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A LOWER BOUND FOR THE ZEROS OF RIEMANN'S ZETA FUNCTION ON THE CRITICAL LINE

[following N. LEVINSON]

by Enrico BOMBIERI

I. Introduction

Let $N(T)$ be the number of zeros of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, in $0 < t \leq T$, and let $N_0(T)$ be the number of such zeros with real part $\sigma = \frac{1}{2}$. It is well-known that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

and the Riemann hypothesis says that we have $N_0(T) = N(T)$.

It was proved for the first time by Hardy [1] in 1914 that $\zeta(s)$ has infinitely many zeros on the critical line $\sigma = \frac{1}{2}$, thus

$$N_0(T) \to \infty \qquad \text{as } T \to \infty .$$

Hardy's qualitative result was given a quantitative form

$$N_0(T) > A\,T$$

for some $A > 0$, by Hardy and Littlewood [2] in 1921, and later, with an explicit value of A, the same result was obtained by Siegel [3] in 1932 with a rather different method.

The next essential progress on the problem of getting a lower bound for $N_0(T)$ was done by Selberg [4] in 1942, who succeeded in proving the result

$$N_0(T) > A\,T \log T$$

for some $A > 0$.

Recently, Selberg's result has been put into a quantitative form by Levinson [5] who obtained

THEOREM.- For $T \geq T_0$, we have

$$N_0(T) > (0.34) \frac{T}{2\pi} \log T .$$

While Selberg's method followed the Hardy and Littlewood approach, Levinson's method is close to Siegel's ideas. In both cases the improvement over the previous results has been obtained using Selberg's fundamental idea of the use of "mollifiers" to dampen the oscillations of $|\zeta(\frac{1}{2} + it)|$ on the critical line.

In this exposé, because of the complexity of calculations, I will limit myself to a presentation of the main ideas, referring to the original work for details.

II. The basic idea

Let $h(s) = \pi^{-(s/2)}\Gamma(\frac{s}{2})$. The functional equation for $\zeta(s)$ can be expressed in the form

$h(\frac{1}{2} + it)\zeta(\frac{1}{2} + it)$ is real for real t.

It was shown by Riemann in his unpublished notes (see Siegel [3]), that one has a formula

$$h(s)\zeta(s) = h(s)f(s) + h(1 - s)\bar{f}(1 - s) \tag{1}$$

where $f(s)$ is the entire function

$$f(s) = \frac{1}{2\pi i} \int_L e^{\pi i w^2} w^{-s} \frac{\pi}{\sin \pi w} \, dw ,$$

where L is the line with slope 1 through $w = \frac{1}{2}$ with $\operatorname{Im} w$ decreasing. By moving the contour to the right, one then finds that inside the critical strip $f(s)$ is approximated by a finite sum

$$f(s) = \sum_{n \leq (t/2\pi)^{\frac{1}{2}}} n^{-s} + O(t^{-\sigma/2}) , \tag{2}$$

which gives rise to the so-called approximate functional equation for $\zeta(s)$.

This formula can be used to get information about $N_0(T)$ in the following way (Siegel [3]). Clearly the zeros of $h(s)\zeta(s)$ on the line $\sigma = \frac{1}{2}$ will occur when

$$\operatorname{Re}\,(h(s)f(s)) = 0$$

or in other words when

$$\arg(h(s)f(s)) \equiv \frac{\pi}{2} \pmod{\pi} .$$

If we denote by $\Delta_C \arg$ the variation of the argument from $\frac{1}{2}$ to $\frac{1}{2} + iT$, it will follow easily that

$$N_o(T) \geq \frac{1}{\pi}\Delta_C \arg(h(s)f(s)) - 1$$
$$= \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(1) + \frac{1}{\pi}\Delta_C \arg f(s)$$

since Stirling's formula yields

$$\frac{1}{\pi}\Delta_C \arg h(s) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O\left(\frac{1}{T}\right).$$

