{n=1}^{\infty}$ is isomorphic to the Orlicz space $\ell_{x^2(\log x)^2}$. So Z_2 is another solution to the three space problem. We are not going to use this property of Z_2, the only properties we are going to use are

(1) Z_2 is reflexive (this follows from a computation of the dual space,see [3])

(2) Z_2 does not contain ℓ_∞^n's uniformly (by [1] Z_2 has type $2-\varepsilon$ and cotype $2+\varepsilon$ for every $\varepsilon > 0$).

In fact it is even true that Z_2 is super-reflexive.

The proof that Z_2 does not have G.L lust consists of three parts. the first one (Proposition 1), which may be of general interest, states that a space with properties (1) and (2) above which also has an unconditional finite dimensional decomposition is complemented in a space with an unconditional basis in a very special form. The second part (Proposition 2) shows that there are only few operators on R^2 which, when repeated on each $[e_n, f_n]$, form a bounded operator on Z_2. The last part shows that if Z_2 would have been complemented in a space with an unconditional basis as in Proposition 1 then there were more such diagonal operators.

<u>Proposition 1</u>: <u>Let</u> X <u>be a Banach space with the following properties</u>

(1) X is complemented in its second dual

(2) X does not contain ℓ_∞^n's uniformly

(3) X has an unconditional finite dimentional decomposition,

$$X = \sum_{n=1}^{\infty} \oplus E_n .$$

Then X has G.L. lust if and only if there exists a Banach Space Y with an unconditional basis $\{y_{i,n}\}_{i=1,\ n=1}^{k_n\ \ \ \infty}$ such that

(i) X is a subspace of Y and $E_n \subset [y_{i,n}]_{i=1}^{k_n}$

(ii) There exists a projection $P : Y \overset{\text{onto}}{\to} X$ with $P([y_{i,n}]_{i=1}^{k_n}) = E_n$.

Proof: The if part is trivial. Assume now that X has G.L. lust, By [2], we may assume that X is complemented in a Banach Lattice L and, again by [2], one may assume that L does not contain ℓ_∞^n's uniformly. In particular L can be considered as a space of functions on [0,1] with $L_\infty \subset L \subset L_1$ and L_∞ dense in L (see [5] Th. 1.b.14). A simple perturbation argument shows also that one may assume that each E_n, n=1, 2,... consists of simple functions. Thus, for each n one can find disjoint functions $y_{1,n}, \dots, y_{k_n, n}$ in L such that $[y_{i,n}]_{i=1}^{k_n} \supset E_n$.

Recall that Rad X is the subspace of $L_2(X)$ spanned by $\{r_n(\cdot)\cdot x\}_{n=1,\ x \in X}^{\infty}$. It is easily checked that if Q is the projection from L onto X then $\tilde{Q}$ defined by

$$\tilde{Q}\,(\Sigma r_n(\cdot) y_n) = \Sigma\, r_n(t) Q y_n$$

is a bounded projection from Rad L onto Rad X.

Now, $\overline{\text{span}}\{r_n(\cdot)\cdot E_n\}_{n=1}^{\infty}$ is complemented is Rad X, there are two ways to see this:

(i) to use a diadonal argument like in Prop. 1.c.8 in [4], or (ii) to use the Maurey - Khinchine inequality (for $\{r_n(t)\, r_m(s)\}_{n,m=1}^{\infty}$ see [5] Th.1.d.6) to show that $\{r_n(\cdot)\, E_m\}_{n,m=1}^{\infty}$ is an unconditional f.d.d for Rad X.

