SÉMINAIRE N. BOURBAKI

G. VAN DIJK

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Séminaire N. Bourbaki, 1971, exp. nº 387, p. 89-106
<http://www.numdam.org/item?id=SB_1970-1971__13__89_0>

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Séminaire BOURBAKI
23e année, 1970/71, n° 387 Novembre 1970

HARMONIC ANALYSIS ON REDUCTIVE $\mathfrak{p}$-ADIC GROUPS

(after HARISH-CHANDRA [4 (c)])

by G. VAN DIJK

In [7], Mautner gives a method for constructing irreducible unitary representations of the $\mathfrak{p}$-adic group PGL(2) , whose matrix-coefficients are square-integrable functions. For this purpose he starts with an irreducible unitary representation τ of the open compact subgroup K , being the canonical image in PGL(2) of the group of integer matrices with determinant a unit in GL(2) , whose restriction to the subgroup generated by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ does not contain the identity representation. The unitary representation of PGL(2) induced by τ , decomposes into a direct sum of finitely many irreducible representations, whose matrix-coefficients with respect to a suitable orthonormal base, are continuous functions with compact support.

These representations are special cases of so-called supercuspidal representations. They are defined as follows.

Let Ω be a $\mathfrak{p}$-adic field, i.e. a locally compact field with a non-trivial discrete valuation. We start with a connected, reductive (linear) algebraic group $\mathbb{G}$ defined over Ω and we denote by G its subgroup of Ω-rational points. Then G is a locally compact, separable and unimodular group. Let $\mathbb{P}$ be a parabolic subgroup of $\mathbb{G}$, defined over Ω , with unipotent radical $\mathbb{N}$. Then $\mathbb{N}$ is defined over Ω as well. We put $P = \mathbb{P} \cap G$, $N = \mathbb{N} \cap G$ and we call P a parabolic subgroup of G with unipotent radical N . P and N determine $\mathbb{P}$ and $\mathbb{N}$ completely. By $\mathbb{Z}$ we denote the maximal Ω-split torus in the center of $\mathbb{G}$. We write $Z = \mathbb{Z} \cap G$. Let f be a continuous function on G with compact support mod Z . For any parabolic subgroup P of G with unipotent radical N , put

$$f^P(x) = \int_N f(xn)dn \qquad (x \in G)$$

where dn is a fixed Haar measure on N .

A continuous (complex-valued) function f on G is said to be a <u>supercusp form</u> if

(i) Supp f is compact mod Z ;

(ii) $f^P = 0$ for all parabolic subgroups $P \neq G$.

Let π be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$. We call π <u>supercuspidal</u> if there exist $\varphi , \psi \in \mathcal{H} - (0)$ such that the function $x \longmapsto (\varphi, \pi(x)\psi)$ $(x \in G)$ is a supercusp form.

In fact, Mautner's method of construction for this class of representations in case $PGL(2)$, can be generalized to semisimple G by taking a "good" open compact subgroup of G (given by Bruhat-Tits) and a so-called supercuspidal representation of K , i.e. an irreducible unitary representation of K whose matrix-coefficients, considered as functions on G , are supercusp forms. It is an open problem if one obtains all supercuspidal representations of G by means of the above construction. There is another, more serious, motivation for studying supercuspidal representations. It originates from the "philosophy of cusp forms" which appeared to be very fruitful in harmonic analysis on reductive groups ([2], [4(b)]). Furthermore we have to mention that the supercuspidal representations occur naturally in the study of automorphic forms (cf. [5]).

Of course our main goal here is the Plancherel formula for G . Apart from technical difficulties (no differential operators, no good "exp" in positive characteristic) we are faced with a fundamental problem : given any irreducible unitary representation of G , does its character exist ?

For supercuspidal representations the answer is affirmative. In the next paragraphs we propose to describe some of the results concerning supercuspidal representations. Proofs are omitted almost everywhere. They can be looked up in [4(c)].

§ 1. Reductive $\mathfrak{p}$-adic groups.

We recall some standard results about algebraic groups (cf. [1]). Let Ω be a field. By an Ω-group we mean a linear algebraic group defined over Ω. Let $\mathbb{G}$ be a connected and reductive Ω-group. By a parabolic subgroup $\mathbb{P}$ of $\mathbb{G}$ we mean an algebraic subgroup which contains a Borel subgroup of $\mathbb{G}$. We say that $\mathbb{P}$ is Ω-parabolic if it is parabolic and defined over Ω. Let $\mathbb{N}$ denote the unipotent radical of $\mathbb{P}$. Then $\mathbb{N}$ is an Ω-subgroup and $\mathbb{P}$ is the normalizer of $\mathbb{N}$ in $\mathbb{G}$. By a Levi Ω-subgroup $\mathbb{M}$ of $\mathbb{P}$ we mean a reductive Ω-group such that the mapping $(m,n) \mapsto mn$ $(m \in \mathbb{M}, n \in \mathbb{N})$ defines an Ω-isomorphism of the algebraic varieties $\mathbb{M} \times \mathbb{N}$ and $\mathbb{P}$. Such a subgroup $\mathbb{M}$ always exists and is connected. Fix $\mathbb{M}$ and let $\mathbb{A}$ be a maximal Ω-split torus lying in the center of $\mathbb{M}$. Then $\mathbb{A}$ is unique and $\mathbb{M}$ is the centralizer of $\mathbb{A}$ in $\mathbb{G}$. We call $\mathbb{A}$ a split component of $\mathbb{P}$. Let N be the group of Ω-rational points of $\mathbb{N}$. For any split component $\mathbb{A}'$ of $\mathbb{P}$ there exists a unique element $n \in N$ such that $\mathbb{A}' = \mathbb{A}^n = n\mathbb{A}n^{-1}$.

