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REPORT ON THE FIXED POINT FORMULA

by Raoul BOTT (1)

The work to be reported on concerns a generalization of the Lefschetz theorem for elliptic complexes. As Michael Atiyah and I have - alas - already reported on the applications of this extension in various places ([1] , [2], ...) I will here principally discuss one of our methods of proof which by now has become nearly embarrassingly transparent. It is also clear that special instances of our procedure abound in the literature, and every lecture seems to uncover new ones. As far as applications go, let me remark though that during the recent visit of F. Hirzebruch at Oxford we seem to have been able to derive the recent results of Langlands ([5]) along a road suggested by Borel some time ago from a generalized form of the fixed point theorem, and the proportionality principle of Hirzebruch.

The question is the following one. We are given a compact smooth manifold X, a sequence of smooth vector bundles $E = \{E_i\}$, $i = 0, \dots n$, over X, together with a differential operator $d : \Gamma(E_i) \to \Gamma(E_{i+1})$, subject to the two conditions :

(i) $d^2 = 0$

(ii) the symbol sequence of d :

$$0 \to E_o \overset{\sigma(d,\xi)}{\to} E_1 \to \dots \overset{\sigma(d,\xi)}{\to} E_n \to 0$$

is exact at <u>each nonzero cotangent</u> vector ξ on X.

The ellipticity conditions (ii) then have as a consequence that the homology groups

$$H^i(E) = \text{Ker } d \cap \Gamma(E_i)/\text{Im}.d \cap \Gamma(E_i)$$

are finite dimensional, so that, in particular, the Euler number

(1) This is a report on joint work with M.F. Atiyah.

$\chi(E) = \sum(-1)^i \dim H^i(E)$ is a well-defined invariant of the complex E. The determination of this number is the generalised R.R or "index" problem, and was recently solved by Atiyah and Singer.

Suppose now that $T = \{T_i\}$ is a sequence of endomorphisms of $\Gamma(E_i)$, for which $d\circ T_i = T_{i+1}\circ d$ so that they induce homomorphisms $H^i(T) : H^i(E) \to H^i(E)$, and define the Lefschetz number of T as the expression

$$L(T) = \sum(-1)^i \text{ Tr. } H^i(T).$$

The generalized "Lefschetz" Problem is now to evaluate $L(T)$ when T has suitable geometric properties.

For instance where E is the de Rham complex $\Lambda = \{\Lambda_i\}$ of differential forms on X and T is induced by the differential of a smooth map, $t : X \to X$, then the classical Lefschetz theorem evaluates $L(T)$ as a sum of certain indexes or "weights" attached to the fixed point sets of t, and in the special case when the graph of t is transversal to the diagonal these weights are all ± 1.

In general, this leads one to define T to be a geometric endomorphism of E, if there is a smooth map $t : X \to X$, together with vector bundle map $\phi_i : \Gamma(t^{-1} E_i) \to \Gamma(E_i)$, so that

$$(1.1) \qquad T_i s(x) = \phi_i(x)\circ s\{t(x)\}, \qquad x \in X.$$

Finally, such a T is called transversal if the graph of t is transversal to the identity map, and our first problem is to compute $L(T)$ for a geometric and transversal endomorphism T in terms of the fixed point set of t. This is then the question most removed from the index theorem and as one would expect, it is a more elementary one.

2. The solution of this problem which we like best at the moment is based on the observation that although the space of C^∞ sections $\Gamma(E_i)$ is ∞-dimensional, T_i has a trace on $\Gamma(E_i)$ in a certain very natural sense - precisely because it is so far removed from the identity. Suppose for a moment that $T : E \to E$ (we suppress the suffix i here and will act as if E were a trivial bundle. The general case introduces only technical complications) were a transformation given by a smooth kernel K_T :

(2.1) $$Ts(x) = \int_X K_T(x, y)\, s(y)\, dy$$

Then clearly the trace of T should be taken to be $\int_X K_T(x, x)\, dx$.

