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KAM TECHNIQUES IN PDE

by **Ricardo PÉREZ-MARCO**

We present a partial account of recent application of KAM techniques in the context of PDEs. We don't address several other topics in Hamiltonian PDE as for instance those related to invariant Gibbs measures or Nekhoroshev bounds of diffusion for solutions of non-linear PDEs. We adopt a Dynamical Systems point of view (this just reflects the background and motivation of the author).

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1. INTRODUCTION

1.1. Ancient historical motivation

The quasi-periodic motions of the planets in our Solar System was observed long, very long time ago, back when Astronomy, Physics and Mathematics were a single science. There is ample evidence from cuneiform clay tablets of Babylonian observations that go as far as 2000 before the Christian era (see the compilations by O. Neugebauer [NEU1] and also [NEU2]). One can also find traces of some accurate Babylonian observations in Ptolomy's Almagest [PTO]. The Babylonians were using the quasi-periodic evolution to forecast future positions of the planets. From a rich data of observations, it is fairly simple to notice the periodic character of the positions, plus an error which is itself periodic, which in turn has a much smaller periodic error, and so on...

It is revealing that this simple observation remains still unproved! This is certainly the oldest open problem in Mathematics.

1.2. Old and new K.A.M. theory

First came Newton's theory of gravitation. The next progress in the above historical problem was only achieved in the XXth century with the emergence of K.A.M. theory. A.N. Kolmogorov [KOL] discovered the persistence of invariant tori in Hamiltonian systems near completely integrable ones. This was one of the major achievements of Dynamical Systems in the XXth Century. K.A.M. theory, named after its founders Kolmogorov, Arnold and Moser, was developed during the late fifties and sixties. We refer to the comprehensive Bourbaki seminar by J.-B. Bost [BOS] for a survey and bibliography of this classical topic. Later, V.K. Melnikov [MEL] announced the persistence not only of mid-dimensional tori, but also of low dimensional tori. The first proofs appeared only in the late eighties by H. Eliasson [ELI1], J. Pöschel [POS1] and S.B. Kuksin [KUK1]. These results opened the door to the application of KAM techniques to infinite dimensional Hamiltonian systems (the approach of Kuksin is indeed infinite dimensional). These occur naturally in Hamiltonian PDE, some of which appear as perturbations of completely integrable ones. The introduction of KAM techniques in this field showed the existence of quasi-periodic solutions for non-linear and non-integrable PDE.

Unfortunately (or fortunately) there hasn't been any Bourbaki seminar in this topic, so there are indeed several good surveys available. The first book of S. Kuksin [KUK1] and, more recently, the one of W. Craig [CRA] and the second book of Kuksin [KUK3] cover largely the developments of the theory. Other surveys where one can find useful material are [BOU2], [KUK2], [POS2]. For this reason, after a preliminary introduction in the first sections, we concentrate on the more recent results and techniques developed by J. Bourgain, some of which are not yet published [BOU5]. Starting from a technique devised by W. Craig and E. Wayne to find periodic solutions, Bourgain pushed it to get quasi-periodic solutions and indeed a whole new approach to K.A.M. theory. It presents a certain number of advantages, for example, it handles the Schrödinger equation in higher dimension where the difficulties related to arithmetic approximation has stopped any progress for a long time. Recent simplifications using local uniformizations for semi-algebraic sets, and central multiscale arguments (inspired from the theory of the discrete Schrödinger equation) to control the inverse of high dimensional matrices with critical sites, solve these problems and yield a unified approach to classical results, for example those on low dimensional tori. We will illustrate this new approach in this simpler, finite dimensional setting. Applications to PDEs do not differ substantially.

We start with a brief survey on persistence of low dimensional tori in finite dimension, followed by some examples of classical PDE where the techniques have been applied. Next we describe Bourgain's new techniques, following [BOU5], and how they are used to prove the existence of low dimensional tori.

2. LOW DIMENSIONAL TORI

2.1. Introduction to Hamiltonian systems and K.A.M. theory

We consider a Hamiltonian system

$$\dot{q}_k = +\frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

defined in $\mathbf{T}^n \times \mathbf{U} \subset \mathbf{T}^n \times \mathbf{R}^n \subset \mathbf{R}^{2n}$ where $U \subset \mathbf{R}^n$ is a bounded domain. In what follows all Hamiltonian systems are supposed to be real analytic (but most of the results persist with lower regularity).

