A. W. Goldie

Non-commutative Dedekind rings

Séminaire Dubreil. Algèbre et théorie des nombres, tome 21, nº 2 (1967-1968), exp. nº 20, p. 1-4

<http://www.numdam.org/item?id=SD_1967-1968__21_2_A11_0>

Séminaire DUBREIL-PISOT
(Algèbre et Théorie des nombres)
21e année, 1967/68, n° 20

27 mai 1968

NON-COMMUTATIVE DEDEKIND RINGS

by A. W. Goldie

An account is given of work of J. C. ROBSON ([2]) in which the development of the theory of maximal orders due to ASANO is generalised to the case when the order is not bounded. Thus, for example, this brings within the scope of the work the case of simple rings which are right noetherian and right hereditary.

A number of definitions is needed at the outset, many of these are known (see N. JACOBSON [1]), but are given for convenience.

(D1) A <u>quotient ring</u> is a ring with 1 in which every regular element is a unit.

(D2) A <u>right order</u> R in a quotient ring Q is a subring of Q such that each element of Q has the form

$$ab^{-1}\ ,\quad a\ ,\ b \in R\ ,\quad b \text{ is regular in } R\ .$$

(D3) Right orders R , S in Q are <u>equivalent</u> $(R \sim S)$ if units $c , d , e , f \in Q$ exist with $cRd \subset S$, $eSf \subset R$.

(D4) Right orders R , S in Q are <u>right equivalent</u> $R \overset{r}{\sim} S$ if units $a , b \in Q$ exist with $aR \subset S$, $bS \subset R$.

<u>Left equivalence</u> $(\overset{\ell}{\sim})$ is defined likewise and requires

$$Ra \subset S\ ,\qquad Sb \subset R\ .$$

LEMMA 1. - *Let* R , S *be right orders in* Q *and* $R \subset S$, $R \sim S$. *Then right orders* T , T' *exist with*

$$R \subset T \subset S\ ,\qquad R \overset{\ell}{\sim} T \overset{r}{\sim} S\ ,$$
$$R \subset T' \subset S\ ,\qquad R \overset{r}{\sim} T' \overset{\ell}{\sim} S\ .$$

<u>Proof</u>. - Set $T = R + aS + RaS$, where $ab^{-1} S\, cd^{-1} \subset R$, $a , b , c , d \in R$.

(D5) A right order which is not properly contained in an equivalent order is said to be <u>maximal equivalent</u> or <u>maximal</u> $\sim$. Similarly for <u>maximal</u> $\overset{r}{\sim}$ and $\overset{\ell}{\sim}$.

(D6) R a right order in Q . An R-submodule $I \subset Q_R$ is a <u>fractional right R-ideal</u> if :

(1) I contains a unit of Q ;

(2) $bI \subset R$ for some unit $b \in Q$.

If $I \subset R$, then I is an integral right R-ideal. Thus an ordinary ideal of R and an integral ideal of R may differ in that the latter must have a regular element.

(D7) I a right R-ideal, set

$$O_r(I) = \{q \in Q \; ; \; I_q \subset I\} \; ; \qquad \text{the right order of } I \; ,$$

$$O_\ell(I) = \{q \in Q \; ; \; qI \subset I\} \; ; \qquad \text{the left order of } I \; .$$

LEMMA 2. - $O_r(I)$ and $O_\ell(I)$ are right orders equivalent to R .

Thus, a maximal $\sim$ order must have 1 , using lemma 2.

(D7') The inverse I^{-1} of I is defined by

$$I^{-1} = \{q \in Q \; ; \; IqI \subset I\} = \{q \in Q \; ; \; Iq \subset O_\ell(I)\} = \{q \in Q \; ; \; qI \subset O_r(I)\} \; .$$

Thus I^{-1} is a left $O_r(I)$-ideal and a right $O_\ell(I)$-ideal.

All above definitions, etc., can be applied to left orders, left ideals, etc.

LEMMA 3. - Let R be a right order in Q , and I a right R-ideal. Then $O_\ell(I) = II^{-1}$ if, and only if, I is a projective right $O_r(I)$-ideal. In this case I is finitely generated over $O_r(I)$.

Proof. - See ROBSON (loc. cit., and for later proofs) :

LEMMA 4. - Let R be a maximal right order in Q .

(i) Let I be a right R-ideal, then I is projective over R if, and only if, $II^{-1} = O_\ell(I)$, and then I is finitely generated over R .