Interpreted differently, this trace is the value of the diagonal distribution Δ in $X \times X$ on the smooth kernel $K(x, y)$. Now suppose Δ_n in a sequence of smooth distributions on $X \times X$ which tend to Δ in the distribution topology. Such a sequence serves to extend this notion of trace to a possibly larger class of endomorphisms T. Indeed by the Schwartz Kernel Theorem every continuous map $T : \Gamma(E) \to \Gamma(E)$ defines a distribution k_T on $X \times X$, which on product functions $\phi(x) \cdot \chi(y)$ is given by

$$k_T(\phi(x).\chi(y)) = \int \{T\, \phi\}(x).\chi(x)\, dx.$$

This "kernel of T" has a definite value on Δ_n, and one can say that T has a "trace relative to $\{\Delta_n\}$", if $\lim_{n\to\infty} k_T(\Delta_n)$ exists. Of course this is a rather ad hoc extension for a general sequence Δ_n. However the geometry of the situation suggests a method of approximating the diagonal by squeezing down on it along the normal directions in a C^∞ fashion. More precisely, let (u, v) be a set of coordinates on the open set U in $X \times X$, with u normal to Δ in U so that $\Delta \cap U = u^{-1}(0)$. Also let δ_n be a sequence of values in the u-variables tending to the δ function at 0, and whose supports also tend to zero; then a sequence of the type $\delta_n \times g$ where g is C^∞ on $X \times X$ and $g(o, v) = 1$ on U is called an elementary flat approximation to Δ on U. A sequence $\{\Delta_n\}$ tending to Δ is called flat if for sufficiently small U, the restriction of Δ_n to n is a finite linear combination of flat approximations. Now then, it is quite easy to show the following

THEOREM 2.2 *Let T be a geometric endomorphism of E, given by a lifting ϕ : $\Gamma(t^{-1} E) \to \Gamma(E)$ of a transversal map $t : X \to X$.*

Then for every flat approximation Δ_n of Δ, graph $T(\Delta_n)$ tends to a limit which is independent of the flat approximation used, and this limit, which we will call the flat trace of T, is given by the formula :

(2.3) $$\text{Trace}^{b}(T) = \sum_P \text{trace}\, \phi_i(P) / |\det\{1 - dt_P\}|$$

where P ranges over the fixed points of t, and dt_P denotes the differential of t at P¶ .

In view of this result one is naturally led to conjecture the following

FIXED POINT THEOREM. *Let* T *be a transversal geometric endomorphism of the elliptic complex* E.

Then,

$$\sum (-1)^i \text{ Trace } H^i(T) = \sum (-1)^i \text{ Trace}^b(T_i). \tag{2.4}$$

In view of the relation (2.3), the formula (2.4) solves our problem. Indeed we have evaluated $L(T)$ in terms of a weighted sum over the fixed points :

$$L(T) = \sum_P \sigma(P) \tag{2.5}$$

the weights being given by :

$$\sigma(P) = \sum_i (-1)^i \frac{\text{trace } \phi_i(P)}{|\det(1-dt_P)|} \tag{2.6}$$

The formula (2.4) expresses an additivity property of the flat trace. In order to prove it,it is expedient to compare this trace with the usual one furnished by linear space theory. For this purpose let E again denote a single vector bundle over X and let T be an endomorphism of E. Assume further that E is equipped with a hermitian structure, and X with a volume. Further, let ∇ be a positive definite selfadjoint operator relative to this structure. By the spectral theorem, ∇ then decomposes $\Gamma(E)$ into *finite dimensional* eigenspaces $\Gamma_\lambda(E)$, where ∇ has the real positive eigenvalues λ, and the Hilbert space into which $\Gamma(E)$ may be completed is completely decomposed by these spaces :

$$\overline{\Gamma(E)} = \sum_\lambda \Gamma_\lambda(E).$$

For each λ let $T(\lambda,\lambda)$ be the composition $\Gamma_\lambda(E) \xrightarrow{T} \Gamma(E) \xrightarrow{\pi_\lambda} \Gamma_\lambda(E)$ where π_λ is the orthogonal projection on $\Sigma_\lambda(E)$, so that the trace of T in the *linear space sense* should then be given by $\sum_\lambda$ trace $T(\lambda,\lambda)$. Of course, this expression will in general be meaningless. To remedy this, consider rather the Zeta-series :

¶ Note that the transversality of t ensures the nonvanishing of the determinantal factor on the right. The bars denote absolute value.