Now let Ω be a $\mathfrak{p}$-adic field. Let G denote the group of all Ω-rational points of $\mathbb{G}$. We shall call G simply, a reductive $\mathfrak{p}$-adic group. As before, by a parabolic subgroup (or cuspidal subgroup) P of G, we mean a subgroup of the form $P = G \cap \mathbb{P}$, where $\mathbb{P}$ is an Ω-parabolic subgroup of $\mathbb{G}$. P determines $\mathbb{P}$ completely. By a split component A of P, we mean a subgroup of the form $\mathbb{A} \cap G$, where $\mathbb{A}$ is a split component of $\mathbb{P}$. $\mathbb{A}$ is completely determined by A, since Ω is an infinite field. We call (P,A) a parabolic (or cuspidal) pair in G. Once A is fixed, we have the corresponding Levi Ω-decompositions $\mathbb{P} = \mathbb{M}\mathbb{N}$ and $P = MN$ where $M = \mathbb{M} \cap G$. We shall call N the unipotent radical of P (as above).

§ 2. Existence of characters for the discrete series.

Let G be a locally compact unimodular group with center Z_G. Let Z be a subgroup of Z_G such that Z_G/Z is compact. As usual, $\hat{Z}$ will denote the set of unitary characters of Z. Fix Z and $\chi \in \hat{Z}$. A unitary representation π of G is called a χ-representation if $\pi(z) = \chi(z).1$ for all $z \in Z$.

Let π be an irreducible (no non-trivial invariant closed subspaces) χ-representation on a Hilbert space $\mathcal{H}$. We say that π is square-integrable mod Z

if

$$\int_{G/Z} |(\varphi\,,\,\pi(x)\psi)|^2\,dx^* < +\infty$$

for some (hence for all) $\varphi\,,\,\psi \in \mathcal{H} - (0)$.

Obviously this notion is invariant under equivalence of representations. The Schur orthogonality relations are still true : if both $(\pi,\mathcal{H})$ and $(\pi',\mathcal{H}')$ are χ-representations, which are square-integrable mod Z , then

$$\int_{G/Z} \overline{(\varphi_1\,,\,\pi(x)\psi_1)}(\varphi_2\,,\,\pi(x)\psi_2)dx^* = d(\pi)^{-1}\,\overline{(\varphi_1,\varphi_2)}(\psi_1,\psi_2)$$

$(\varphi_i\,,\,\psi_i \in \mathcal{H}\,,\ i = 1\,,2)$, where $d(\pi)$ is a positive number only depending on π and the normalization of the Haar measure on G/Z . Moreover, if π is not equivalent to π' , then

$$\int_{G/Z} \overline{(\varphi_1\,,\,\pi(x)\psi_1)}(\varphi_2\,,\,\pi'(x)\psi_2)dx^* = 0 \quad \text{for all } \varphi_i \in \mathcal{H}\,,\ \psi_i \in \mathcal{H}'\ (i=1\,,2)\ .$$

Denote by $E(G)$ the set of equivalence classes of irreducible unitary representations of G and by $E(G,\chi)$ the subset of $E(G)$, consisting of those classes which contain χ-representations. Let $E_2(G)$ be the subset of $E(G)$ consisting of the classes which contain square-integrable representations mod Z and put $E_2(G,\chi) = E_2(G) \cap E(G,\chi)$. We call $E_2(G)$ the discrete series of G . It is independent of the choice of Z .

Observe that $d(\pi)$ only depends on the class ω of π . We shall write $d(\omega)$ as well as $d(\pi)$.

Let K be any compact subgroup of G . By $E(K)$ we denote the set of equivalence classes of irreducible unitary representations of K . If $\omega \in E(G)$ and $\underline{d} \in E(K)$, we define $[\omega:\underline{d}]$ as follows. Fix a representation $\pi \in \omega$ and let π_K denote the restriction of π to K . Then $[\omega:\underline{d}]$ is the multiplicity of $\underline{d}$ in π_K . Since K is compact, every irreducible representation of K is finite-dimensional. We write $d(\underline{d})$ for the degree of a representation in the class $\underline{d}$.

THEOREM 1.- *Let* K *be an open compact subgroup of* G . *Normalize the Haar measures* dx *and* dz *on* G *and* Z *such that the total measures of* K *and* $K \cap Z$ *are* 1 . *Normalize the Haar measure* dx^* *on* G/Z *such that* $dx = dx^*\,dz$. *Then for any*

$\underline{d} \in E(K)$ and $\chi \in \hat{Z}$,

$$\sum_{\omega \in E_2(G,\chi)} d(\omega)\,[\omega:\underline{d}] \leq d(\underline{d}) .$$

This is a simple consequence of the Schur orthogonality relations, mentioned above.

