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G. SCHECHTMAN

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ÉCOLE POLYTECHNIQUE

CENTRE DE MATHÉMATIQUES

PLATEAU DE PALAISEAU - 91128 PALAISEAU CEDEX

Téléphone : 941.82.00 - Poste N°

Télex : ECOLEX 691 596 F

S E M I N A I R E

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A SIMPLE PROOF OF TWO THEOREMS CONCERNING BASES OF C(0,1)

G. SCHECHTMAN

(Ohio State University (Columbus))

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In [4] a simple construction of a complementably universal basis was introduced. In this short note we report an observation, made by W.B. Johnson and myself, which shows that the same construction can be used to prove some other known results.

We first need some definitions.

Definition 1 : A basis $\{b_i\}_{i=1}^{\infty}$ (of some Banach space) is said to be (complementably) universal for a family of bases $\mathfrak{B}$ if

(a) $\{b_i\}_{i=1}^{\infty} \in \mathfrak{B}$, and

(b) every basis in $\mathfrak{B}$ is equivalent to a subsequence of $\{b_i\}_{i=1}^{\infty}$ (on which the canonical projection is bounded).

Definition 2 : A basis $\{b_i\}_{i=1}^{\infty}$ of a Banach space B is said to be K-reproducible if, whenever B is isometrically embedded into a Banach space X with a basis $\{x_i\}_{i=1}^{\infty}$ and for every $\varepsilon > 0$, $\{b_i\}_{i=1}^{\infty}$ is $K+\varepsilon$ equivalent to a block basis of $\{x_i\}_{i=1}^{\infty}$. $\{b_i\}_{i=1}^{\infty}$ is said to be reproducible if it is K-reproducible for some $K \geq 1$.

Definition 3 : A (unconditional) basis $\{b_i\}_{i=1}^{\infty}$ of a Banach space B is said to be (unconditionally) reproducing if every (unconditional) basic sequence in B is equivalent to a block basis of $\{b_i\}_{i=1}^{\infty}$.

The following theorem is essentially due to Pełczynski [3] :

Theorem 1 : The set of all bases and the set of all unconditional bases (of separable Banach spaces) both admit complementably universal elements.

In [4] (see also [2] p. 92) a simple proof was given to this theorem. We are going to show that the same construction proves also the following two theorems, due respectively to Lindenstrauss and Pełczynski [1] and to Pełzcynski [3].

Theorem 2 : Every basis of C(0,1) is reproducible. Moreover, if the basis constant is K, it is K-reproducible.

Theorem 3 : Every basis of C(0,1) is reproducing.

We first recall the main construction of [4]. Let T be the set of all finite sequences of positive integers. For $i_1,\dots,i_n$ positive integers we define $e_{i_1,\dots,i_n} : T \to \mathbb{R}$ by

$$e_{i_1,\dots,i_n}(j_1,\dots,j_m) = \begin{cases} 1 & \text{if } n=m \text{ and } i_k = j_k \text{ for } k=1,\dots,m \\ 0 & \text{otherwise .} \end{cases}$$

Let $\{y_i\}_{i=1}^{\infty}$ be a dense sequence in $C(0,1)$. Define a norm on the linear span of $\{e_{i_1,\dots,i_n}\}$ by

$$\Big\|\sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} e_{i_1,\dots,i_n}\Big\| = \sup \sup_{1\le n<\infty} \Big\|\sum_{k=1}^{n} a_{i_1,\dots,i_k} y_{ik}\Big\|$$

where the outer sup is taken over all subsets of T of the form $\{(i_1),(i_1,i_2),(i_1,i_2,i_3)\dots\}$ (we call such subsets branches), and all but finitely many of the $a_{i_1,\dots,i_n}$ are zero. We denote the completion of this normed space by U.

It is easy to check now that under a certain order $\{e_{i_1,\dots,i_n}\}$ is a complementably universal basis for the set of all bases.

We proceed now with the proof of theorems 2 and 3.

Let X be a Banach space with a basis $\{x_i\}_{i=1}^{\infty}$ and assume that X contains $C(0,1)$ isometrically . Let $\{b_i\}_{i=1}^{\infty}$ be a basic sequence in $C(0,1)$ with basis constant K (i.e. $\|\sum_{i=1}^{n} \alpha_i b_i\| \le K \|\sum_{i=1}^{m} \alpha_i b_i\|$ for all $n \le m$ and all $\{\alpha_i\}_{i=1}^{m}$). We consider the space U as being a subspace of $C(0,1)$ (as we may by the Banach-Mazur theorem), and thus as a subspace of X.

Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence of positive numbers to be specified later. We are going to find a subsequence $i_1,i_2,\dots$ of the integers and a block basis $\{z_k\}_{k=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ such that

(a) $$\|b_k - y_{i_k}\| < \varepsilon_k$$

(b) $$\|e_{i_1,\dots,i_k} - z_k\| < \varepsilon_k \qquad k = 1,2,\dots$$

once this is done it is clear that $\{y_{i_k}\}_{k=1}^{\infty}$ is equivalent to a basic sequence equivalent to $\{b_k\}_{k=1}^{\infty}$ with constant as good as we wish provided the ε_k are small enough. The definition of the norm in U implies then that $\{y_{i_k}\}_{k=1}^{\infty}$ and, thus $\{b_k\}_{k=1}^{\infty}$, is K+ε equivalent to $\{e_{i_1,\dots,i_k}\}_{k=1}^{\infty}$ (where ε is a positive number as small as we wish provided the ε_k are small). Finally (b) will imply that $\{b_k\}_{k=1}^{\infty}$ is equivalent to $\{z_k\}_{k=1}^{\infty}$ with a good constant.

It is clear how to choose i_1 and z_1. Assume that we have chosen $i_1,\dots,i_{n-1}$ and $z_1,\dots,z_{n-1}$ to satisfy (a) and (b). There are infinitely many i such that $\|b_n - y_i\| < \varepsilon_n$;denote this sequence if i by $\mathbb{N}'$. $\{e_{i_1,\dots,i_{n-1},i}\}_{i\in\mathbb{N}'}$ is equivalent to the unit vector basis of c_o thus tends weakly to zero. So we can find an $i \in \mathbb{N}'$, call it i_n, and a vector $z_n \in X$ which, together with $z_1,\dots,z_{n-1}$ forms a finite block basis of $\{x_i\}_{i=1}$ and $\|z_n - e_{i_1,\dots,i_n}\| < \varepsilon_n$. □

<u>Remark</u> : It is also known that every unconditional basis of $L_p(0,1)$, $1 < p < \infty$, is both reproducible and unconditionnally reproducing. Also the family of all unconditional basic sequences in $L_p(0,1)$ admits a complementably universal element.

<u>Problem</u> : Does the family of all basic sequences in $L_p(0,1)$, $1 \le p < \infty$, admit a universal element ?

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