(2.7) $$\zeta(T, s) = \sum_{\lambda \geq 0} \text{trace } T(\lambda, \lambda) / \lambda^s$$

For large Re(s) this series is well known to represent a holomorphic function of s.

Now the compatibility relation which will yield (2.4) quite easily is the following :

CONTINUATION THEOREM. The Zeta-series $\zeta(T, s)$ extends to a holomorphic function on the entire s-plane, and its value at 0 agrees with the flat trace :

(2.8) $$\zeta(T, 0) = \text{Trace}^b(T).$$

To prove (2.4) from this last fact one proceeds as follows. We choose hermitian structures on the vactor-bundles E_i, and a volume on X. The adjoint d^* of d is then well defined, and we may set

$$\nabla = 1 + d\,d^* + d^*\,d.$$

This operator preserves the bundles individually, is self adjoint and positive. We therefore have a decomposition of each $\Gamma(E_i)$ into eigenspaces $\Gamma_\lambda(E_i)$ of ∇, and the differential operator d then induces a differential operator d_λ :

(2.9) $$0 \to \Gamma_\lambda(E_0) \xrightarrow{d_\lambda} \Gamma_\lambda(E_1) \xrightarrow{d_\lambda} \dots \to \Gamma_\lambda(E_n) \to 0$$

because ∇ commutes with d. Further, the Hodge Theory implies that this sequence is exact for $\lambda > 1$, while d_1 vanishes identically and $\Gamma_1(E_i)$ may be identified with $H^i(E)$. Hence trace $H^i(T)$ may be identified with trace $T_i(1, 1)$.

Consider now the alternating sum $\chi(s) = \sum(-1)^i \zeta(T_i; s)$ for large s. The eigenspace of 1 contributes precisely L(T) to this expression as we have just seen. Further for larger λ, the exactness of (2.9) implies that

$$\sum(-1)^i \text{ trace } T_i(\lambda, \lambda) = 0$$

Hence $\chi(s)$ is in fact constant and equal to L(T). Setting $s = 0$ one obtains the result.

3. The proof which I have just sketched throws the burden of the work on the continuation theorem. On may in turn deduce that result from the theorem that

the s^{th} powers of a positive definite selfadjoint differential operator, D, form an analytic family of pseudo-differantial operators, and from the quite straightforward fact that if P_s is an analytic family of such operators and T is a transversal geometric map, then $\text{trace}^b\ (P_s . T)$ depends analytically on s. Indeed, once these facts are granted, it is clear that $\text{trace}^b\{D^{-s}.T\}$ agrees with $\zeta(T; s)$ for large s, and therefore defines an analytic extension of $\zeta(T, s)$ to the whole plane.

The theorem that D^s can be realised as a pseudo-differential operator was, so to speak, commissioned by us from Hörmander and Seeley.
That the ζ function $\zeta(T, s)$ has a meromorphic extension when T is the identity, was proved in a special instance by Minakshi-Sundram and Plejel in [3].

Actually for our purposes one could get away with the weaker statement of Kotake and Narasimhan [4] to the effect that the kernel of D^s is completely regular. Also one could use a different method of smoothing - that is to say, study the expression $\text{Trace}^b\{f_s\ (\nabla) . T\}$ for suitable functions of the Laplacian - for example when $f_s(t) = e^{-ts}$.

Indeed if one wishes to minimise the number of things one assumes about differential equations, then one may deduce our formula by quite straightforward homological methods, using only that $(1 + \nabla)^{-1}$ is given by a pseudo-differential operator.

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