COROLLARY.- *Fix* $\omega \in E_2(G)$. *Under the conditions of Theorem 1,*

$$[\omega:\underline{d}] \leq d(\omega)^{-1}\, d(\underline{d})$$

for all $\underline{d} \in E(K)$.

Now let G be a reductive $\mathfrak{p}$-adic group with split component Z . Let G_o be an open set in G and denote by $C_c^\infty(G_o)$ the space of all locally constant complex-valued functions on G_o with compact support (no topology). By a *distribution* on G_o we mean simply a linear mapping of $C_c^\infty(G_o)$ into the complex numbers.

THEOREM 2.- *Let* $\omega \in E_2(G)$ *and fix* $\pi \in \omega$. *Then for any* $f \in C_c^\infty(G)$, *the operator*

$$\pi(f) = \int_G f(x)\pi(x)dx$$

is of the trace class (even of finite rank).

Since G contains arbitrarily small open compact subgroups, this is an immediate consequence of the corollary of Theorem 1. Define $\theta_\pi(f) = \mathrm{tr}\ \pi(f)$ $(f \in C_c^\infty(G))$. Then θ_π , the character of π , is a distribution on G which depends only on the class ω of π . Hence we may denote it by θ_ω . Then θ_ω is invariant and the mapping $\omega \mapsto \theta_\omega$ $(\omega \in E_2(G))$ is injective.

Let ${}^oE(G)$ denote the set of all classes in E(G) which contain supercuspidal representations. Obviously ${}^oE(G) \subset E_2(G)$. Consequently, supercuspidal representations have a character.

§ 3. A theorem.

Let G be a reductive $\mathfrak{p}$-adic group with split component Z. For $\chi \in \hat{Z}$ we denote by $C_c^\infty(G,\chi)$ the space of locally constant complex-valued functions f on G with compact support, satisfying

$$f(xz) = f(x)\chi(z) \qquad (x \in G\ ,\ z \in Z)\ .$$

By ${}^0C_c^\infty(G,\chi)$ we mean the space of supercusp forms in $C_c^\infty(G,\chi)$ Given two cuspidal pairs (P_i,A_i) $(i = 1, 2)$ in G, we write $(P_1,A_1) \prec (P_2,A_2)$ if $P_1 \subset P_2$ and $A_1 \supset A_2$. A cuspidal pair is called mincuspidal if it is minimal with respect to this partial order.

Let (P,A) be a cuspidal pair in G, $P = MN$ the corresponding Levi decomposition. Let f be a continuous complex-valued function with compact support. We define

$$f^P \sim 0 \quad \text{if} \quad \int_M f^P(xm)\overline{\varphi(m)}\, dm = 0 \quad (x \in G)$$

for all $\varphi \in {}^0C_c^\infty(M,\chi)$ and all $\chi \in \hat{A}$.

Suppose P is given. Then this definition is independent of the choice of A and the normalization of the occuring Haar measures.

LEMMA 1.- _Let_ (P,A) _be a cuspidal pair in_ G, $P = MN$ _the corresponding Levi decomposition. There is a one-to-one correspondence between cuspidal pairs_ $({}^*P,{}^*A)$ _in_ M _and those cuspidal pairs_ (P',A') _in_ G _for which_ $(P',A') \prec (P,A)$. _The correspondence is given as follows_ :

$${}^*P = {}^*M{}^*N\ ,\quad P' = M'N'\ ,\quad M' = {}^*M\ ,\quad A' = {}^*A\ ,\quad N' = {}^*NN\ ,\quad {}^*N = N' \cap M\ .$$

THEOREM 3.- _Let_ f _be a continuous complex-valued function on_ G, _with compact support. Suppose_ $f^P \sim 0$ _for all_ P _(including_ $P = G$). _Then_ $f = 0$.

The proof rests upon Lemma 1 and is very similar to the case of reductive groups over a finite field (cf. [2], C, § 3).

§ 4. A relation between supercusp forms and supercuspidal reprentations.

In order to state the results of this paragraph, we have to recall a recent theorem of Bruhat and Tits. It is of great importance in the proofs as well. We refer to [10].

Let (P_i, A_i) $(i = 1, 2)$ be two cuspidal pairs in G. Let $w(A_1, A_2)$ denote the set of all bijections $s : A_1 \to A_2$ with the following property. There is an element $y \in G$ such that $a^s = a^y$ $(= yay^{-1})$ for all $a \in A_1$. It is known that $w(A_1, A_2)$ is a finite set. Fix $s \in w(A_1, A_2)$. We say that $y \in G$ is a representative of s in G if $a^s = a^y$ for all $a \in A_1$. In case $A_1 = A_2 = A$ we write $w(A)$ instead of $w(A,A)$. Then $w(A)$ is a finite group. Let (P_o, A_o) be a mincuspidal pair in G and $P_o = M_oN_o$ the corresponding Levi decomposition. For any root α of (P_o, A_o), let ξ_α be the corresponding character of A_o. Let A_o^+ be the set of all points $a \in A_o$ where $|\xi_\alpha(a)| \geq 1$ for every root α of (P_o, A_o).