When the Hamiltonian $H = H_0$ is independent of the angular variables $\mathbf{q}$, the system is completely integrable. In that case, the solutions of the system are periodic or quasi-periodic solutions

$$\mathbf{q}(t) = t\ \lambda + \mathbf{q}(0), \quad \lambda = \Big(\frac{\partial H_0}{\partial p_1}, \dots, \frac{\partial H}{\partial p_n}\Big)$$

fill densely tori contained in the fibers $\{\mathbf{p} = \mathbf{p}(0)\}$. When the frequency vector λ is purely irrational, that is

$$\dim_{\mathbf{Q}}(1, \lambda) = n + 1,$$

then the invariant tori have maximal dimension n.

Many natural systems appear in nature as a small perturbation of completely integrable ones. A fundamental example of completely integrable system in Celestial Mechanics is the two-body problem. The three-body planar with a third small mass, and its different versions (planar, restricted) appear as a perturbation of the completely integrable system. Kolmogorov [KOL] announced in 1954 the persistence of many of these maximal tori for a perturbed system $H = H_0 + \varepsilon H_1$ where ε is small (H_1 may also depend on ε). More precisely, under the "twist" condition

$$\det \left[\frac{\partial^2 H_0}{\partial p_k \partial p_l}\right] \neq 0,$$

any torus with a frequency λ with good arithmetic will persist under a sufficiently small (depending on λ) perturbation. The condition on the arithmetic of the frequency ensures that $\langle \mathbf{k}, \lambda \rangle$ is not two small (modulo 1) depending on the size of $\mathbf{k} \in \mathbf{Z}^n$ and is necessary in this type of Small Divisors problem. The small divisors appear when one attempts to write down and solve the equation of the invariant tori (Moser's approach, see for example [S-M], section 32) for the simpler situation of invariant curves of the annulus) or write down approximate first integrals (Arnold's approach [ARN]).

We point out for later reference that the above Hamiltonian system can be written in a convenient form using complexified variables

$$u_k = p_k + iq_k, \quad v_k = \overline{u}_k = p_k - iq_k\,;$$

considering the real analytic Hamiltonian $H = H(\mathbf{u}, \overline{\mathbf{u}})$ the above equations read

$$i\dot{u}_k = 2\,\frac{\partial H}{\partial \overline{u}_k}.$$

2.2. Low dimensional tori

When the frequency λ is not purely irrational, the solution of the completely integrable system fills a low dimensional torus. These in general are unstable due to its normal degenerate character (nevertheless see [CHE] and [ELI2]). But one may consider the general problem of persistence of low dimensional tori with non-degenerate normal part. In general, mixed hyperbolic and elliptic character may be present in the normal direction (according to when the spectrum of the Hamiltonian vector field in the normal direction has non purely imaginary ($\notin i\mathbf{R}$) or purely imaginary ($\in i\mathbf{R}$) eigenvalues respectively). For a complete account in this situation, and state of the art theorems with minimal twist condition, we refer to the extensive article of H. Rüsmann [RUS]. The main new feature in the persistence of lower dimensional tori is that, in absence of external parameters, the frequencies of the dynamics on the persistent tori cannot be prescribed. At the opposite, these frequencies are employed as parameters to locate the persistent tori.

When the normal behavior is normally hyperbolic, the problem is simpler. The first result of this type is A.M. Liapounov's center theorem [LIA] of preservation of periodic solutions (or tori of dimension 1). For the higher dimensional results see [GRA], [MOS] and [ZEH].

We consider in what follows the purely elliptic situation with tangential frequencies $(\lambda_1, \ldots, \lambda_n)$ (that are used as parameters) and normal frequencies $(\mu_1, \ldots, \mu_m)$, for the linearization of the unperturbed system. Melnikov announced [MEL] during the sixties, and at the end of the eighties Eliasson [ELI1], Kuksin [KUK1], and Pöschel proved the persistence of these tori under the non-resonance conditions,

$$\langle \mathbf{k}, \lambda \rangle - \mu_j \neq 0$$

for all $k \in \mathbf{Z}^n$ and $1 \leqslant j \leqslant m$, and

$$\langle \mathbf{k}, \lambda \rangle + \mu_{j_1} - \mu_{j_2} \neq 0$$

for all $k \in \mathbf{Z}^n$ and $1 \leqslant j_1, j_2 \leqslant m$, $j_1 \neq j_2$. The second condition plays a role in the reduction to the preliminary normal form. J. Bourgain got rid of it using the Lyapunov-Schmidt approach we describe below that does not require this preliminary normal form (at the expense of having to control the inverses of non-diagonal linear operators). Since then J. You [YOU] has obtained an improvement of the original KAM approach by using a different normal form that also yields non-diagonal linearized operators.