Now let R be the closed rectangle with vertices at c, $c+iT$, $\frac{1}{2}+iT$, $\frac{1}{2}$ where $c > 1$. The variation of the argument of $f(s)$ on $\partial R - C$ is easily bounded by $O(\log T)$ and thus it follows that

$$\frac{1}{\pi}\Delta_C \arg h(s) = -2\,N_R(T;f) + O(\log T)$$

where $N_R(T;f)$ is the number of zeros of $f(s)$ in the rectangle R, zeros on ∂R being counted with half multiplicity. Thus

(3) $$N_o(T) \geq \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} - 2N_R(T;f) + O(\log T)$$

and the problem becomes that of obtaining a good upper bound for the number of zeros of $f(s)$ in the rectangle R. It is also clear that our choice of $f(s)$ is not limited to the one appearing in the classical Riemann-Siegel formula ; the only thing we need is (1) and a good bound for $\Delta_{\partial R-C} \arg f(s)$, which is usually not difficult to obtain.

III. The upper bound for $N_R(T;f)$

Here one uses a familiar lemma of Littlewood :

PROPOSITION.- Let R_1 be a rectangle. Then for f regular in R_1, we have

$$\operatorname{Re}\left(\frac{1}{2\pi i}\int_{\partial R_1} \log f(s)\,ds\right) = \sum_{\rho} \text{dist}$$

where the summation is over the zeros of $f(s)$ in R_1 and dist denotes the distance of a zero from the left-side of R_1. The argument of the log is obtained by continuous variation starting at the lower left corner and going counter-clockwise.

If we take for R_1 the rectangle with vertices at c, $c+iT$, $a+iT$, a where $a < \frac{1}{2}$, and use simple estimates on $\log|f(s)|$ or $\arg f(s)$ on the horizontal and right-side of R_1, we easily get a bound

$$\left(\tfrac{1}{2}-a\right)N_R(T,f) \leq \frac{1}{2\pi}\int_0^T \log|f(a+it)|\,dt + O(\log T).$$

The last integral is estimated using a convexity inequality, and the problem is reduced to give good estimates for mean values of the type

$$\frac{1}{T}\int_0^T |f(a+it)|^2 \frac{dt}{(t+1)^b}$$

for suitable b . If however we work on this suggestion, for example with the function $f(s)$ in the Riemann-Siegel formula, the final result will be only

$$N_0(T) \geq \frac{T}{2\pi e} + o(T) ,$$

which is achieved only if we work with asymptotic estimates.

The new idea that has to be introduced at this stage is due to Selberg. If we note that

$$N_R(T,f) \leq N_R(T,\psi f)$$

for every ψ regular in the rectangle R , we may as well estimate the integral

$$\int_0^T \log|\psi f|dt \qquad \text{instead of} \qquad \int_0^T \log|f|dt ,$$

and this in turn means that we will have to deal with mean values

$$\frac{1}{T}\int_0^T |\psi(a+it)f(a+it)|^2 \frac{dt}{(t+1)^b} ,$$

and we may hope to choose ψ so that the integral will be made much smaller. Of course, we have to check that the introduction of the mollifier ψ will not change the evaluation of $\int \log(\psi f)\, dt$ on the horizontal and right-side of R_1 , and in order to this it is sufficient to have ψ near to 1 on the right side of R_1 , and have ψ of polynomial growth in the critical strip. Hence a finite Dirichlet polynomial

$$\psi(s) = \sum_1^X \frac{b_n}{n^s} ,$$

with $b_1 = 1$, $|b_n| = O(1)$, $X = O(T^A)$, will be an admissible choice.