THEOREM (Bruhat-Tits).- _We can choose an open and compact subgroup_ K _of_ G _with the following properties._

(i) $G = KP_o$.

(ii) $G = KA_o^+ \omega_{M_o} K$, _where_ ω_{M_o} _is a finite subset of_ M_o.

(iii) _Every element of_ $w(A_o)$ _has a representative in_ K.

(iv) _If_ $(P,A) \succ (P_o, A_o)$ _is a cuspidal pair and_ $P = MN$ _the corresponding Levi decomposition, then_ $P \cap K = (M \cap K)(N \cap K)$.

(v) _Put_ $K_M = K \cap M$ _and_ ${}^*P_o = M \cap P_o$. _If we replace_ (G, P_o, A_o, K) _by_ $(M, {}^*P_o, A_o, K_M)$, _the above four conditions are again fulfilled._

Let P be any parabolic subgroup of G. There is a minimal parabolic subgroup contained in P. Since the minimal parabolic subgroups are Ω-conjugate, we can find an element k in G, and by (i) even in K, such that $P^k \supset P_o$. We then obviously have $G = KP$. Moreover, we can choose a split component A of P in such a way that $(P^k, A^k) \succ (P_o, A_o)$. Let $P = MN$ be the corresponding Levi decomposition. Then $P^k = M^kN^k$ and by (iv), $P^k \cap K = (M^k \cap K)(N^k \cap K)$, hence $P \cap K = (M \cap K)(N \cap K)$.

Let F be a finite subset of $E(K)$. Define

$$\alpha_F = \sum_{\underline{d} \in F} \alpha_{\underline{d}}$$

where $\alpha_{\underline{d}}(k) = d(\underline{d})\ \mathrm{tr}\ \underline{d}(k) \quad (k \in K)$, $\alpha_{\underline{d}}(x) = 0$ if $x \in {}^cK$. Then $\alpha_F \in C_c^\infty(G)$. By $C_c(G,\alpha_F)$ we denote the convolution algebra of the complex-valued continuous functions f with compact support, satisfying $\alpha_F * f * \alpha_F = f$. Furthermore, put ${}^oC_c(G,\chi,\alpha_F) = {}^oC_c^\infty(G,\chi) \cap C_c(G,\alpha_F)$, where $\chi \in \hat{Z}$. We have the following important lemma.

LEMMA 2.- <u>The elements of</u> ${}^oC_c(G,\chi,\alpha_F)$ <u>are Hecke-finite</u>: <u>Fix</u> $f \in {}^oC_c(G,\chi,\alpha_F)$. <u>Let</u> J_f <u>be the space spanned by all functions of the form</u> $\alpha * f * \beta \quad (\alpha, \beta \in C_c(G,\alpha_F))$. <u>Then</u> $\dim J_f < \infty$.

The following theorem has a real analogue. The proofs are mutatis mutandis the same (cf. [4(a)], lemma 77). The main burden of the proof rests upon Lemma 2.

Let $L_2(G,\chi)$ denote the completion of $C_c^\infty(G,\chi)$ with respect to the norm $\|f\| = \{\int_{G/Z} |f(x)|^2\, dx^*\}^{\frac{1}{2}} \quad (f \in C_c^\infty(G,\chi))$. Let λ be the left-translation on $L_2(G,\chi)$: $\lambda(x)f(t) = f(x^{-1}t) \quad (x, t \in G)$. Then λ is a continuous unitary $\bar{\chi}$-representation of G on the Hilbert space $L_2(G,\chi)$, which we call the left regular representation of G on $L_2(G,\chi)$.

THEOREM 4.- <u>Fix</u> $f \in {}^oC_c^\infty(G,\bar{\chi})$, $f \neq 0$. <u>Let</u> λ <u>denote the left regular representation of</u> G <u>on</u> $L_2(G,\bar{\chi})$. <u>Let</u> $\mathcal{H}_f$ <u>be the smallest closed subspace of</u> $\mathcal{H} = L_2(G,\bar{\chi})$, <u>which is stable under</u> λ <u>and which contains</u> f . <u>Then there exists</u> $r \in \mathbb{N}$ <u>such that</u> $\mathcal{H}_f = \sum_{1 \le i \le r} V_i$, <u>where</u> V_i <u>are closed, mutually orthogonal</u> λ-<u>stable irreducible subspaces of</u> $\mathcal{H}$. <u>Let</u> λ_i = <u>restriction of</u> λ <u>on</u> V_i . <u>Then</u> λ_i <u>is a supercuspidal</u> χ-<u>representation</u>.

COROLLARY.- <u>Let</u> $f \in {}^oC_c^\infty(G,\chi)$. <u>Then</u> f <u>is a finite sum of matrix-coefficients of supercuspidal</u> χ-<u>representations</u>.