Once understood the conditions that are required for the persistence of lower dimensional tori, the techniques were ready to be used for infinite dimensional Hamiltonian systems. In that way Kuksin established the first results for hamiltonian PDE's. His method is exposed in his book [KUK1].

3. SOME HAMILTONIAN PDEs

The methods presented below apply to a wide array of Hamiltonian PDEs. Sometimes part of the difficulty consists in finding a suitable Birkhoff normal form, in which the equation appears as a perturbation of a completely integrable PDE.

– A remarkable and extensively studied completely integrable PDE is Korteweg-de Vries (KDV) equation,

$$\partial_t u = -\partial_{xxx} u + 6u\partial_x u.$$

We refer to the upcoming book of T. Kappeler and J. Pöschel [K-P] for K.A.M. on KDV.

– The non-linear Schrödinger equation (NLS)

$$i\partial_t u - \partial_{xx} u + g(x)u + \varepsilon\partial_{\overline{u}} H(u, \overline{u}) = 0$$

which can be considered in the context of periodic or Dirichlet boundary conditions.

– The non-linear wave equation

$$\partial_{tt} u = \partial_{xx} u + g(x)u + \varepsilon h(x, u)$$

which can also be considered with periodic or Dirichlet boundary conditions.

We refer to chapter 2 of [CRA] for more examples and precisions, and more information on the Hamiltonian character and Birkhoff normal forms of these equations near an equilibrium position.

4. PERSISTENCE OF LOW DIMENSIONAL TORI

We use this problem to illustrate the techniques in [BOU5]. We follow section XVII of [BOU5]. This approach to Melnikov's theorem, without the simplifications of [BOU5], appeared first in [BOU1]. The application to the construction of quasi-periodic solutions of the non-linear Schrödinger equation and non-linear wave equation *in arbitrary dimension* can be found in sections XVIII and XIX of [BOU5] and are based on the same ideas.

4.1. Setup of the problem

The problem of persistence of low dimensional tori can be reduced to the perturbation of a linear Hamiltonian system (see [BOU2] chapter 6) by a standard procedure of writing the system in the appropriate Birkhoff normal form.

Complexifying coordinates we are led to find invariant tori for the perturbed system, $1 \leqslant j \leqslant N$ (real dimension $2N$),

$$\frac{1}{i}\dot{z}_j = \frac{\partial H}{\partial \overline{z}_j}$$

where

$$H(z,\overline{z})=\sum_{j=1}^{N}\lambda_j|z_j|^2+\varepsilon H_1(z,\overline{z})$$

We assume for simplicity that H_1 is independent of ε, but this is irrelevant. It is also assumed that the non-linear perturbation is polynomial. This plays a role on the argument with semi-algebraic sets, but the proof should go through in the general analytic case.

We consider $\lambda=(\lambda_1,\ldots,\lambda_n)$ with $1\leqslant n<N$ and the solution of the unperturbed system,

$$z_j(t)=a_je^{i\lambda_jt}$$

with $a_j=0$ for $n<j\leqslant N$. When λ is purely irrational, this solution fills densely an n-dimensional torus. Note also that in this case we can assume the amplitudes (a_j) to be real and positive (shift t). For a substantial (when $\varepsilon\to 0$) set of values of λ, this torus survives the perturbation, but we may have to change slightly the frequency of the dynamics on it. We express this using the Fourier expansion of the solution. The theorem to prove is the following:

