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COHERENCE OF 3-MANIFOLD FUNDAMENTAL GROUPS

by John STALLINGS

G. P. Scott [3] proved that fundamental groups of 3-dimensional manifolds, closed or not, possess a remarkable quality : every finitely generated subgroup is finitely presented. We call this quality coherence. It should be remarked that P. Shalen also discovered this result independently, and that G. A. Swarup [5] had an important partial result along these lines.

To delineate the nature of coherence, we note that free groups and fundamental groups of 2-manifolds are coherent : if A is such a group, then any finitely generated subgroup is again such a group and hence finitely presented. Incoherent groups can be constructed using Higman's embedding theorem [1].

As a more explicit example, the direct product of two non-cyclic free groups is incoherent :

Let $A = \{a, b\} \times \{c, d\}$ be the direct product of two free groups of rank two. Consider the subgroup N generated by a , c , and bd . It is easy to see that N is a normal subgroup of A , with infinite cyclic quotient group $Z = A/N$. If we consider homology with coefficients the field Q of rational numbers, it turns out that $H_2(N)$ cannot be finitely generated over Q and so N cannot be finitely presented. This is an exercise which can be performed with the help of the spectral sequence of the extension, together with the fact that $H_0(Z; M)$ and $H_1(Z; M)$ have the same rank over Q , if M is a QZ -module which is finitely generated over Q .

It follows from Scott's theorem therefore that this group A cannot be the fundamental group of any 3-manifold.

Scott's proof follows this pattern : reduce to the case of a group S which is freely indecomposable (i.e. indecomposable into a free product). Show that any indecomposable, finitely generated group is the image of an indecompo-

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sable, finitely presented group such that all intermediate groups are indecomposable. Finally, apply some topological ideas due to Swarup to finish the proof.

THEOREM.- Let M be a 3-manifold, and let S be a subgroup of $\pi_1(M)$, such that S can be generated by n elements, $0 \leq n < \infty$. Then S is finitely presentable.

(a) The proof is by induction on n . The theorem is clear if $n = 0$ or 1 . If S can be decomposed into a free product $A * B$ with neither factor trivial, then Grushko's theorem [4] shows that both A and B can be generated by fewer than n elements. Inductively both A and B are finitely presented and thus so is S .

(b) The problem is now narrowed down to the case in which S is indecomposable into a free product, and in which every subgroup of S generated by fewer than n elements is finitely presentable. At this point some very interesting points of group theory come into play.

(c) Let $G = G_1 * G_2 * \dots * G_r * F$

$$H = H_1 * \dots * H_t$$

be free products ; where F is a free group ; let $\varphi : G \rightarrow H$ be an epimorphism such that for $i = 1, \dots, r$, $\varphi(G_i)$ is conjugate to a subgroup of some factor $H_{j(i)}$. Then there is a free decomposition

$$G = K_1 * \dots * K_t$$

such that $\varphi(K_j) = H_j$ for all j .

This improvement on Grushko's theorem (to which it reduces for $r = 0$) from [2] can also be proved topologically along the lines of [4].

(d) Suppose G is a finitely generated group. Then it can be decomposed as a free product

$$G = G_1 * G_2 * \dots * G_r * F_s$$

where F_s is a free group of rank s , and each G_i is neither infinite cyclic

nor decomposable into a free product. The existence of such a decomposition follows from Grushko's theorem ; the Kurosh subgroup theorem implies that the G_i are unique up to order and conjugacy within G , and that s is determined by G . The pair $c(G) = (r + s, s)$ is called the complexity of G ; complexities are compared lexicographically.

If G is as above, a homomorphism $\varphi : G \rightarrow H$ will be called semi-injective if $\varphi|_{G_i}$ is injective for all i . It is easy to see that this notion is independent of the particular decomposition of G , and that if any free decomposition of G is given such that φ is injective on all the non-free factors, then φ is semi-injective.

A major lemma can now be stated :

(e) Let $\varphi : G \rightarrow H$ be a semi-injective epimorphism, where G is as in (d). Then if φ is not an isomorphism, $c(G) > c(H)$.

To see this, factor H into indecomposable factors

$$H = H_1 * \dots * H_t * Z * \dots * Z$$

with u infinite cyclic factors Z . Thus $c(G) = (r + s, s)$ and $c(H) = (t + u, u)$. We suppose G factored as in (d). Then, since $\varphi(G_i)$ is indecomposable, not infinite cyclic, by Kurosh's theorem $\varphi(G_i)$ is conjugate to a subgroup of some H_j . Now apply (c). The result is to factor G into

$$G = K_1 * \dots * K_t * K_{t+1} * \dots * K_{t+u}$$

where each term maps onto the corresponding factor of H ; thus, each factor K_j is non-trivial, and since G can be decomposed into only $r + s$ factors, we find

$$r + s \geq t + u .$$

If $r + s = t + u$, the factors K_j have to be indecomposable. Thus all G_i occur, up to conjugacy, in the list of K_j . If K_j is conjugate to G_i , then by the semi-injective hypothesis $\varphi|_{K_j}$ is one-to-one, and so $\varphi(K_j)$ is not Z . This shows that, if $r + s = t + u$,

$$r \leq t , \quad \text{i.e. } s \geq u .$$

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If $s = u$, then $\varphi|_{K_i} : K_i \approx H_i$ for $i \leq t$, and for $i > t$, $\varphi(K_i) =$ one of the Z factors; with the fact that $K_i = Z$ for $i > t$, we would have shown that φ is an isomorphism on each K factor into the corresponding H factor, and so φ would be globally an isomorphism. Since we are assuming φ is not an isomorphism, this means that if $r + s = t + u$, then $s > u$; knowing $r + s \geq t + u$, we have proved $(r + s, s) > (t + u, u)$, Q.E.D.