§ 5. Existence of characters in general.

Let (P,A) be a cuspidal pair in G with corresponding Levi decomposition $P = MN$. Let ρ be an irreducible unitary representation of M on a Hilbert space V, whose class belongs to ${}^{o}\mathcal{E}(M)$. Extend ρ to a representation of P by putting $\rho(p) = \rho(mn)$ $(p \in P\ ;\ m \in M\ ,\ n \in N\ ;\ p = mn)$. Denote by π the unitary representation of G induced by ρ in the sense of Mackey. The definition is as follows. Let H be the space of all continuous functions $f : G \to V$ with compact support mod P and satisfying the following condition :

$$f(xp) = \Delta_P^{\frac{1}{2}}(p)\ \rho(p^{-1})f(x) \qquad (x \in G\ ,\ p \in P)$$

where $\Delta_P(p)$ is given by the relation $d_\ell(qp) = \Delta_p(p)d_\ell q$, $d_\ell q$ being a left Haar measure of P. We provide H with the scalar product

$$(f_1, f_2) = \int_K \langle f_1(k)\ ,\ f_2(k)\rangle\ dk$$

(K given by Bruhat-Tits). Let $\mathcal{H}$ be the completion of H w.r.t. the norm of this scalar product. Then π is the (continuous) unitary representation of G on $\mathcal{H}$, given by

$$\pi(x)f(y) = f(x^{-1}y) \qquad (f \in \mathcal{H}\ ;\ x\ ,\ y \in G)\ .$$

The set of all π, obtained by this method, is a complete set of representations : let f be a continuous complex-valued function with compact support and suppose

$$\pi(f) = \int_G f(x)\ \pi(x)\ dx = 0$$

for all π, then $f = 0$. The main tool in the proof is the corollary of Theorem 4.

Observe that it suffices to consider standard parabolic subgroups (w.r.t. some mincuspidal pair).

<u>Assumption</u>.- <u>Let</u> G <u>be a reductive</u> $\mathfrak{p}$-<u>adic group</u>. <u>Let</u> K <u>be an open compact subgroup of</u> G <u>given by Bruhat and Tits</u>. <u>Then</u>

$$\sup_{\omega \in {}^{o}\mathcal{E}(G)} [\omega:\underline{d}] < \infty$$

<u>for all</u> $\underline{d} \in \mathcal{E}(K)$.

Now fix $\underline{d} \in \mathcal{E}(K)$. Let π be as above. Denote by $[\pi:\underline{d}]$ the multiplicity of $\underline{d}$ in the restriction of π to K. Then it is easily checked that $[\pi:\underline{d}]$ is bounded if π runs over the complete set of representations introduced above. By a wellknown result of Godement ([3], Lemma 4) this yields :

$$\sup_{\omega \in E(G)} [\omega:\underline{d}] < \infty .$$

It is clear that this result implies the existence of the character for every $\omega \in E(G)$.

At this point we take the opportunity to state a conjecture. Keeping in mind the corollary of Theorem 1, the above assumption leads naturally to the following :

<u>Conjecture</u> $\qquad \inf_{\omega \in {}^{O}E(G)} d(\omega) > 0 .$

In view of results of Shalika for $SL(2,\Omega)$ (cf. [8]), one is inclined to believe that, at least for semisimple G , the "formal degrees" $d(\omega)$ $(\omega \in {}^{O}E(G))$ are integers, up to a constant depending on the choice of the Haar measures.

As shown by Shalika in case the residual characteristic of Ω is not 2 , every $\pi \in {}^{O}E(G)$ is induced by an irreducible representation τ of some maximal compact subgroup of G $(G = SL(2,\Omega))$. Assuming this, it is an easy exercise to prove that $d(\omega) = d(\underline{d})$, where $\underline{d}$ is the class of τ , provided the Haar measure on G is so normalized that the total measure of K is one. The general case is however rather misty and no general result in this direction has been obtained.

§ 6. <u>Characters are functions.</u>

Let π be a supercuspidal representation of G . Define $\theta_\pi(f) = \mathrm{tr}\, \pi(f)$ $(f \in C_c^\infty(G))$.

Let $\ell = \mathrm{rank}\, \mathbb{G}$. Denote by $D(x)$ $(x \in G)$ the coefficient of t^ℓ in the polynomial $\det(t + 1 - Ad(x))$ in the indeterminate t . We call x <u>regular</u> if $D(x) \neq 0$. Denote by G' the set of regular elements of G . Then G' is an open, invariant and dense subset of G , whose complement is of Haar measure zero. We have the following theorem.

THEOREM 5.- Let π be a supercuspidal representation of G . There exists a function F_π on G with the following properties.

(i) F_π is locally constant on G' and

$$\theta_\pi(f) = \int_G f(x)F_\pi(x)dx$$

for all $f \in C_c^\infty(G)$ with $\operatorname{Supp} f \subset G'$.

(ii) If $\operatorname{char} \Omega = 0$, then in addition F_π is locally summable on G and

$$\theta_\pi(f) = \int_G f(x)F_\pi(x)dx \qquad (f \in C_c^\infty(G)) .$$

The starting point for the proof of the above result is the following theorem.