THEOREM. — *Let $\Omega\subset\mathbf{R}^n$. Given $\varepsilon_0>0$ sufficiently small, for each ε, $0<|\varepsilon|<\varepsilon_0$, there is a compact set $\Omega_\varepsilon\subset\Omega$ with $|\Omega-\Omega_\varepsilon|\to 0$ when $\varepsilon\to 0$, and a smooth map $\Lambda_\varepsilon:\Omega\to\mathbf{R}^n$, $\lambda\to\Lambda_\varepsilon(\lambda)=\lambda'$, such that for each $\lambda\in\Omega_\varepsilon$ there is a quasi-periodic solution*

$$z(t)=\sum_{k\in\mathbf{Z}^n}\widehat{z}(t)e^{i\langle k,\lambda'\rangle t}$$

such that (we denote $(e_1,\ldots,e_n)$ the canonical base of $\mathbf{Z}^n$, and $c>0$)

$$\begin{aligned}\widehat{z}_j(e_j)&=a_j, &&\forall 1\leqslant j\leqslant n,\\ |\widehat{z}_j(k)|&\leqslant e^{-c|k|}, &&\forall 1\leqslant j\leqslant n,\ \forall k\in\mathbf{Z}^n\\ \sum_{(j,k)\notin\mathcal{R}}|\widehat{z}_j(k)|&\leqslant\sqrt{\varepsilon},\end{aligned}$$

where $\mathcal{R}=\{(j,e_j);j=1,\ldots,n\}$ is the resonant set.

4.2. Lyapunov-Schmidt decomposition

The idea of the proof is not to try to localize the invariant torus, following Moser, nor to try to seek the domains where approximate first integrals exist, following Arnold. The new idea is to catch the solution directly using its Fourier expansion. The invariant torus is then obtained as the closure of this quasi-periodic solution. In some sense, this torus is constructed from the inside.

This is a very natural idea and it is not surprising that it yields a powerful approach.

How to find the perturbed solution? Again, the natural idea is to expand in Fourier series the equation

$$\frac{1}{i}\dot{z}_j - \lambda_j z_j + \varepsilon \frac{\partial H_1}{\partial \overline{z}_j} = 0,$$

then identify the Fourier coefficients of both sides. This gives for $k \in \mathbf{Z}^n$ and $1 \leqslant j \leqslant N$,

$$(\langle k, \lambda' \rangle - \lambda_j)\widehat{z}_j(k) + \varepsilon \widehat{\frac{\partial H_1}{\partial \overline{q}_j}}(k) = 0.$$

We use the first n frequencies as a parameter $\lambda' \in \mathbf{R}^n$.

If we request $\widehat{z}_j(e_j) = a_j$ as specified by the theorem for $1 \leqslant j \leqslant n$, we get the Q-equations

$$\lambda'_j - \lambda_j + \frac{\varepsilon}{a_j} \widehat{\frac{\partial H_1}{\partial \overline{q}_j}}(e_j) = 0.$$

The other equations, for the non-resonant indexes, are called the P-equations. The division of the equations into P and Q equations constitutes the Lyapounov-Schmidt decomposition. W. Craig and E. Wayne [C-W] used this method to find periodic solutions (1-dimensional tori) for PDE's. This already involves dealing with Small Divisors. The method is as follows. One first solves the P-equations, which consists in inverting a non-singular infinite dimensional linear operator to get an approximate solution $z(\lambda, \lambda')$. Then, one plugs this solution into the Q-equations to determine λ' in function of λ using the implicit function theorem.

4.3. Main ideas in the proof

As expected, things are not as simple as described. First, solving the P-equations with proper estimates is not an easy task. The P-equations correspond to non-resonant indexes and do have a formal solution, but small divisors appear in the inversion of the linear operator, so one must control the arithmetic of λ' to get the desired bounds. These restrictions on the arithmetic of λ' make that after hard work the good bounds are only obtained on a closed set Ω_ε with empty interior. To use the implicit function theorem in such a set demands careful justification. The procedure, well known in K.A.M., consists in finding successive approximations $z^{(l)}$, regular everywhere, of the P-equations. We can then solve the Q-equation at each step to get and approximate values of λ'.

In order to find the successive approximations $z^{(l)}$ one uses a rapidly convergent Newton scheme. At each step the P-equations are truncated by considering only those k's such that $||k||_\infty \leqslant N_l$, where the sequence of scales (N_l) increases geometrically. Thus, at each step we are faced with the inversion of a finite dimensional (of growing dimension) linear operator. More precisely, in order to get good estimates for $z^{(l+1)}$ one needs to control the inverse of the operator

$$T_l = D_l + \varepsilon S_l\,,$$

where D_l is a diagonal operator with eigenvalues $\pm\langle k, \lambda'\rangle - \lambda_j$ (some, the critical sites, are small and give small divisors), and S_l is a self-adjoint operator with exponentially decaying off-diagonal entries. More precisely, $z^{(l+1)}$ is defined by

$$z^{(l+1)} = z^{(l)} - T_l^{-1}(F(z^{(l)})),$$

where $F(z^{(l)})$ is the data on the P-equation when $z^{(l)}$ is plugged in.