IV. The final estimate

We start with (1) taking for $f(s)$ a function which has an asymptotic expansion

$$f(s) \sim \sum_1^X \frac{a_n}{n^s}$$

in the critical strip. Here X is not too large (in the final choice, $X \sim \frac{T}{2\pi}$)

and a_n is near to 1 , say

$$a_n = 1 - \frac{\log n}{\log X} \ ;$$

this is essentially Levinson's choice. Since we want ψf to be "small", we take for ψ an approximate inverse of $f(s)$, and noting that for $\sigma > 1$ we have

$$\left(\sum_1^\infty \frac{1}{n^s}\right)^{-1} = \sum_1^\infty \frac{\mu(n)}{n^s}$$

where $\mu(n)$ is the Möbius function, it is natural to choose

$$\psi(s) = \sum_1^Y \frac{b_n}{n^s}$$

where

$$b_n = \mu(n)\left(1 - \frac{\log n}{\log Y}\right) n^{-(\frac{1}{2}-\sigma)} .$$

Here the factor $1 - \dfrac{\log n}{\log Y}$ appears for "smoothing" reasons, while the factor $n^{-(\frac{1}{2}-\sigma)}$ is essentially 1 since we will work on $\sigma = a$ with $a = \frac{1}{2} - \dfrac{\lambda}{\log T}$, λ bounded. The final choice of the parameter Y is somewhat smaller than $T^{\frac{1}{2}}$.

It remains to estimate

$$(4)\quad (\tfrac{1}{2} - a)\, 2N_R(T,f) \leq \frac{1}{\pi}\int_0^T \log|\psi f|\,dt + O(\log T)$$
$$\leq \frac{T}{2\pi}\log\left(\frac{1}{T}\int_0^T |\psi f(a+it)|^2\,dt\right) + O(\log T) .$$

Levinson's result is :

<u>Lemma</u>.- If $a = \frac{1}{2} - \dfrac{\lambda}{\log T}$ then

$$\frac{1}{T}\int_0^T |\psi f(a+it)|^2\,dt \sim F(\lambda)$$

where

$$F(\lambda) = e^{2\lambda}\left(\frac{1}{2\lambda^3} + \frac{1}{24\lambda}\right) - \frac{1}{2\lambda^3} - \frac{1}{\lambda^2} - \frac{25}{24\lambda} + \frac{7}{12} - \frac{\lambda}{12} .$$

It should be pointed out at this stage of the argument that without the introduction of the mollifier ψ one would have obtained on estimate which would have been larger by a factor $\log T$, and this would have been of no use for our purposes. It is also worthwhile to remark that in Selberg's proof of $N_0(T) > A\,T\log T$ it is only the gain of the factor $\log T$ which matters, while in Levinson's proof one should also end up with a not too large value for $F(\lambda)$. Most of the difficulties in the proof arise from the fact that we want to compute $F(\lambda)$ explicity.

Using Levinson's Lemma and the basic inequalities (3) and (4), we find

$$N_0(T) \geq [1 - \frac{1}{\lambda} \log F(\lambda) + o(1)] \frac{T}{2\pi} \log T$$

for every fixed $\lambda > 0$. We choose $\lambda = 1.2869$ and find

$$N_0(T) \geq (0.34276) \frac{T}{2\pi} \log T$$

for $T \geq T_0$, as we wanted.

We conclude with the remark that, while it is certainly possible to improve Levinson's explicit estimate by using other choices of functions $f(s)$ and mollifiers $\psi(s)$, it seems very difficult to reach the asymptotic result $N_0(T) \sim \frac{T}{2\pi} \log T$. The main reason seems to be in the method we have used to estimate $N_R(T,f)$, using Littlewood's lemma and a convexity inequality for the estimation of $\int \log|f(s)|dt$. Here an interesting point arises. In fact, the success of the introduction of a mollifier shows that something has been lost in the convexity estimate, probably only partially restored by the factor ψ. On the other hand, if we deal with functions which like $\zeta(s)$ satisfy functional equations, but have no Euler products, while it is still possible to show that $N_0(T) > A\,T$, we have no reason to expect that $N_0(T) > A\,T \log T$ in this case. The existence of an Euler product implies the existence of a good mollifier $\psi(s)$, so that the proof of $N_0(T) > A\,T \log T$ does make use of the arithmetic properties of $\zeta(s)$, though in an imperfect way. A more direct study of the zeros of the functions $f(s)$ arising in the Riemann-Siegel formula (1) is needed in order to reach essentially better results.

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