THEOREM 6.- Let G be a reductive $\mathfrak{p}$-adic group. Let $\omega \in E_2(G)$. Fix $\pi \in \omega$ and denote by $\mathcal{H}_\pi$ the space of π . Then for any $f \in C_c^\infty(G)$,

$$\operatorname{tr} \pi(f)(\varphi,\psi) = d(\omega) \int_{G/Z} dx^* \int_G f(y)(\varphi,\pi(y^x)\psi)dy$$

for all $\varphi , \psi \in \mathcal{H}_\pi$.

This is an easy consequence of the Schur orthogonality relations. Now let $\omega \in {}^oE(G)$ and fix $\pi \in \omega$. Choose a unit vector $\varphi \in \mathcal{H}_\pi$, which is left fixed by some open subgroup of G , and define

$$\theta(x) = d(\pi)(\varphi,\pi(x)\varphi) \qquad (x \in G) .$$

Then θ is a supercusp form and

$$\theta_\pi(f) = \int_{G/Z} dx^* \int_G f(y)\ \theta(y^x)dy \qquad (f \in C_c^\infty(G)) .$$

The idea of the proof of Theorem 5 is clear : one has to interchange integrals.

By a Cartan subgroup of G we mean a subgroup of the form $\Gamma = \boldsymbol{\Gamma} \cap G$, where $\boldsymbol{\Gamma}$ is a maximal Ω-torus in $\mathbb{G}$. Put $\Gamma' = \Gamma \cap G'$. Then Γ' is open and dense in Γ . Put $G_\Gamma = (\Gamma')^G$; G_Γ is open, G-invariant and $G = \cup_\Gamma G_\Gamma$. In the proof of the first part of Theorem 5 (which is valid in arbitrary characteristic) we reduce the problem to Cartan subgroups Γ : the restriction of θ_π to G_Γ is a G-invariant distribution and gives rise to a unique distribution σ_Γ on Γ' . This is not difficult. The basic lemma then reads as follows.

LEMMA 3.- Let K be any open compact subgroup of G . For any compact subset ω_Γ of Γ' , there exists a compact set $\omega \subset G$ such that

$$\int_K \theta(\gamma^{xk})dk = 0$$

for all $\gamma \in \omega_\Gamma$, unless $x \in \omega Z$.

The proof is by contradiction and involves the fact that we are dealing with a supercusp form.

The lemma enables us to conclude that σ_Γ is a function :

$$\sigma_\Gamma(\beta) = \int_\Gamma \beta(\gamma) F_\Gamma(\gamma)d\gamma \qquad (\beta \in C_c^\infty(\Gamma'))$$

where $F_\Gamma(\gamma) = \int_{G/Z} dx^* \int_K \theta(\gamma^{xk})dx \quad (\gamma \in \Gamma')$. The definition of F_Γ is independent of the choice of K . Now let $\gamma \in G'$. Choose a Cartan subgroup Γ such that $\gamma \in \Gamma'$. Γ is unique. Define $F_\pi(\gamma) = F_\Gamma(\gamma)$. It is an easy exercise to show that F_π satisfies the conditions of the first part.

The proof of the second part works only in characteristic zero. This is mainly due to the fact that we often turn to the Lie algebra $\mathfrak{g}$ of G , while the mapping "exp" behaves badly in positive characteristic. The idea of the proof is due to Jacquet and Langlands ([5], § 7). They give a similar proof in case $G = GL(2,\Omega)$ (without any restriction on char Ω). It consists of cutting off the integration over G/Z in the formula

$$\theta_\pi(f) = \int_{G/Z} dx^* \int_G f(y)\, \theta(y^x)dy \qquad (f \in C_c^\infty(G))$$

and giving estimates for the pieces to be able to apply Lebesgue's Theorem on dominated convergence.

For any $a \in \Omega$, we define its absolute value $|a|$ in the usual way so that $d(at) = |a|dt$, where dt is a Haar measure on the additive group of Ω . Let q be the number of elements of the residue class field of Ω . Define $\lambda(x)$ by $q^{\lambda(x)} = |D(x)| \quad (x \in G')$.

Let $M_n(\Omega)$ denote the algebra of all $n \times n$ matrices with coefficients in Ω . Put $|x| = \max_{i,j} |x_{ij}| \quad (x = (x_{ij}) \in M_n(\Omega))$. For $x \in GL(n,\Omega)$, define

$$\|x\| = \max(|x|,|x|^{-1}) .$$

Then $\|xy\| \leq \|x\| . \|y\|$ and $\|x\| \geq 1$. Let $\sigma : GL(n,\Omega) \to \mathbb{N}$ be given by $q^{\sigma(x)} = \|x\|$.