Two main technical simplifications are present in [BOU5] with respect to the original approach [BOU1]. First, a multiscale analysis is performed on the complexification of the matrix T_l in order to obtain good bounds for its inverse. More precisely, the diagonal part is modified by shifting the eigenvalues of D_l into $\pm(\langle k, \lambda'\rangle + \sigma) - \lambda_j$ with a complex parameter σ. Using the analyticity on σ, plus some harmonic analysis estimates, and a multiscale argument incorporated in the induction, it is proved that the inverse has the proper bounds except for a small set of parameters σ. This constitutes the core of the estimate, and the common theme of [BOU5] which focus on the estimation of Lyapounov exponents. This type of multiscale analysis is well known in the theory of the discrete Schrödinger equation. The formal analogy between the two theories is brought a step closer with these common techniques.

The second step consists in showing that in the (λ, λ') plane not much is discarded in order to avoid these bad eigenvalues. One can write (using the Q-equation)

$$\lambda'_l = \lambda + \varepsilon\varphi_l(\lambda).$$

We want to avoid to have $\langle k, \varepsilon\varphi_l(\lambda)\rangle$ in the bad set of σ's, $\mathcal{S}_l$. One can say that in this step we adjust the arithmetic of λ' to avoid the small divisors. In previous works the arithmetic related to the resonant set was a major obstacle. For example, in [BOU4] it prevented the extension of the results to higher dimension, and it required arithmetic lemmas on the grouping and well separation of the critical sites. In [BOU5] these problems are treated with a more conceptual argument. With the previous approach, the main point is to notice that the set to be avoided by (λ, λ') is a semi-algebraic set for which we have an explicit control on the degree. This implies (in numerous senses) that the geometry is controlled. Since the condition becomes

$$\langle k, \varepsilon\varphi_l(\lambda)\rangle \notin \mathcal{S}_l,$$

with k large, the removed measure is small.

These ideas are very general and applicable to all sorts of PDEs, in particular to the non-linear Schrödinger equation and the wave equation in arbitrary dimension as treated in [BOU5].

4.4. Inversion of analytic matrices

We give the following illustrative example ([BOU5], Proposition 13.1; see also [B-G-S]), on how lower scale bad sites cannot affect the inverse for too many complex parameters, or, alternatively, how to use the space of holomorphy around to improve the bound on the bad set of parameters. We denote for $S \subset \mathbf{N}$, R_S the "restriction to indexes in S" linear operator.

THEOREM. — *Let $A(\sigma) \in M_d(\mathbf{C})$ be a real analytic matrix function for $\sigma \in [-\delta, \delta]$, holomorphic in*

$$\{|\Re\sigma| < \delta\} \cap \{|\Im\sigma| < \gamma\},$$

and such that

$$||A(\sigma)|| \leqslant B_1.$$

For each $\sigma \in [-\delta, \delta]$, there is a subset $\Lambda \subset [1, d]$ such that

$$|\Lambda| \leqslant M$$

and

$$|| \left(R_{[1,d]-\Lambda}\ A(\sigma)\ R_{[1,d]-\Lambda}\right)^{-1} || \leqslant B_2$$

We assume also that

$$|\{\sigma \in [-\delta, \delta]; ||A(\sigma)^{-1}|| \geqslant B_3\}| \leqslant 10^{-3}\gamma(1 + B_1)^{-1}(1 + B_2)^{-1}$$

Then for $\kappa < (1 + B_1 + B_2)^{-10M}$, we have

$$|\{\sigma \in [-\delta/2, \delta/2]; ||A(\sigma)^{-1}|| \geqslant \kappa^{-1}\}| \leqslant \exp\left(-\frac{c \log \kappa^{-1}}{M \log(M + B_1 + B_2 + B_3)}\right).$$

Combining this result with the exponential decrease of the off-diagonal terms, one can prove the following type of result that fits into the induction described in the previous section.