(ii) Let T be a two-sided R-ideal. Then T is projective over R as a right ideal if, and only if, $TT^{-1} = R$, and then T is finitely generated over R .

Note. - Let R be a simple ring with 1 , which is a right order in Q . Then R is a maximal $\sim$ order.

For if $S \supset R$, $S \overset{r}{\sim} R$, then $cS \subset R$ for some unit $c \in Q$. Then RcS is an ideal of R , and $RcS = R$. Then $RS = RcS^2 = RcS = R$. However $S \subset RS$, so that $S = R$. A similar argument holds when $S \overset{\ell}{\sim} R$, and the case $S \sim R$ is a composition of $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$.

(D8) A right order R with 1 is an Asano right order if the R-ideals form a group under multiplication.

The main theorem on Asano orders is the following :

THEOREM 5. - *Let* R *be a right order with* 1 *in a quotient ring* Q . *The following properties are equivalent* :

(1) R *is an Asano right order* ;

(2) R *is a maximal right order, and every integral* R-*ideal is a projective right* R-*ideal* ;

(3) *For every integral* R-*ideal* T , *there exists an* R-*ideal* T^* *with*

$$TT^* = T^* T = R \quad ;$$

(4) *The* R-*ideals form an abelian group under multiplication*.

Proof. - See ROBSON (loc. cit.).

Examples show that for many purposes the concept of an Asano right order is not precise enough, since it permits of an analysis of the ideal structure, but not of the one-sided ideal structure. Accordingly it has nothing to say for simple rings. The ideas are next made more precise by the following definition.

(D9) A *right Dedekind right order* is a maximal right order R in which every integral right ideal is projective.

For brevity refer to this type as an RD-order.

Example : The polynomial ring $F[x , y]$, F a field of zero characteristic, such that $xy - yx = 1$ is a noetherian, hereditary domain and is simple. It is an RD-order, but is not a bounded order in the sense of Asano.

We next look at the structure of an RD-order R in a quotient ring Q .

LEMMA 6. - *If* I *is a right* R-*ideal, then* $O_\ell(I)$ *is an* RD-*order. If* S *is a maximal right order and* $S \sim R$, *then* $S = O_\ell(I)$ *for some right* R-*ideal* I .

THEOREM 7. - *Let* R *be an* RD-*order. The class of all right* S-*ideals, where* S *runs through all maximal right orders equivalent to* R , *forms a Brandt groupoid under multiplication where defined*.

For Brandt groupoïds, see argument N. JACOBSON (loc. cit.).

In the special case when R is a right order in a simple Artin Q , we see that the integral right ideals are the essential right ideals by Goldie's theorem and each right ideal of R is projective and finitely generated. Thus an RD-order, which is a prime ring, is just a right noetherian, right hereditary, prime ring. The main result is as follows.

THEOREM 8. - The following conditions on a ring R are equivalent :

(1) R is a right Dedekind prime ring ;

(2) $R \cong eK_n e$, $n > 0$, $e = e^2 \in K_n$;

(3) R is Morita equivalent to K ;

(4) $R \cong \mathrm{End}_K(P)$ for some finitely generated projective K-module P ;

(5)
$$R \cong \begin{pmatrix} 0_\ell(I_1) & I_1 I_2^{-1} & \cdots & I_1 I_m^{-1} \\ I_2 I_1^{-1} & 0_\ell(I_2) & \cdots & \\ \vdots & & & \\ I_m I_1^{-1} & & \cdots & 0_\ell(I_m) \end{pmatrix}$$

for right ideals $I_1 , \dots , I_m$ of K . Here K is a right Dedekind domain.

Example : Set $K = \mathbb{Z}[\sqrt{-5}]$. It is a Dedekind domain. Now

$$I = 2K + (1 + \sqrt{-5})K$$

is a non-principal ideal of K . Take the K-module $M = I \oplus K$, and set $R = \mathrm{End}_K(M)$. M is projective, so that R is a right Dedekind prime ring.

R is not a complete matrix ring, for it would have to be of the form $R = S_2$ and then $M \cong J \oplus J$ for some $J \subset K$ and $S \cong \mathrm{End}_K(J)$. Then $J^2 \cong IK = I$, and J^2 is always a principal ideal of K ; it is a contradiction.

BIBLIOGRAPHY

[1] JACOBSON (Nathan). - The theory of rings. - New York, American mathematical Society, 1943 (Mathematical Surveys, 2).

[2] ROBSON (J. C.). - Non-commutative Dedekind rings, J. of Algebra, t. 9, 1968, p. 249-265.