For any $T \geq 1$, let Ω_T denote the set of all $x \in G$ such that $1 + \sigma(x) \leq T$. Denote by Ω_T^* the image of Ω_T under the canonical projection $G \to G^* = G/Z$ and write Φ_T for its characteristic function. Then

$$\theta_\pi(f) = \lim_{T \to \infty} \int_{G/Z} \Phi_T(x^*)dx^* \int_G f(y)\theta(y^x)dy$$

$$= \lim_{T \to \infty} \int_{G/Z} f(y)\theta_T(y)\, dy \qquad (f \in C_c^\infty(G))$$

where $\theta_T(y) = \int_{G/Z} \Phi_T(x)\theta(y^x)dx^*$ $(y \in G)$.

We want estimates for θ_T . As above, we restrict θ_T to $G_\Gamma = (\Gamma')^G$, Γ being any Cartan subgroup of G . Let A denote the maximal split torus of Γ .

Fix a compact set ω in G such that $\mathrm{Supp}\ \theta \subset \omega Z$, $\mathrm{Supp}\ f \subset \omega$. Let ω_Γ be the set of all $\gamma \in \Gamma$ such that $\gamma^x \in \omega Z$ for some $x \in G$. Then an easy observation shows that ω_Γ is relatively compact mod Z . The first estimate is the following :

$$|\theta_T(\gamma^y)| \leq c.(1 + |\lambda(\gamma)|)^{4\ell} \int_{G/A} |\theta(\gamma^x)|d\bar{x} \quad (\gamma \in \omega_\Gamma \cap G')$$

where $\ell = \dim A/Z$ and c is a positive constant. Moreover we have

$$\lim_{T \to \infty} \theta_T(x) = F_\pi(x) \qquad \text{for } x \in \omega \cap G' .$$

This needs considerable preparation, too much to explain here, and involves a refinement of Lemma 3 : the compact set ω of Lemma 3 must be known in terms of $\gamma \in \omega_\Gamma$. This can be realized since we are in $\mathrm{char}\ \Omega = 0$.

Let Γ be as above. Let h be a locally constant function with compact support mod Z . We define

$$F_h(\gamma) = |D(\gamma)|^{\frac{1}{2}} \int_{G/Z} h(\gamma^x)d\bar{x} \qquad (\gamma \in \Gamma') .$$

This integral exists.

THEOREM 7.- (char $\Omega = 0$). Let Γ be any Cartan subgroup of G and let ω_Γ be a compact subset of Γ. Then

$$\sup_{\gamma \in \omega_\Gamma \cap G'} |F_h(\gamma)| < \infty$$

for all locally constant functions h with compact support mod Z.

The proof is very long and by no means trivial. It is mainly carried out on the Lie algebra of G.

By Theorem 7 we obtain a constant $c_1 > 0$ such that

$$|\theta_T(\gamma^y)| \leq c_1 . |D(\gamma)|^{-\frac{1}{2}} (1 + |\lambda(\gamma)|)^{4\ell}$$

for all $\gamma \in \Gamma$ and $y \in G$ such that $\gamma^y \in \omega \cap G'$. Since there are only finitely many non-conjugate Cartan subgroups in G, there exists a constant $c_2 > 0$ such that

$$|\theta_T(x)| \leq c_2 |D(x)|^{-\frac{1}{2}} (1 + |\lambda(x)|)^{4\ell}$$

for all $x \in \omega \cap G'$ and all $T \geq 1$.

THEOREM 8.- (char $\Omega = 0$). There exists $\varepsilon > 0$ such that the function $x \longmapsto |D(x)|^{-\frac{1}{2}-\varepsilon}$ is locally summable with respect to the Haar measure on G.

First of all, the proof is reduced to the case of semisimple groups. Let Γ be a Cartan subgroup of G. Let dx, $d\gamma$ denote the Haar measures on G and Γ respectively. Let $\tilde{\Gamma}$ be the normalizer of Γ in G and define $W_\Gamma = \tilde{\Gamma}/\Gamma$. W_Γ is a finite group with $[W_\Gamma]$ elements.

LEMMA 4.- Let $d\bar{x}$ be the invariant measure on $\bar{G} = G/\Gamma$ such that $dx = d\bar{x}d\gamma$. Then

$$\int_{G_\Gamma} f(x)dx = [W_\Gamma]^{-1} \int_\Gamma |D(\gamma)| d\gamma \int_{G/\Gamma} f(\gamma^x) d\bar{x} \qquad (f \in C_c^\infty(G_\Gamma)) .$$

By means of this Lemma the proof of Theorem 8 is reduced to :

LEMMA 5.- Let Γ be a Cartan subgroup of G. There exists $\varepsilon > 0$ such that $\gamma \longmapsto |D(\gamma)|^{-\varepsilon}$ is locally summable on Γ.

This is proved, by going over to the Lie algebra of Γ, with induction on the dimension of $\mathfrak{g}$.

By Theorem 8, the function

$$x \mapsto |D(x)|^{-\frac{1}{2}}(1 + |\lambda(x)|)^{4\ell}$$

is locally summable on G .

By Lebesgue's Theorem we have now

(i) F_π is locally summable on G ,

(ii) $\theta_\pi(f) = \lim_{T \to \infty} \int_G f(y)\, \theta_T(y)dy = \int_G f(y)F_\pi(y)dy$.

§ 7. Some consequences.