Now we return to the proof of the theorem, recalling the situation at (b).

(f) Let S be generated by n elements and be indecomposable into a free product. Suppose that every subgroup of S generated by fewer than n elements is finitely presented. Then there is an indecomposable finitely presented group H and an epimorphism

$$\varphi : H \twoheadrightarrow S$$

such that if H' is any intermediate group, i.e.

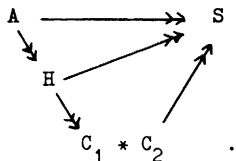
$$\begin{array}{ccc} H & \xrightarrow{\varphi} & S \\ & \searrow & \nearrow \\ & H' & \end{array}$$

then H' is also indecomposable.

To see this, consider the class of all finitely presented groups generated by n elements, which can be mapped onto S by a semi-injective epimorphism. The free group of rank n is in this class, and their complexities constitute a well-ordered set. Let then $A \twoheadrightarrow S$ be a semi-injective epimorphism with A generated by n elements and having minimal complexity. In the uninteresting case that $A \twoheadrightarrow S$ is an isomorphism, we take $H = A \approx S$.

Otherwise let H be A factored by one additional relation ρ in the kernel of $A \twoheadrightarrow S$. Suppose H' is an intermediate group; we shall show that H' is indecomposable. If, to the contrary, $H' = B_1 * B_2$ non-trivially, since H' has n generators, Grushko's theorem implies that B_1 and B_2 have fewer generators. Then the images C_i of B_i in S are, by hypothesis, finitely presented groups, and $H \twoheadrightarrow S$ factors through $C_1 * C_2$. Note that $C_1 * C_2$ is

finitely presented, and that $C_1 * C_2 \twoheadrightarrow S$, being injective on each factor, is semi-injective :



The top arrow is semi-injective, and so $A \twoheadrightarrow C_1 * C_2$ is semi-injective ; and in the case we are considering, the kernel of $A \twoheadrightarrow C_1 * C_2$ contains a non-trivial element ρ . Thus by (e), $c(C_1 * C_2) < c(A)$ contradicting the choice of A . Thus, H' is indecomposable.

(g) If S is a subgroup of $\pi_1(M^3)$, by passing to a covering space, we can arrange to have $S = \pi_1(M)$. By parts (a), (b) and (f), the proof of the theorem will be finished if we can show this :

Let $\pi_1(M) = S$ (a non-cyclic group), and suppose there is a finitely presented group H and an epimorphism $\varphi : H \twoheadrightarrow S$ such that any group intermediate to H and S is freely indecomposable. Then there is a compact submanifold N of M with $\pi_1(N) \approx \pi_1(M)$. (The fundamental group of a compact manifold is always finitely presented.)

To see this, first find a finite 2-dimensional complex K with $\pi_1(K) = H$ and a map $f : K \rightarrow M$ inducing φ . A neighborhood of $f(K)$ in M can be taken to be a compact 3-manifold L with boundary $\text{Bd } L$. If H' is the image of $\pi_1(K)$ in $\pi_1(L)$, then H' is intermediate to H and S and therefore freely indecomposable.

We shall change L by surgery on its boundary within M . Throughout, there will be a subgroup H' of $\pi_1(L)$ which maps by inclusion onto $\pi_1(M) = S$, and which is intermediate to H and S .

Exterior surgery : a 2-cell $D \subset M$ with $D \cap L = \text{Bd } D$ non-contractible in $\text{Bd } L$. Change L to $L \cup (\text{thickened } D)$. H' is changed to the image of the old H' in the new $\pi_1(L)$.

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Interior surgery : a 2-cell $D \subset L$, with $Bd D = D \cap Bd L$ non-contractible in $Bd L$. Change L to $L' = L \setminus (\text{thickened } D)$. This removes a 1-handle from L , and thus $\pi_1(L) = \pi_1(L') * Z$ (*). The old H' is an indecomposable subgroup of $\pi_1(L)$, not cyclic, and so is conjugate to a subgroup of $\pi_1(L')$ which becomes the new H' .

Eventually, $Bd L$ becomes "incompressible" and so, by the theorems of Papakyriakopoulos $\pi_1(L) \rightarrow \pi_1(M)$ is a monomorphism. It is also an epimorphism since H' in $\pi_1(L)$ maps onto $\pi_1(M)$. Let N be this final L . That completes the proof.

(*) Or, $\pi_1(L) = \pi_1(L') * \pi_1(L'')$.

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