The space $\overline{\text{span}}\{r_n(\cdot)E_n\}_{n=1}^{\infty}$ is clearly isomorphic to X and is contained in $\overline{\text{span}}\{r_n(\cdot)\, y_{i,n}\}_{i=1,\ n=1}^{k_n\ \ \infty}$, so the only thing we still have to prove is that $\{r_n(\cdot)\, y_{i,n}\}_{i=1,\ n=1}^{k_n\ \ \infty}$ is an unconditional basic sequence. But, by the Maurey-Khinchine inequality

$$\left(\int \Big\| \sum_n \sum_i a_{i,n}\, r_n(t)\, y_{i,n}\Big\|^2\right)^{1/2} \approx \Big\|\Big(\sum_n \big(\sum_i a_{i,n}\, y_{i,n}\big)^2\Big)^{1/2}\Big\|$$

$$= \Big\|\Big(\sum_n \sum_i a_{i,n}^2\, y_{i,n}^2\Big)^{1/2}\Big\| \qquad \square$$

<u>Remarks</u> (1) From the proof one gets that Y also does not contain ℓ_∞^n's uniformly.

(2) If $\sup_n \dim E_n < \infty$, one can get with some additional work, that $\{f_{i,n}\}_{i=1}^{k_n}$ are uniformly equivalent to the unit vector basis of $\ell_1^{k_n}$ $n = 1,2, \ldots.$ (this was our first approach to the problem, but it is not needed in the present proof).

Given an operator T on $\mathbb{R}^2$ we consider it also as an operator on each of the two dimentional spaces $[e_n, f_n]$ in a natural way (using e_n, f_n as a basis). We now define formaly an operator $\tilde{T}$ on Z_2 by

$$\tilde{T}\left(\sum_{n=1}^{\infty} a_n e_n + b_n f_n\right) = \sum_{n=1}^{\infty} T(a_n e_n + b_n f_n)$$

<u>Proposition 2:</u> $\tilde{T}$ <u>is bounded if and only if</u>

$$T(ae_n + bf_n) = (a\alpha + b\beta)e_n + b\alpha f_n$$

<u>i.e.</u> <u>if and only if the matrix of</u> T <u>is of the form</u>

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}.$$

<u>Proof:</u> The if part is again very simple. To prove the only if part assume that $\tilde{T}$ is bounded and that the matrix of T is

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

If $\gamma \neq 0$ then

$$\tilde{T}\left(\sum_{n=1}^{\infty} a_n e_n\right) = \alpha \sum_{n=1}^{\infty} a_n e_n + \gamma \sum_{n=1}^{\infty} a_n f_n .$$

Let $\{a_n\}_{n=1}^{\infty}$ be such that $\sum_{n=1}^{\infty} a_n^2 = 1$ then $\|\sum_{n=1}^{\infty} a_n e_n\| < \infty$.

This implies that $\|\sum_{n=1}^{\infty} a_n f_n\|$ is bounded, so we get that

$$\left(\sum_{n=1}^{\infty}(a_n \log|a_n|)^2\right)^{1/2} \leq K\left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2}$$

which of course is false. Thus $\gamma = 0$ and

$$\tilde{T}\left(\sum_{n=1}^{\infty} a_n e_n + b_n f_n\right) = \sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)e_n + \delta b_n f_n .$$

For $\{b_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} b_n^2 = 1$ we get

$$|\delta| + \left(\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n - \delta b_n \log|b_n|)^2\right)^{1/2}$$

$$\leq \|\tilde{T}\|\left(1 + \left(\sum_{n=1}^{\infty}(a_n - b_n \log|b_n|)^2\right)^{1/2}\right)$$

If $\{b_n\}_{n=1}^{\infty}$ are also chosen to satisfy

$$\left(\sum_{n=1}^{\infty}(b_n \log|b_n|)^2\right)^{1/2} \geq N$$

and we choose $a_n = b_n \log|b_n|$ we get

$$|\delta| + |\alpha - \delta| \cdot N - |\beta| \leq \|\tilde{T}\|$$

which is a contradiction for large N unless $\alpha = \delta$. □

We are ready now to prove the main result.

Theorem: Z_2 does not have G.L. lust.