We assume $\operatorname{char} \Omega = 0$. Let G be as defined in § 1. A Cartan subgroup Γ of G is called elliptic if Γ/Z is compact. Such subgroups exist (cf. [6], § 15).

The Selberg principle. Let Γ be any Cartan subgroup of G . Let θ be a supercusp form. Then

$$\int_{G/\Gamma} \theta(\gamma^x)d\bar{x} = 0 \qquad (\gamma \in \Gamma')$$

unless Γ is elliptic.

The proof is easy.

Let $B_1,\dots,B_r$ be a maximal set of non-conjugate elliptic Cartan subgroups of G . As usual, put $G_{B_i} = (B_i')^G$ $(1 \le i \le r)$. Define

$$G_e = \bigcup_{1 \le i \le r} G_{B_i} .$$

Then G_e is an open subset of G . We call it the elliptic set. We normalize the Haar measures d_ib^* on B_i/Z in such a way that

$$\int_{B_i/Z} d_ib^* = 1 \qquad (1 \le i \le r) .$$

Notice that in the real case r is at most 1 . In the $\mathfrak{p}$-adic case r can be larger than 1 as already the group $G = GL(2)$ shows, where $r = 3$. For $\omega \in {}^0E(G)$, denote by θ_ω the character of ω . Fix $\chi \in \hat{Z}$. For $\omega \in {}^0E(G,\chi)$, put

$$\Phi_{\omega,B_i}(b) = |D(b)|^{\frac{1}{2}}\, \theta_\omega(b) \qquad (b \in B_i' ;\ 1 \le i \le r) .$$

THEOREM 9 (cf. [4(a)], Corollary 1 of Lemma 81 and [5], § 15).- Let $\omega_1, \omega_2 \in {}^{0}E(G,\chi)$. Then

$$\sum_{1 \le i \le r} [W_{B_i}]^{-1} \int_{B_i/Z} \overline{\Phi_{\omega_1,B_i}(b)}\, \Phi_{\omega_2,B_i}(b)\, d_i b^* = \begin{cases} 0 & \text{if } \omega_1 \ne \omega_2 \\ 1 & \text{if } \omega_1 = \omega_2 . \end{cases}$$

The proof of the orthogonality relations rests upon Theorem 5, the Selberg principle and Lemma 4.

COROLLARY.- Fix $\chi \in \hat{Z}$ and let

$$\theta = c_1\theta_{\omega_1} + \dots + c_n\theta_{\omega_n}$$

where c_i are complex numbers and $\omega_1, \dots, \omega_n \in {}^{0}E(G,\chi)$. If $\theta = 0$ on G_e, then $\theta = 0$.

§ 8. A conjecture.

If one tries to prove the results of [4(a)], Part II, for $\mathfrak{p}$-adic groups, one is naturally lead to the following problem. It also appears in Harish-Chandra's theory of Eisenstein integrals [4(b)].

Let V be a finite-dimensional complex Hilbert space and let τ be a unitary representation of K (given by Bruhat-Tits) on V. Denote by $H(\tau)$ the convolution algebra of the mappings $\beta : G \to \mathrm{End}(V)$ with compact support, satisfying

$$\beta(k_1 x k_2) = \tau(k_1)\beta(x)\tau(k_2) \qquad (k_1, k_2 \in K,\ x \in G).$$

Let (P,A) be any cuspidal pair in G, $P = MN$ the corresponding Levi decomposition. Put

$$V_P = \text{subspace of } v \in V \text{ such that } \tau(n)v = v \text{ for all } n \in N \cap K .$$

Let E_P be the orthogonal projection of V on V_P. Put

$$\tau_M(m) = \tau(m)E_P \qquad (m \in K \cap M) .$$

Then τ_M may be regarded as a representation of $K \cap M$ on V_P and so we can consider the algebra $H(\tau_M)$. Define Δ_P as in § 5. For $\beta \in H(\tau)$, put

$$\beta^{(P)}(m) = \Delta_P(m)^{-\frac{1}{2}} \int_N \beta(mn)\,dn \qquad (m \in M) .$$

Then $\beta^{(P)}(m) = E_P\beta^{(P)}(m)E_P$ for all $m \in M$. Moreover $\beta^{(P)} \in H(\tau_M)$. The mapping $\mu_P : \beta \mapsto \beta^{(P)}$ is actually a homomorphism of $H(\tau)$ into $H(\tau_M)$.

Conjecture (Problem). $H(\tau_M)$ is a finite right-module over $\mu_P(H(\tau))$, i.e. there exist $p \geq 1$ and $\alpha_1,\ldots,\alpha_p \in H(\tau_M)$ such that

$$H(\tau_M) = \sum_{1 \leq i \leq p} \alpha_i * \mu_P(H(\tau)) .$$

It is obviouly enough to state the conjecture for standard parabolic pairs with respect to some mincuspidal pair (P_o,A_o) . Satake has proved the conjecture in case char $\Omega = 0$, $\mathbf{G}$ is simply connected, τ = identity representation of K and P is minimal. In this case $H(\tau_M)$ is a free module of finite rank over $\mu_P(H(\tau))$ (cf. [9], ch. II).