THEOREM 4.1 ([BOU5], 13.31). — *Let A be an $N \times N$ with exponential off-diagonal decay,*

$$|A(n, n')| \leqslant e^{-c_0|n-n'|}.$$

We consider the sub-scale $\overline{N} = N^\tau$, $0 < \tau < 1$.

We assume that for all intervals $J \subset [1, N]$, $|J| \geqslant \overline{N}$,

$$||(R_J A R_J)^{-1}|| \leqslant e^{L^b},$$

with $0 < b < 1$, $\tau + b < 1$.

An $\overline{N}$-interval J is said to be good if moreover

$$|A_J^{-1}(n, n')| \leqslant e^{-c|n-n'|}$$

for $n, n' \in J$ with $|n - n'| \geqslant \overline{N}/10$ (with $0 < c < c_0/10$).

Assume that there are at most N^b disjoint bad $\overline{N}$-intervals.

Then for $|n - n'| \geqslant N/10$,

$$|A^{-1}(n, n')| \leqslant e^{-c'|n-n'|}$$

where $c' = c - N^{-\kappa}$, $\kappa = \kappa(\tau, b) > 0$.

REFERENCES

[ARN] V.I. Arnold – Proof of a theorem of A.N. Kolmogorov on the conservation of quasi-periodic motions under a small change of the Hamiltonian function, *Uspekhi Mat. Nauk.* **18** (1963), no. 5, p. 13–40, Russ. Math. Surv. **18** (1963), no. 5, p. 9-36.

[BOS] J.-B. Bost – Tores invariants des systèmes dynamiques hamiltoniens (d'après Kolmogorov, Arnold, Moser, Rüssmann, Zehnder, Herman, Pöschel, ...), in *Sém. Bourbaki 1984-85*, Astérisque, vol. 133-134, Soc. Math. France, Paris, 1985, exp. n° 639, p. 113–157.

[BOU1] J. Bourgain – Construction of quasi-periodic solutions of Hamiltonian perturbations of linear equations and applications to nonlinear PDE, *International Math. Res. Notices* **11** (1994), p. 475–497.

[BOU2] ______, Non-linear Schrödinger equations, Park City Lectures, July 1995.

[BOU3] ______, On Melnikov's persistence problem, *Mathematical Research Letters* **4** (1997), p. 445–458.

[BOU4] ______, Quasi-periodic solutions of Hamiltonian perturbations of 2D-linear Schrödinger equations, *Ann. Math.* **148** (1998), p. 363–439.

[BOU5] ______, Green's function estimates for lattice Schrödinger operators and applications, Manuscript, 2001.

[B-G-S] J. Bourgain, M. Goldstein & W. Schlag – Anderson localization for Schrödinger operators on $\mathbf{Z}^2$ with quasi-periodic potential, Preprint, 2000.

[CHE] C.-Q. Chen – Lower dimensional invariant tori in the region of instability for nearly integrable Hamiltonian systems, *Comm. Math. Phys.* **203** (1999), no. 2, p. 385–419.

[CRA] W. Craig – *Problèmes de petits diviseurs dans les équations aux dérivées partielles*, Panoramas et Synthèses, vol. 9, Soc. Math. France, Paris, 2000.

[C-W] W. Craig & C.E. Wayne – Newton's method and periodic solutions of non-linear wave equations, *Comm. Pure Appl. Math.* **46** (1993), p. 1409–1498.

[ELI1] L.H. Eliasson – Perturbations of stable invariant tori for Hamiltonian systems, *Ann. Sco. Norm. Sup. Pisa, Sci. Fis. Mat., Ser. IV* **XV** (1988), no. 1, p. 115–147.

[ELI2] ______, Biasymptotic solutions of perturbed integrable Hamiltonian systems, *Bol. Soc. Mat. Bras.* **25** (1994), no. 1, p. 57–76.

[GRA] S.M. Graff – On the continuation of hyperbolic invariant tori for Hamiltonian systems, *J. Diff. Eq.* **15** (1974), p. 1–69.

[K-P] T. Kappeler & J. Pöschel – KDV and KAM, Book to appear in Springer-Verlag.

[KOL] A.N. Kolmogorov – On the conservation of conditionally periodic motions for a small change in Hamilton's function, *Dokl. Acad. Nauk. SSSR* **98** (1954), p. 525–530.

[KUK1] S.V. Kuksin – *Nearly integrable infinite-dimensional systems*, LNM, vol. 1556, Springer-Verlag, 1991.