Proof Assume that it has. We may assume that we are in the situation of Proposition 1, with $E_n = [e_n, f_n]$, $n = 1,2,\ldots$. By Proposition 2 it is enough to produce an operator $\tilde{T}$ such

(i) $\tilde{T}([e_n,f_n]) \subset [e_n,f_n]$ and

(ii) $\inf_n d\left(\tilde{T}_{|[e_n,f_n]}, \operatorname{span}\left\{\begin{pmatrix}\alpha & \beta \\ 0 & \alpha\end{pmatrix}\right\}_{\alpha,\beta}\right) > 0$

($\tilde{T}_{|[e_n, f_n]}$ is considered as a matrix and the norm in (ii) is the operator norm). Indeed, if such an operator exists it is easy, using the fact that $\Sigma \oplus E_n$ is a "symmetric" decomposition and a simple compactness argument, to construct a diagonal operator which does not satisfy the conclusion of Proposition 2.

The operator $\tilde{T}$ that we'll use is of the form $\tilde{T}(x) = P(\sum_{i,\ n \in A} y^*_{i,n}(x)\ y_{i,n})$ for some set A. It is clearly enough to find, for each n, a set $A \subset (1, \ldots, k_n)$ such that T_A defined by

$$T_A(x) = P(\sum_{i \in A} y^*_{i,n}(x)\, y_{i,n})$$

has the property

$$d\left(T_A, \operatorname{span}\left\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}\right\}_{\alpha,\beta}\right) \geq \mu$$

for some absolute constant $\mu > 0$.

Fix n. For each $1 \leq i \leq k_n$ define

$$T_i = P(y^*_{i,n}(x)\, y_{i,n}), \quad x \in [e_n, f_n].$$

This is a one dimensional operator thus, has the form

$$\begin{pmatrix} a_i & b_i \\ \alpha_i a_i & \alpha_i b_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ a_i & b_i \end{pmatrix}$$

in the second case define $\alpha_i = 1$ (this comes to simplify the notations). Also, being a one dimensional operator

$$d(T_i, \text{ span } I) > \lambda \| T_i \|$$

where I is the identity operator, λ an absolute constant. For convenience we use here and elsewhere the ℓ_∞^4 norm in $B(\mathbb{R}^2)$ rather than the operator norm (of course, they are equivalent). Since $\sum_{i=1}^{k_n} T_i = I$ and the sum is unconditional we get

$$\sum_{i=1}^{k_n} \alpha_i b_i = 1, \quad \sum_{i=1}^{k_n} |b_i| \le K$$

where K is the unconditionality constant of $\{y_{i,n}\}$ times $\| P \|$.

Define

$$B = \{i, \ | \alpha_i | > \frac{1}{2K}\}$$

then

$$\sum_{i \in B} \alpha_i b_i > \frac{1}{2} .$$

For $i \in B$ define

$$S_i = \begin{pmatrix} a_i & 0 \\ \alpha_i a_i & \alpha_i b_i \end{pmatrix} \quad (S_i = T_i \text{ in the second case})$$

$d(S_i, I)/\|S_i\|$ is still bounded away from zero. So, for each $i \in B$ at least one of the following four possibilities occurs (μ an absolute constant)

(i) $\alpha_i a_i > \mu|\alpha_i b_i|$

(ii) $-\alpha_i a_i > \mu|\alpha_i b_i|$

(iii) $a_i - \alpha_i b_i > \mu|\alpha_i b_i|$ ($-\alpha_i b_i > \mu|\alpha_i b_i|$ in the second case)

(iv) $\alpha_i b_i - a_i > \mu|\alpha_i b_i|$ ($\alpha_i b_i > \mu|\alpha_i b_i|$ in the second case).

Thus, there exists $A \subset B$ such that $\sum_{i \in A} \alpha_i b_i > \frac{1}{8}$ and one of the four possibilities holds for all $i \in A$ simultaneously.

It is easily seen that $d(\sum_{i \in A} S_i, \text{span } I)$ and thus also $d(\sum_{i \in A} T_i, \text{span } I)$ are bounded away from zero.

$\square$

In [3] Kalton and Peck proved that any infinite dimentional complemented subspace of Z_2 contains an isomorphic copy of Z_2. Using their proof we were able to show that any such subspace contains a complemented subspace isomorphic to Z_2. Thus no complemented subspace of Z_2 has G. L. lust. As far as we know this is the first example of this kind.

Finally, the index 2 has no special role here. One can consider instead any of the spaces Z_p $1 < p < \infty$.

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